AN AID TO SOLVING PROBLEMS IN STRENGTH OF MATERIALS

ИЗДАТЕЛЬСТВО • МИР •

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ПОСОБИЕ
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AN AID TO SOLVING PROBLEMS IN STRENGTH OF MATERIALS

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CONTENTS

NOTATION

A = reaction at support; amplitude of forced vibrations.

a = length of portion of rod; dimension of cross section; acceleration of centre of gravity of a mass.

B = reaction at support; width of cross section; width of cross plates in built-up columns.

b = length of portion of rod; width of cross section.

C = stiffness (rigidity) of bar or system.

°C = degrees, Centigrade.

D, d = diameter.

E = Young's modulus of elasticity in tension or compression.

e = distance determining the position of the shear (flexural) centre; distance from neutral axis to centre of gravity of a curved beam in bending.

F, F_x = cross-sectional area of rod.

F_{sh}, F_{br} = area subject to shear, to bearing action.

f, f_x = deflection of beam.

G = modulus of elasticity in shear or rigidity modulus.

g = acceleration of gravity.

h = height of rectangular cross section; height of drop of load.

i_z, i_y, i_u, i_v = radii of inertia with respect to the z-, y-, u-, v-axis.

i = minimum radius of inertia of a cross section.

I_z, I_y, I_u, I_v = axial moments of inertia of the area of a figure with respect to the z-, y-, u-, v-axis.

I_p = polar moment of inertia of the area of a figure.

$I_{\substack{max \\ min}}$ = principal moments of inertia of the area of a figure.

I_t = geometrical characteristic of torsional rigidity of a section.

I_m = moment of inertia of mass with respect to axis of rotation.

I_{zy} = product of inertia of the area of a figure with respect to the z- and y-axis.

k = shape factor taking into account the effect of the shape of a cross section on the strain energy due to a transverse force; coefficient taking into account the curvature of a coil of a helical spring and the effect of a transverse force.
k_d = dynamic factor.
k_m = mass reduction factor.
l = length of bar (or its portion).
l_r = reduced length of a compressed bar.
Δl = total change in bar length.
l_0 = free length of the portion of a built-up column between cross plates.
M = moment of an external concentrated couple of forces.
M_t = torque (twisting moment).
M, M_x = bending moment.
$M_{\max}$ = maximum bending moment (in absolute value).
M_{f_x} = bending moment of fictitious beam.
M_z, M_y = bending moments with respect to the principal centroidal axes of inertia z and y of a cross section.
M_{eqI}, M_{eqII}, M_{eqIII}, M_{eqIV}, M_{eqV} = equivalent (design) bending moments according to the various strength theories.
$\overline{M}$, $(\overline{M}_t)$ = bending (twisting) moment due to a unit generalized force.
M_r = reduced moment in beams of variable cross section.
M_d = dynamic moment.
m = linear moment of external force couples uniformly distributed over a length; mass of load, of bar.
m_r = reduced mass.
N, N_x = axial force; power in horsepower, watts, kilowatts; oscillation frequency (1/sec); number of cycles.
$\overline{N}$ = effort exerted by a unit generalized force.
N_d = dynamic axial force.
n = revolutions per minute (rpm); factor of safety.
n_y and n_u = safety factors based on the yield point and ultimate strength.
$[n]$ = allowable factor of safety.
n_{st} = stability factor of safety.
P = concentrated force.
P_{cr} = critical force.
P_i = generalized forces.
P_f = fictitious generalized force.
P_d = dynamic force.

P_{dis} = disturbing force.
P_0 = amplitude of disturbing force.
p = intensity of distributed force over an area; pressure; total (resultant) stress.
p_0 = octahedral resultant stress; contact pressure between built-up cylindrical tubes.
p_{max}, p_{min} and p_m = maximum, minimum and mean stress in a cycle.
p_a = amplitude of cycle.
p^r_{max}, p^r_m, p^r_a = maximum stress, mean stress and amplitude of cycle at the fatigue limit.
p_r = fatigue limit.
p_{-1} = fatigue limit in a symmetrical cycle.
$[p_r]$ = permissible stress in a cycle with the asymmetry factor r.
$[p_{-1}]$ = permissible stress in a symmetrical cycle.
Q, Q_x = weight of bar, load; shearing force; transverse force in a cross section.
Q_f = transverse force of a fictitious beam.
Q_y, Q_z = transverse forces acting along axes y, z.
$\overline{Q}$ = transverse force due to unit generalized force.
Q_0 = weight of body subject to impact.
q, q_x = intensity of load distributed over a length.
q = factor of sensitivity of material to stress concentration.
r = radius; radius of curvature of neutral layer of a curved beam; asymmetry factor of a cycle.
S_z, S_y (S) = static moments of the area of a figure with respect to (neutral) axes z, y.
s = arc length; length of a curved line.
ds = elementary length of geometric axis of a portion of a rod.
T = period of vibration; kinetic energy.
t° = Centigrade temperature of a system or bar.
t = time.
U = elastic strain energy.
u, u_{dist}, u_{vol} = elastic strain energy per unit volume; its portions spent on change of shape (distortion) and of volume.
V, v = initial volume; axis.
ΔV, $\frac{\Delta V}{V}$ = total, relative change in volume.
W_p = polar section modulus of round and annular cross sections.
W_t = section modulus of a cross section in torsion.
W (W_1, W_2) = section modulus of the cross section of a beam with

respect to neutral axis (axial section moduli for stretched, compressed fibre).

$X_1, X_2, \ldots, X_n$ = redundant unknowns.
x, y, z = axes; coordinates.
α = angle; coefficient of linear expansion of material; coefficient of distribution of stresses of a curved beam in flexure; stress-concentration factor.
α_{eff} = effective stress-concentration factor.
β = angle; oscillation built-up factor.
γ = weight of unit volume of a material; unit shearing deformation.
$\gamma_1, \gamma_2, \gamma_3$ = principal angles of shear.
γ_0 = octahedral shear.
Δ = linear magnitude denoting error in the manufacture of an elastic element of a system; interference in the components of built-up tubes.
δ = elastic generalized displacement; static displacement.
$\delta_z, \delta_y, \delta_v, \delta_h$ = projections of displacement on the z- and y-axis on the vertical and horizontal.
$[\delta]$ = allowable displacement of a point.
$\delta_{ip}, \delta_{ii}, \delta_{ik}$ = generalized displacements; coefficients in equations of the strength method.
δ_d = dynamic displacement.
δ_0 = generalized displacement of the point of suspension of a load due to statical effect of a disturbing force.
ε = linear deformation (unit strain), (unit elongation).
$\varepsilon_1, \varepsilon_2, \varepsilon_3$ = principal linear deformations (strains).
ε_t' = unit lateral deformation (strain).
ε_{se} = size effect factor.
ε_{sf} = surface quality factor.
θ, θ_x = angle of rotation of cross section of a beam.
θ_d = dynamic angle of rotation.
K = bulk modulus.
χ = coefficient of compressibility of a material.
λ = slenderness ratio of a bar.
λ_c, λ_l = slenderness ratio of a column, component (leg) of built-up column.
μ = Poisson's ratio for a material; length reduction factor for a compressed bar.

ρ = distance from centre; radius of curvature of the geometric axis of a curved beam.

ρ_t, ρ_m = radii of curvature of circumferential and meridional cross sections of the wall of a vessel.

σ, σ_x, σ_α = normal stresses.

σ_1, σ_2, σ_3 = principal stresses at the point being considered.

σ_l = proportional limit.

σ_y = yield point.

σ_u = ultimate strength (tensile strength).

$[\sigma]$ = allowable stress in tension and compression.

σ_0 = octahedral normal stress.

$[\sigma_t]$, $[\sigma_c]$, $[\sigma_f]$, $[\sigma_{br}]$ = allowable normal tensile, compressive, flexural and bearing stresses.

σ_{eq_I}, $\sigma_{eq_{II}}$, ..., σ_{eq_V} = equivalent (design) stresses according to the strength theories (first, second,..., and fifth theory) (Mohr's hypothesis).

σ_{cr} = critical stress.

$[\sigma_{st}]$ = allowable ensuring stability stress.

σ_r, σ_t = radial and tangential normal stresses in a thick-walled tube.

σ_t, σ_m = circumferential and meridional stresses in thin-walled vessels.

σ_r = fatigue limit.

$\sigma_{\max}$, σ_m, σ_a = maximum, mean normal stresses and amplitude of a cycle.

σ_{-1} = endurance limit in symmetrical flexure (completely reversed cycle).

σ_{-1p} = fatigue limit for symmetrical axial tension and compression (completely reversed cycle).

$[\sigma_r]$ = allowable normal stress in cycle with asymmetry factor r.

$[\sigma_{-1}]$ = allowable normal stress in symmetrical (completely reversed) cycle.

σ_d = dynamic normal stress.

τ_1, τ_α = shearing stress.

$\tau_{\max}$ = maximum shearing stress.

τ_1, τ_2, τ_3 = extremal shearing stresses.

τ_0 = octahedral shearing stress.

$[\tau]$ = allowable stress in shear.

τ_d = dynamic shearing stress.

τ_r = endurance limit in torsion.

τ_{max}, τ_m,
τ_a = maximum, mean shearing stress and amplitude of a cycle.

τ_{-1} = endurance limit for completely reversed (symmetrical) torsion.

$[\tau_r]$ = allowable stress in torsion for a cycle with asymmetry factor r.

$[\tau_{-1}]$ = allowable stress in symmetrical (completely reversed) cycle.

φ = total angle of twist; allowable stress reduction factor in buckling.

$[\varphi]$ = allowable angle of twist.

ω = area of bending moment diagram; angular velocity; circular frequency of oscillation.

ω_0 = circular frequency of change in the disturbing force.

UNITS OF MEASUREMENT

INTERNATIONAL SYSTEM OF UNITS (SI)

m, kg, sec = metre, kilogram, second; units of length, mass, time (fundamental units).

cm, mm = submultiple units of length (centimetre, millimetre).

N = newton, unit of force (1 N $\cong$ 1/9.81 kgf = 0.102 kgf).

kN, MN,
daN = multiple units of force: kilonewton, meganewton, dekanewton (daN) (1 kN = 10^3 N; 1 MN = 10^6 N; 1 daN = = 10 N $\cong$ 1.02 kgf).

N/m² = unit of stress and pressure (1 N/m² $\cong$ 1.02 $\times$ $\times$ 10^{-5} kgf/cm²).

MN/m² = multiple unit of stress and pressure (1 MN/m² = = 10^6 N/m² = 10.2 kgf/cm²).

J = joule, unit of work (1 J $\cong$ 1/9.81 kgf/m $\cong$ 0.102 kgf/m).

W, kW = watt, kilowatt, units of power (1 kW $\cong$ 102 kgf-m/sec $\cong$ $\cong$ 1.36 hp).

METRE-KILOGRAM (FORCE)-SECOND SYSTEM

m, kgf,
sec = metre, kilogram-force, second (1 kgf $\cong$ 9.81 N).

tnf = ton-force (1 tnf = 10^3 kgf $\cong$ 9.81 $\times$ 10^3 N = 9.81 kN).

kgf/cm² = unit of stress and pressure (1 kgf/cm² $\cong$ 9.81 $\times$ 10^4 N/m² $\cong$ $\cong$ 0.0981 MN/m²).

bar = nonsystem unit of pressure (1 bar = 10^5 N/m² $\cong$ $\cong$ 1.02 kgf/cm²).

kgf-m = unit of work (1 kgf-m $\cong$ 9.81 J).

hp = nonsystem unit of power (metric horse power) (1 hp = = 75 kgf-m/sec $\cong$ 0.736 kW).

GENERAL REMARKS ON THE PROBLEMS

1. The quantities indicated in the drawings for all the problems are considered to be the given ones; the sought-for quantities (wherever necessary) are followed by question marks.
2. Problems stated in the International System of Units (SI) are to be solved in this system.
3. All dimensions in drawings are in millimetres unless otherwise indicated.
4. Elements shown in drawings with two lines and hatched are considered to be absolutely rigid.
5. All elements in compression are considered sufficiently stable (unless otherwise specified).

CHAPTER 1. TENSION AND COMPRESSION

1.1.

Axial Force

The resultant of the normal elastic forces at a section is called the *axial force*. The axial force is determined by the method of sections. The magnitude of the axial force N_x at any cross section of a rod is equal to the algebraic sum of all external axial forces (concentrated ones P and those distributed according to an arbitrary law with the intensity q_x) that act on the rod at one side of the section under consideration. Tensile forces are considered to be positive, and compressive ones negative.

The general formula for determining an axial force at an arbitrary cross section of a rod has the following form:

$$N_x = \sum P + \sum \int q_x \, dx \tag{1}$$

Integration is to be performed over the length of each portion on which a distributed force acts, and summation over all portions situated on one side of the section under consideration.

If the vector of the axial force N_x is directed outwards from the considered section, then the conditions of equilibrium of the cut-away portion of the rod, i.e. formula (1) will give the amount and appropriate sign of the force.

Example 1. Given: $P_1 = P$, $P_2 = 3P$, $P_3 = 2P$; and the distributed load q_x changes according to a linear law from $q = 0$ to $q = \frac{P}{a}$ (Fig. 1).

Plot the diagram for N.

Solution. Cutting the rod at cross sections in each portion and applying formula (1) we obtain the following values of the axial forces:

$$N_1 = -P_1 = -P; \; N_2 = -P_1 + P_2 = -P + 3P = 2P;$$

$$N_3 = -P_1 + P_2 - \int_0^x P \frac{x}{2a^2} \, dx = -P + 3P - P \frac{x^2}{4a^2}$$

$$= P\left(2 - \frac{x^2}{4a^2}\right);$$

$$N_{3_{x=0}} = 2P; \quad N_{3_{x=a}} = \frac{7}{4} P;$$

$$N_4 = -P_1 + P_2 - \int_0^x P\frac{x}{2a^2}\,dx - P_3 = -P\frac{x^2}{4a^2};$$

$$N_{4_{x=a}} = -\frac{P}{4};\quad N_{4_{x=2a}} = -P;$$

$$N_5 = -P_1 + P_2 - \int_0^{2a} P\frac{x}{2a^2}\,dx - P_3 = -P$$

The diagram for N is shown in Fig. 1.

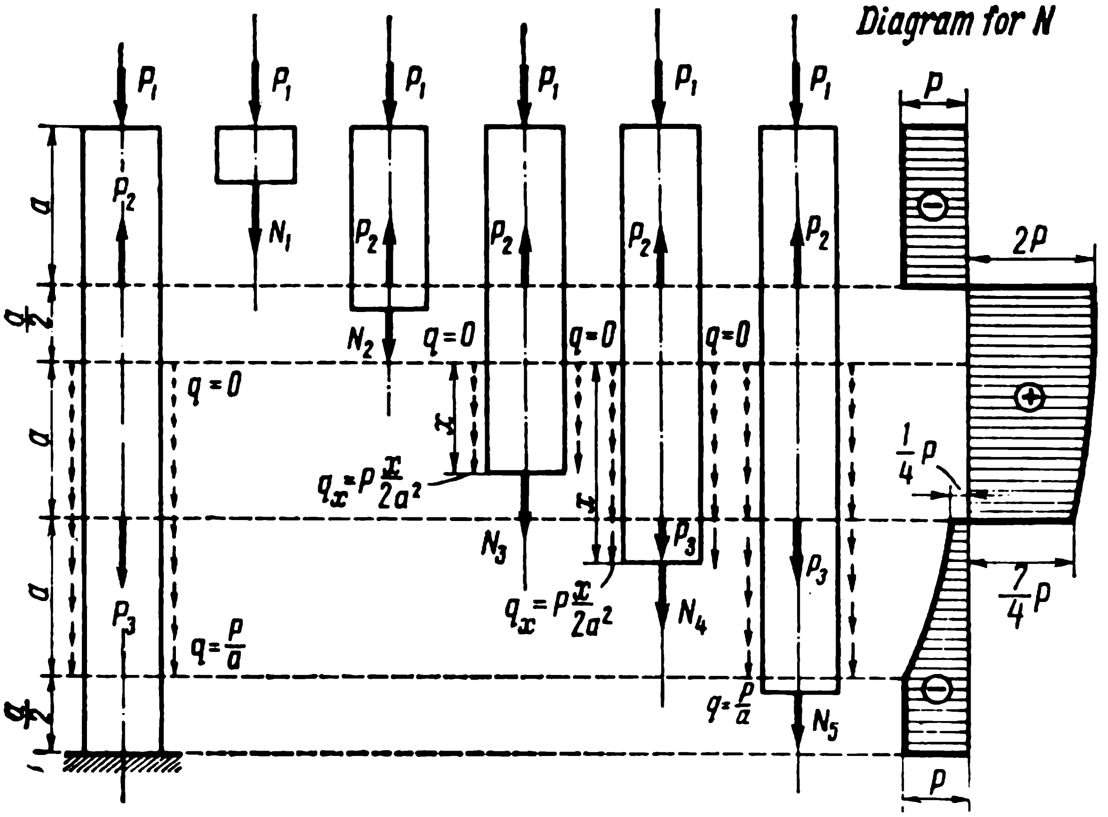
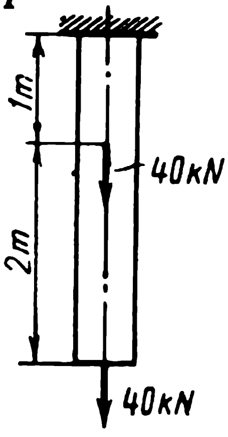


Fig. 1

Problems 1 through 8. Draw diagrams for the axial force N.
In Problems 6, 7 and 8 assume that the intensity of the distributed load q_x varies linearly.

1



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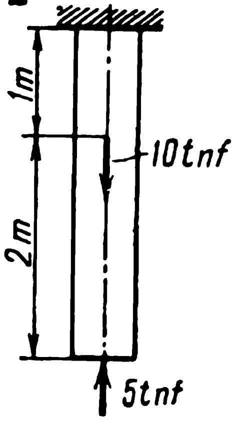

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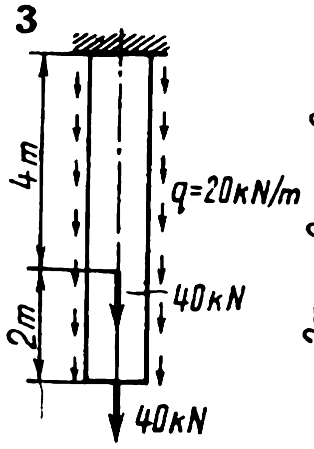

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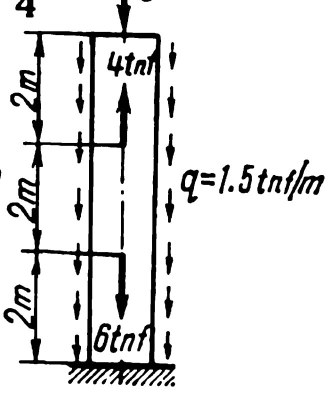
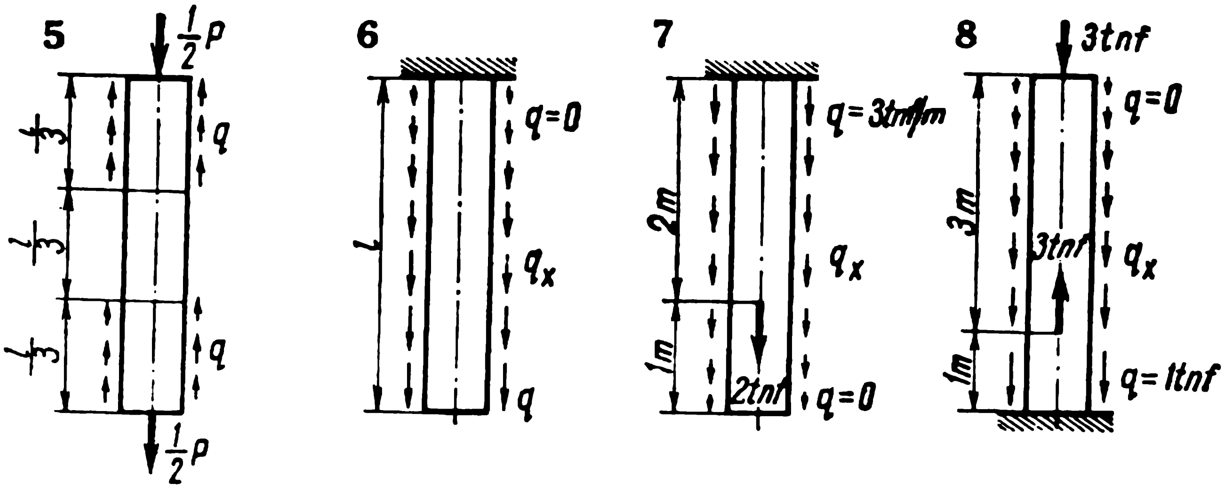




1.2.

Normal Stresses, Total Elongation and Potential Energy

It is assumed that the normal stresses σ_x are uniformly distributed over all cross sections of rods subject to tension or compression (the same assumption is approximately valid for rods of variable cross section).

Therefore the normal stress at an arbitrary cross section of a rod is the ratio of the axial force N_x at this section to its area F_x, i.e.

$$\sigma_x = \frac{N_x}{F_x} \tag{2}$$

Assuming that the material of the rods obeys Hooke's law, we can find the total elongation of a rod by the following general formula:

$$\Delta l = \sum \int \frac{N_x\,dx}{EF_x} \tag{3}$$

where E is the modulus of elasticity (Young's modulus) of the material.

Integration is to be performed over the length of each portion, and summation over all the portions of the rod.

If N and F are constant over the length l of a rod, then

$$\Delta l = \frac{Nl}{EF}$$

The general formula for determining the amount of elastic strain energy U, accumulated in a rod in tension or compression, has the form:

$$U = \sum \int \frac{N_x^2\,dx}{2EF_x} \tag{4}$$

Integration and summation are to be performed here in the same manner as in determining the elongation of a rod.

Since, within the limits of elasticity of a material, the amount of potential energy may be considered to be equal to the work done by external forces, the strain energy for rods in tension or compression due to forces P, applied at the ends, is

$$U=\frac{1}{2}P\Delta l \qquad (5)$$

Example 2. Given: $P = 10$ kN; $l = 0.3$ m; $d = 0.01$ m; $d_x = (0.01 + x^2)$ m; and $E = 2\times 10^5$ MN/m^2 (Fig. 2).

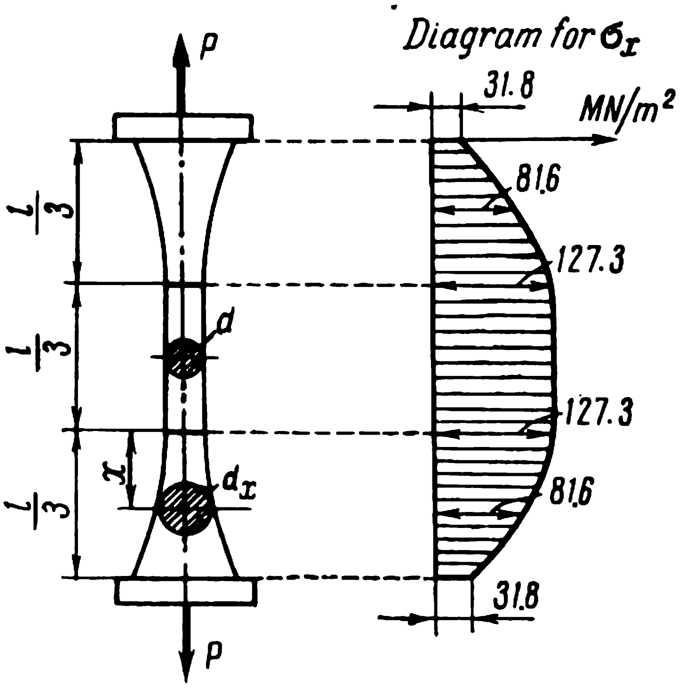


Fig. 2

Construct the diagram for σ_x and determine Δl and U.

Solution. The axial force at any cross section is $N_x = P = 10$ kN. The areas of cross sections are: in the cylindrical portion

$$F=\frac{\pi d^2}{4}=0.25\pi\times 10^{-4}\ \text{m}^2;$$

in the portions of variable diameter

$$F_x=\frac{\pi d^2 x}{4}=\frac{\pi}{4}(0.01+x^2)^2\ \text{m}^2$$

The normal stresses are: in the cylindrical portion

$$\sigma_x=\frac{N_x}{F}=\frac{10^4}{0.25\pi\times 10^{-4}},\ \text{or}\ \sigma_x\cong 1.273\times 10^8\ \text{N/m}^2=127.3\ \text{MN/m}^2;$$

in the portions of variable diameter

$$\sigma_x=\frac{N_x}{F_x}=\frac{4P}{\pi(0.01+x^2)^2}$$

$$=\frac{4\times 10^4}{\pi(0.01+x^2)^2}\ \text{N/m}^2\cong\frac{127.3}{(1+100x^2)^2}\ \text{MN/m}^2;$$

$$\sigma_{x=0}=127.3\ \text{MN/m}^2;\ \sigma_{x=\frac{l}{6}}=\frac{127.3}{(1+100\times 0.05^2)^2}\cong 81.6\ \text{MN/m}^2;$$

$$\sigma_{x=\frac{l}{3}}=31.8\ \text{MN/m}^2$$

The diagram for σ_x is given in Fig. 2.

From formula (3) the total elongation of the rod is:

$$\Delta l=\sum\int\frac{N_x\,dx}{EF_x}=\frac{Pl}{3EF}+\frac{2P\times 4}{\pi E}\int\limits_0^{\frac{l}{3}}\frac{dx}{(0.01+x^2)^2}$$

$$=\frac{Pl}{3EF}+\frac{8P}{\pi E}\left|\frac{x}{2(0.01+x^2)}\times\frac{1}{0.1^2}\right.$$

$$\left.+\frac{1}{2\times 0.1^3}\arctan\frac{x}{0.1}\right|_0^{0.1}=\frac{10\times 10^3\times 0.3}{3\times 2\times 10^{11}\times 0.25\pi\times 10^{-4}}$$

$$+\frac{8\times 10^4}{\pi\times 2\times 10^{11}}\left(\frac{0.1}{2(0.01+0.1^2)\times 0.1^2}+\frac{1}{2\times 0.1^3}\arctan 1\right);$$

$$\Delta l\cong 1.46\times 10^{-4}\ \mathrm{m}=0.0146\ \mathrm{cm}$$

From formula (5), the amount of elastic strain energy in the rod is

$$U=\frac{P\Delta l}{2}=\frac{10^4\times 1.46\times 10^{-4}}{2}=0.73\mathrm{J}\cong 7.45\ \mathrm{kgf\text{-}cm}$$

Problems 9 through 16. Construct diagrams of the normal stresses σ; determine the absolute changes Δl in the rod lengths and find the strain energies U accumulated in the rods.

Assume that $E = 2\times 10^5$ MN/m². In Problems 11 through 14 assume that $E = 2\times 10^6$ kgf/cm².

1.3.

Lateral Deformation and Volume Change

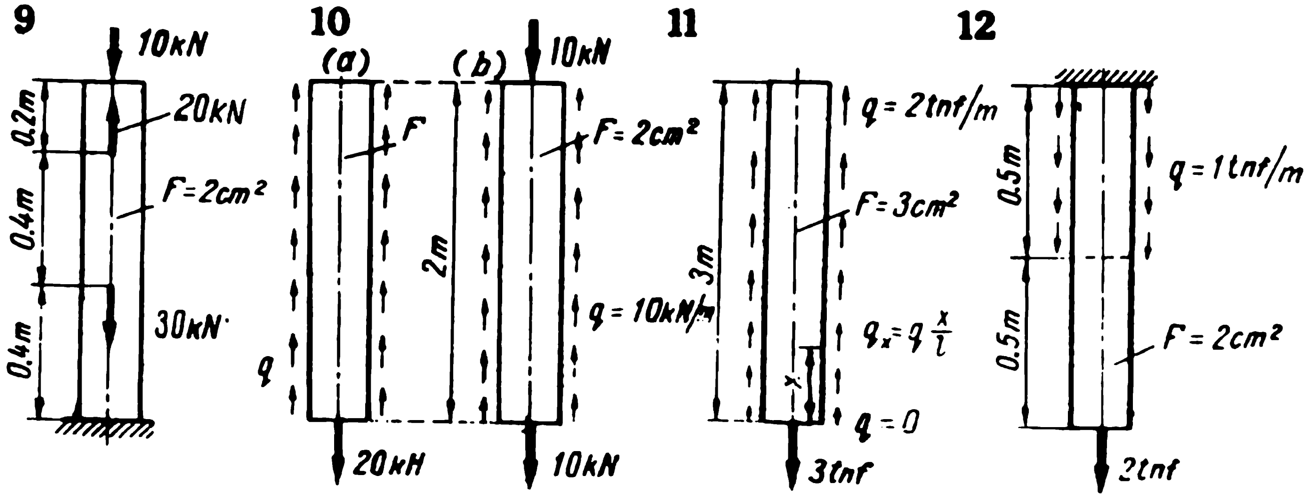
According to Hooke's law, unit axial tensile or compressive strain is equal to

$$\varepsilon=\frac{\sigma}{E}\tag{6}$$

and unit lateral deformation to

$$\varepsilon'=-\mu\varepsilon=-\mu\frac{\sigma}{E}\tag{7}$$

where μ is Poisson's ratio of the material.



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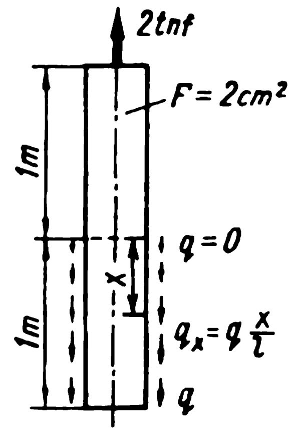

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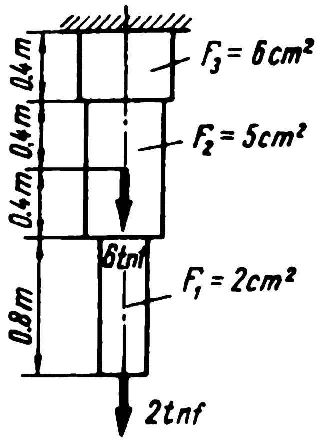

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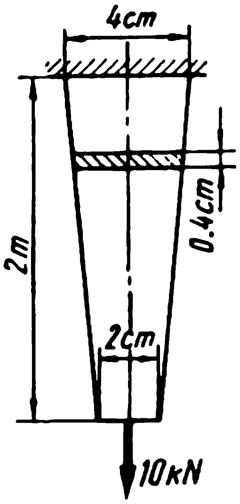

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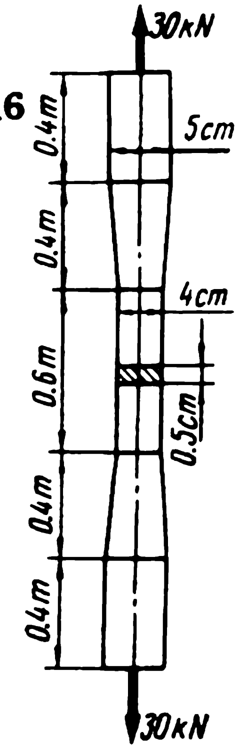

The unit change in the area of a cross section of a rod can be determined by the formula

$$\frac{\Delta F}{F} \cong -2\mu\varepsilon = -2\mu\frac{\sigma}{E} \tag{8}$$

The unit change in the volume of the rod is determined from the expression

$$\Delta V = \frac{(1-2\mu)}{E}\sum\int N_x\,dx \tag{9}$$

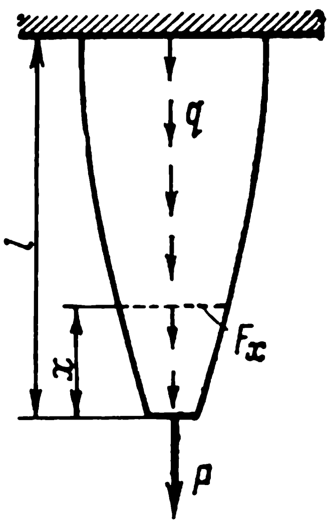

Fig. 3

Integration is to be performed over the length of each portion, summation over all the portions.

If a rod is stretched or compressed by the forces P applied to its ends, then

$$\Delta V = \frac{(1-2\mu)}{E}Pl \tag{10}$$

Example 3. Given: P, q, l, F_x, E and μ (Fig. 3). Find ε_x, $\frac{\Delta F_x}{F_x}$ and ΔV.

Solution. According to formulas (1) and (2) the axial force and normal stress at an arbitrary cross section will be:

$$N_x = P + qx; \quad \sigma_x = \frac{N_x}{F_x} = \frac{P+qx}{F_x}$$

Since, according to Hooke's law, unit elongation is expressed by $\varepsilon_x = \frac{\sigma_x}{E} = \frac{P+qx}{EF_x}$, it follows from formula (8), that the unit change

in the cross-sectional area of the rod equals

$$\frac{\Delta F_x}{F_x} = -2\mu \frac{\sigma_x}{E} = -2\mu \frac{P+qx}{EF_x}$$

Using formula (9), we determine the total volume change of the rod. Thus

$$\Delta V = \frac{(1-2\mu)}{E} \int_0^l N_x \, dx = \frac{(1-2\mu)}{E} \int_0^l (P+qx)\, dx$$

$$= \frac{(1-2\mu)}{E} \left(P + \frac{ql}{2}\right) l$$

Problems 17 through 24. Find the quantities indicated on the accompanying diagrams.

In Problem 24 assume that $E = 2 \times 10^6$ kgf/cm^2 and $\mu = 0.3$ for steel.

1.4.

Displacements of Points in Hinged-End Bar Systems

Elastic displacements of points in a system of hinged-end bars are determined by the following general procedure.

The axial forces in all the elastic elements of the system are found from the equations of statics. The total elongations of the elements are determined by Hooke's law.

Assuming that the elements of the system are not disconnected during deformation, the method of intersections can be used to set up the equations of combined displacements, i.e. the geometrical relations between the displacements of the elements that make up the system. The required displacement is then determined from these relationships.

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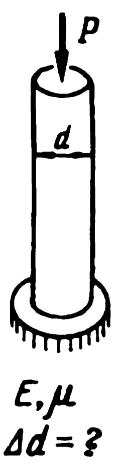

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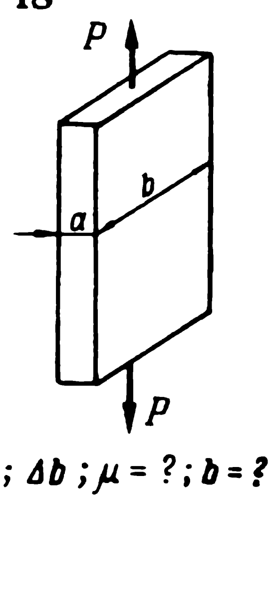

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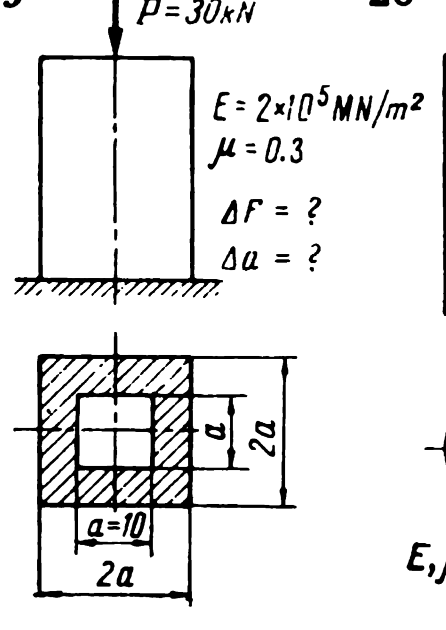

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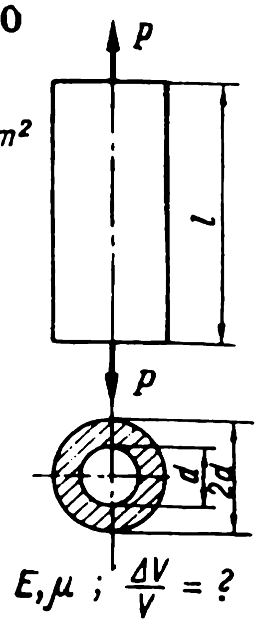

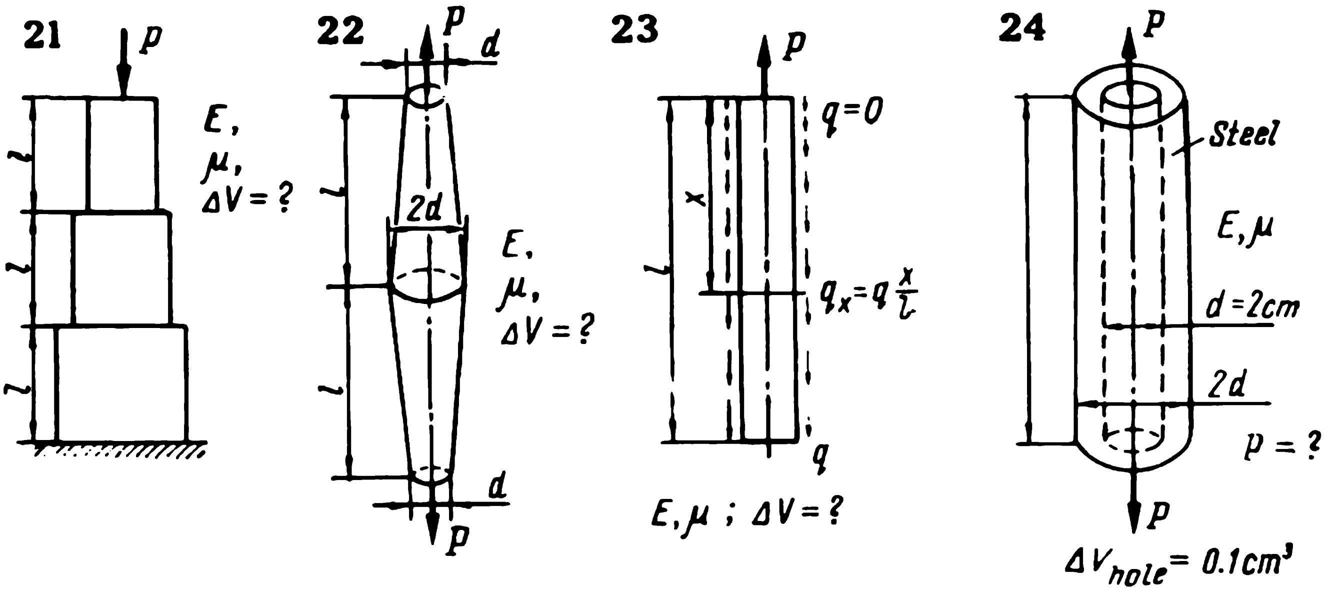

When using the method of intersections (or of arcs) one should bear in mind that each element of a system is subject to axial deformation and can also revolve about the respective hinge. Therefore, each point of an element can be displaced both along the axis of the element and along a circular arc of appropriate radius. These arcs can be replaced by perpendiculars dropped to the radii of revolution since the elastic elongation of the elements is small compared to their lengths.

Example 4. Given: P, a, E_1, F_1, E_2 and F_2 (Fig. 4*a*).

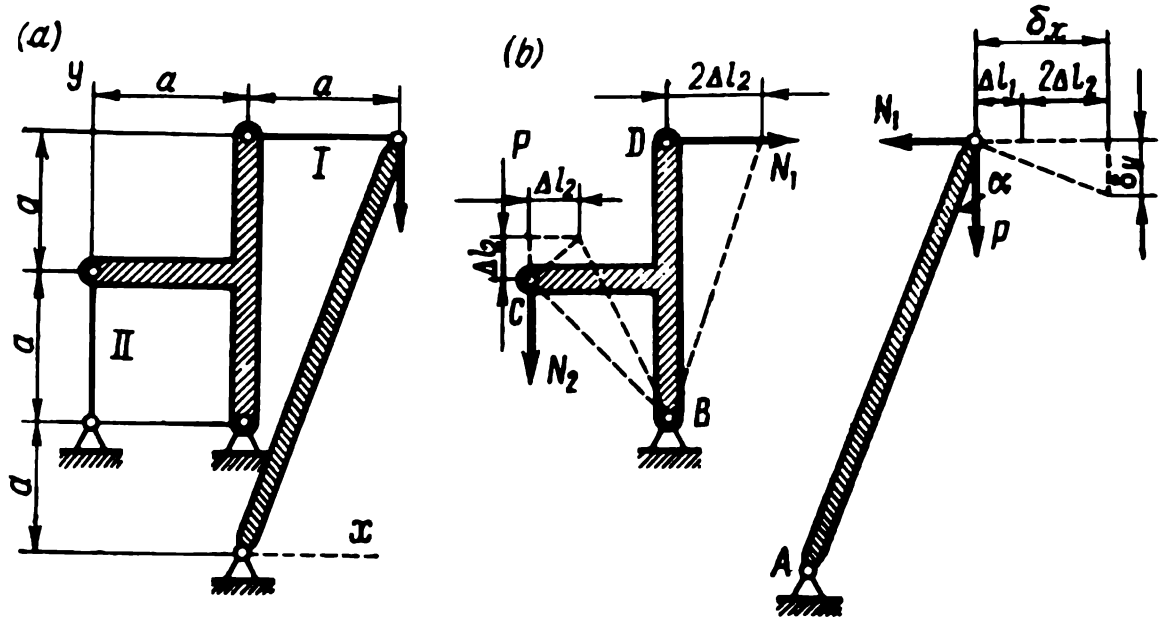

Fig. 4

Find the horizontal δ_x and vertical δ_y projections of the displacement δ of the point of application of force P.

Solution. Dismember the given system into two systems by cutting tie rods *I* and *II* (Fig. 4*b*).

From the conditions of statics: $\sum M_A = 0$, and $\sum M_B = 0$, the forces in the tie rods are $N_1 = \frac{P}{3}$ and $N_2 = \frac{2}{3}P$.

According to Hooke's law, $\Delta l_1 = \frac{Pa_2}{3E_1F_1}$ and $\Delta l_2 = \frac{2Pa}{3E_2F_2}$.

Using the method of intersections (Fig. 4*b*), we can find the horizontal displacement of point C which equals Δl_2 and the displacement of point C perpendicular to the line BC: $\delta_C = \Delta l_2 \sqrt{2}$.

Point D can be displaced only horizontally. This displacement equals $\delta_D = \delta_C \frac{2a}{a\sqrt{2}} = 2\Delta l_2$.

The horizontal displacement of the point at which force P is applied will be the sum of the horizontal displacement of point D and elongation of the first tie rod, i.e.

$$\delta_x = 2\Delta l_2 + \Delta l_1 = \frac{4Pa}{3E_2F_2} + \frac{1}{3}\frac{Pa}{E_1F_1} = \frac{Pa}{3}\left(\frac{4}{E_2F_2} + \frac{1}{E_1F_1}\right)$$

The vertical displacement of the point of application of force P is

$$\delta_y = \delta_x \tan\alpha = \delta_x \frac{a}{3a} = \frac{Pa}{9}\left(\frac{4}{E_2F_2} + \frac{1}{E_1F_1}\right)$$

Problems 25 through 40. Determine the displacements of the points of application of external forces P (or of other points specified in the conditions) and the normal stresses at cross sections of elastic bars.

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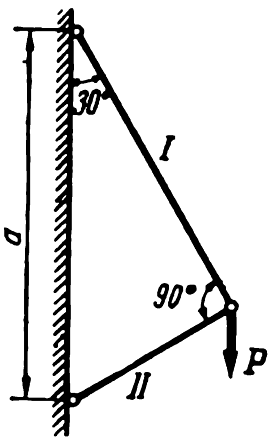

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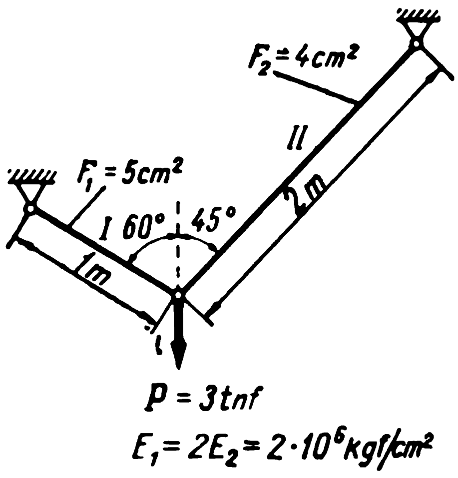

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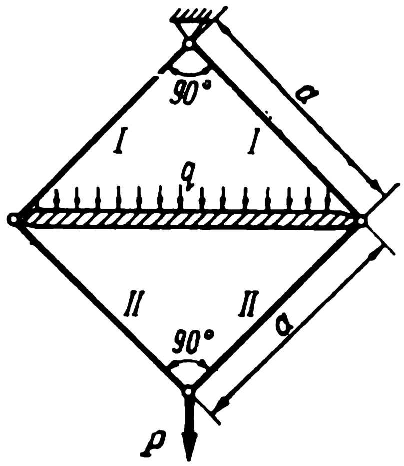

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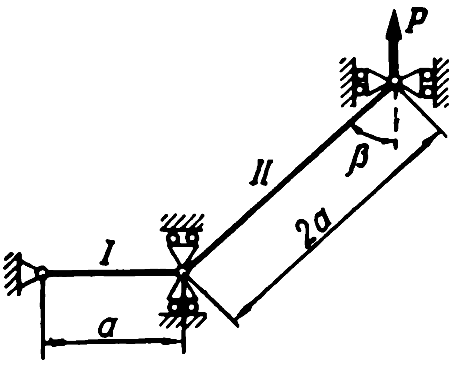

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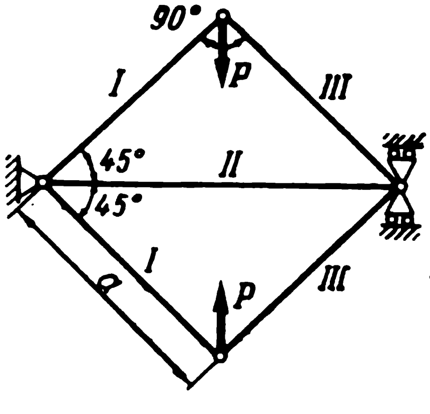

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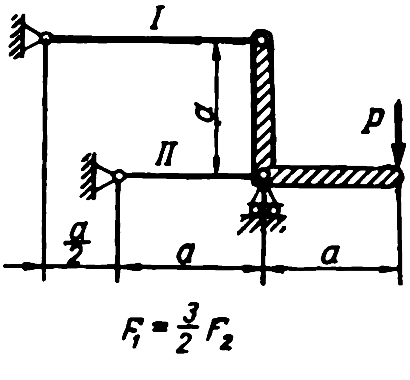

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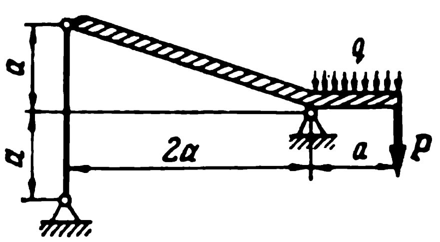

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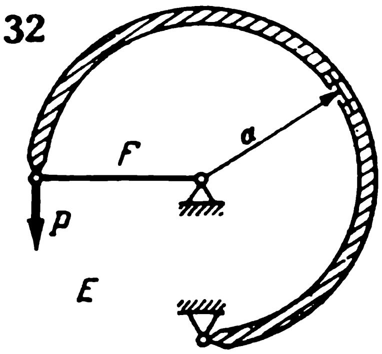

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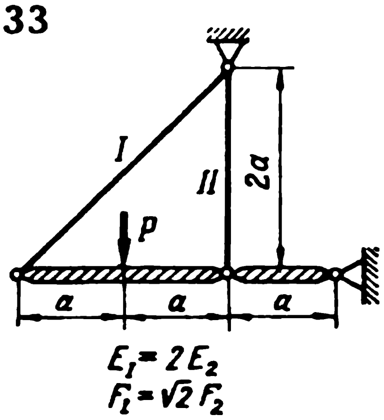

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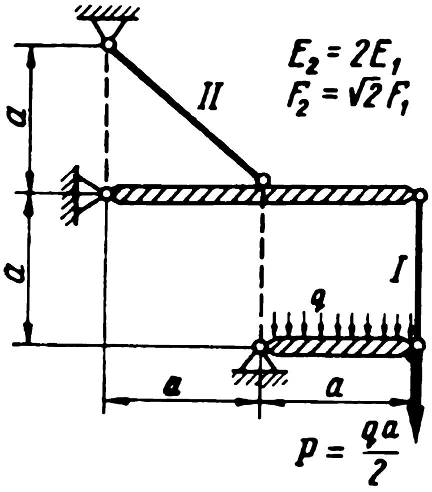

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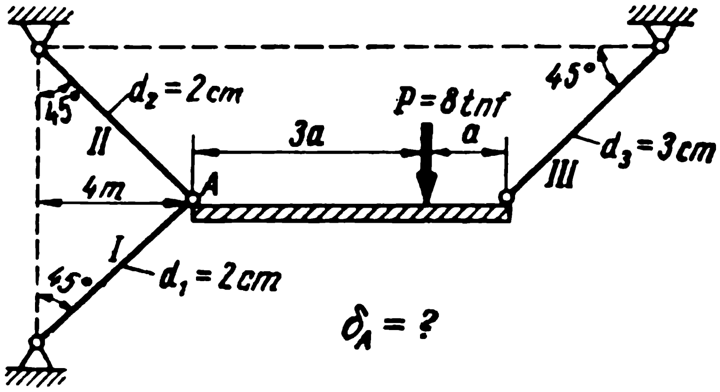

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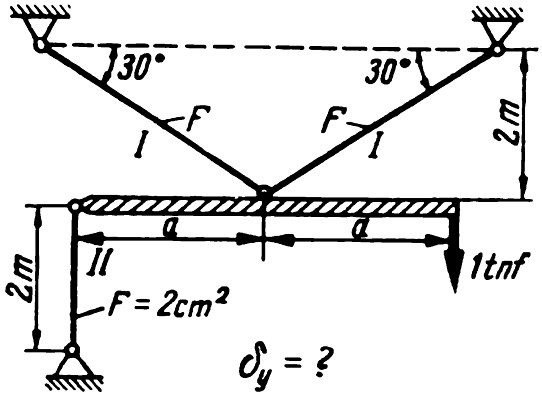

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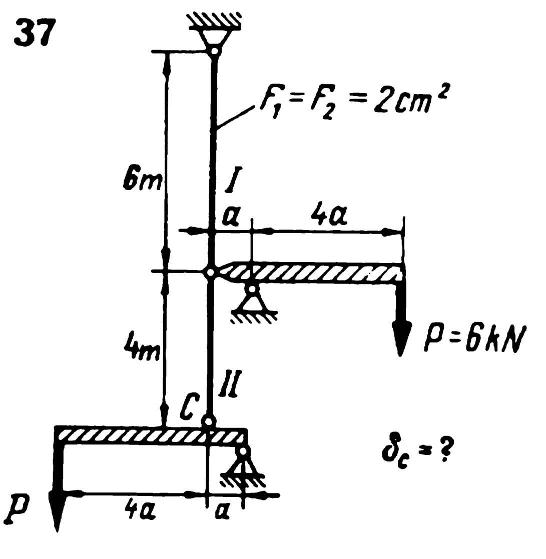

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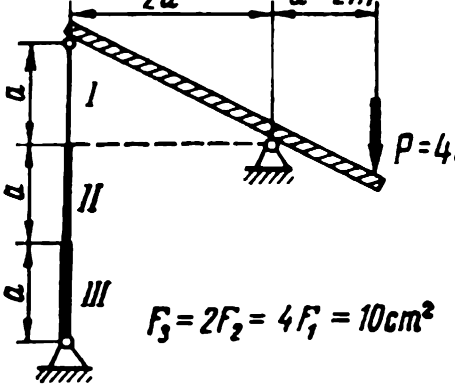

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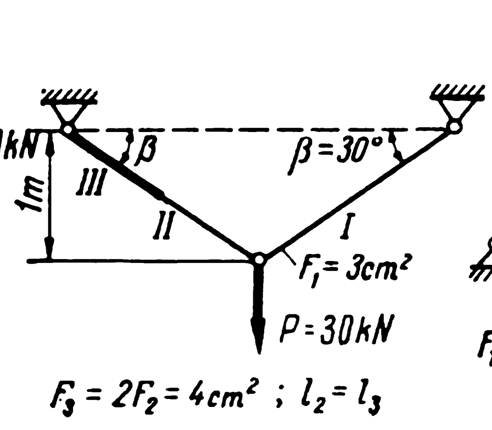

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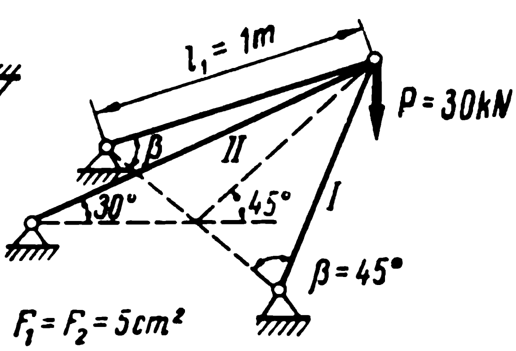

In problems where the conditions are specified by letters and there are no numerical values for E and F, assume that they are known and equal for all elastic elements of the system. In Problems 37 through 40 assume for all bars that $E = 2 \times 10^5$ MN/m². In Problems 35 and 36 assume that $E = 2 \times 10^6$ kgf/cm².

1.5.

Strength and Stiffness

The required cross-sectional area F of a stretched or compressed rod of uniform cross section is calculated from the following formula

$$F = \frac{N_{max}}{[\sigma]} \tag{11}$$

where N_{max} is the maximum absolute value of the axial force in the rod, and $[\sigma]$ is allowable tensile-compressive stress of a material or the allowable tensile stress $[\sigma_t]$ or the allowable compressive stress $[\sigma_c]$.

For plastic materials, which equally withstand tension and compression

$$[\sigma_t] = [\sigma_c] = [\sigma] = \frac{\sigma_y}{n_y} \tag{12}$$

where σ_y is the yield point of the material in tension (or compression), and n_y is the safety factor based on the yield point.

If an additional condition is stipulated which requires that the elastic displacement δ of some point of the system does not exceed the given allowable value $[\delta]$, then the stiffness is checked by the inequality

$$\delta \leqslant [\delta] \tag{13}$$

Example 5. Let $P = 1000$ kgf; $a = 1$ m; $\alpha = 30°$; $[\sigma_t]_I = 1000$ kgf/cm²; $E_1 = 2 \times 10^6$ kgf/cm²; $[\sigma_c]_{II} = 100$ kgf/cm²; $E_2 = 0.1 \times 10^6$ kgf/cm²; and the permissible horizontal $[\delta_x]$ and

vertical $[\delta_y]$ projections of the point of application of force P are $[\delta_x] = [\delta_y] = 1.3$ mm (Fig. 5).

Find F_1 and F_2.

Solution. From the equations of statics (Fig. 5*a*), $\sum X = 0$ and $\sum Y = 0$; $N_1 = N_2 = N$ and $N = P = 1000$ kgf. From formula (11)

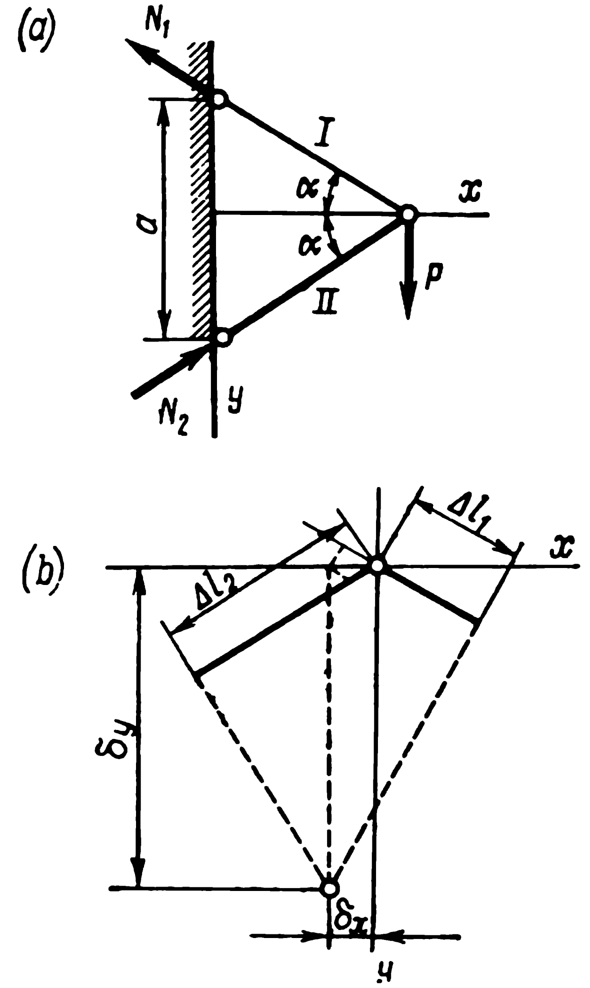

Fig. 5

$$F_1 = \frac{N}{[\sigma_t]_I} = \frac{10^3}{10^3} = 1 \text{ cm}^2;$$

$$F_2 = \frac{N}{[\sigma_c]_{II}} = \frac{10^3}{10^2} = 10 \text{ cm}^2$$

According to Hooke's law

$$\Delta l_1 = \frac{Na}{E_1F_1} = \frac{10^3 \times 10^2}{2 \times 10^6 \times 1} = 0.05 \text{ cm};$$

$$\Delta l_2 = \frac{Na}{E_2F_2} = \frac{10^3 \times 10^2}{0.1 \times 10^6 \times 10} = 0.1 \text{ cm}$$

From the geometrical construction obtained by the method of intersections (Fig. 5*b*), it follows that Δl_1 equals the sum of the projections δ_x and δ_y in the direction of bar *I*, and Δl_2 equals the sum of the projections δ_x and δ_y in the direction of bar *II*, i.e.

$$\Delta l_1 = \delta_y \sin\alpha - \delta_x \cos\alpha;$$

$$\Delta l_2 = \delta_y \sin\alpha + \delta_x \cos\alpha$$

Hence,

$$\delta_y = \frac{\Delta l_1 + \Delta l_2}{2\sin\alpha} = 1.5 \text{ mm}; \qquad \delta_x = \frac{\Delta l_2 - \Delta l_1}{2\cos\alpha} = 0.289 \text{ mm}$$

Since $\delta_y > [\delta_y]$ the cross-sectional areas of the bar should be increased.

Retaining the cross-sectional area of bar *I* which is $F_1 = 1$ cm², find F_2, the required area of bar *II*.

From the stiffness condition

$$\delta_y = \frac{1}{2\sin 30^\circ}(\Delta l_1 + \Delta l_2) \leqslant [\delta_y]$$

or

$$0.05 + \Delta l_2 \leqslant 0.13 \text{ cm}$$

whence

$$\Delta l_2 = \frac{Na}{E_2F_2} \leqslant 0.08 \text{ cm and } F_2 = \frac{Na}{0.08E_2} = 12.5 \text{ cm}^2$$

With such an area the stress in bar *II* is

$$\sigma_{II} = \frac{N}{F_2} = 80 \text{ kgf/cm}^2$$

and the horizontal and vertical displacements of the point of application of force P are

$$\delta_y = 1.33 \text{ mm}; \quad \delta_x = \frac{0.08-0.05}{\sqrt{3}} \cong 0.0173 \text{ cm} = 0.173 \text{ mm}$$

Example 6. A circular ring of inside radius r = 100 mm, outside radius R = 101 mm, and length l is subject to a uniform internal radial pressure p = 20 bars (Fig. 6*a*).

Determine the increase in ring radius Δr and the actual safety factor n_y of the ring if it is made of a material with a Young's modulus $E = 2 \times 10^5$ MN/m² and a yield point $\sigma_y = 300$ MN/m².

Solution. To determine the tensile force N developed in the ring wall, it is cut diametrically (Fig. 6*b*). The condition of equilibrium for the semi-ring is specified as the sum of the projections of forces and stresses (which will be external forces for the semi-ring) on the y-axis. Then

$$N = \int_0^{\frac{\pi}{2}} plr \sin\alpha \, d\alpha = prl$$

The normal stress in the ring wall will be

$$\sigma = \frac{N}{F} = \frac{prl}{(R-r)\,l} = \frac{20 \times 10^5 \times 10 \times 10^{-2}}{0.1 \times 10^{-2}}$$

$$= 2 \times 10^8 \text{ N/m}^2 = 200 \text{ MN/m}^2$$

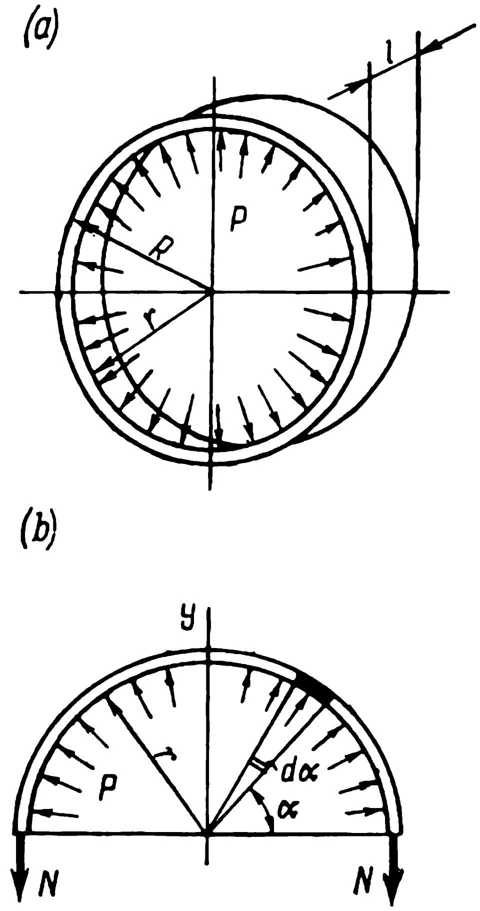

Fig. 6

The safety factor n_y (on the basis of the yield point of the material) will be

$$n_y = \frac{\sigma_y}{\sigma} = \frac{300}{200} = 1.5$$

The total increase in the inside radius of the ring is found by using Hooke's law. Since

$$2\pi(r + \Delta r) - 2\pi r = \frac{N \times 2\pi r}{E\,(R-r)\,l}$$

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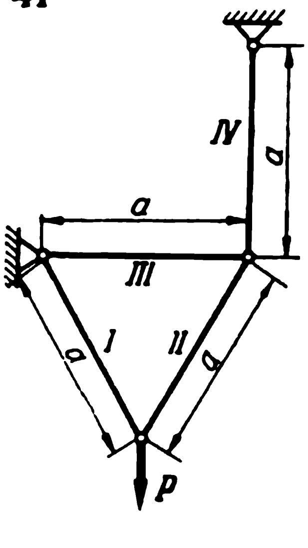

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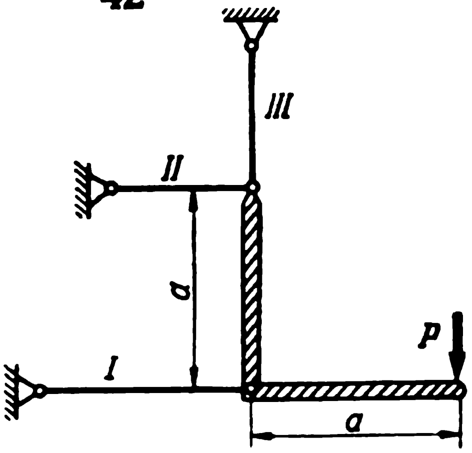

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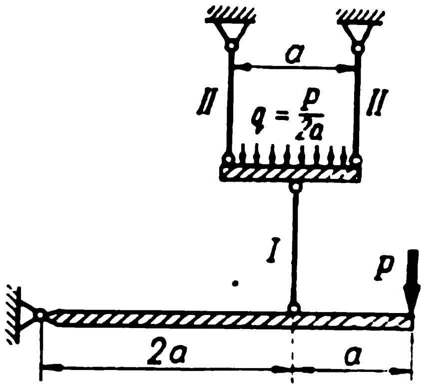

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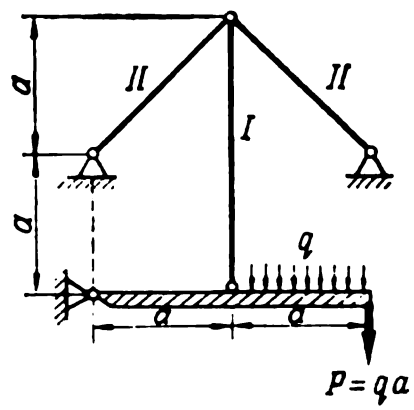

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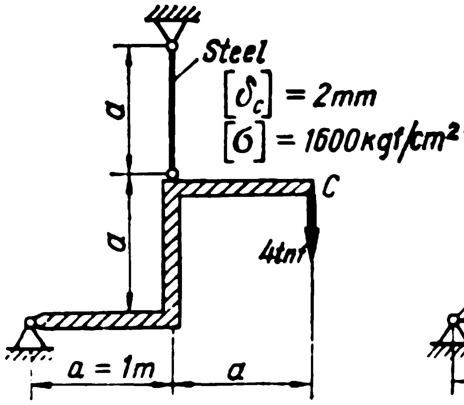

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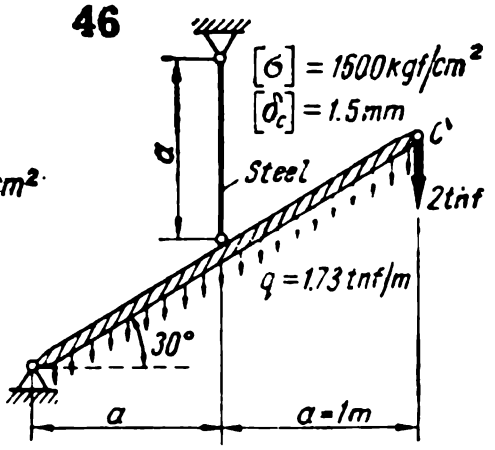

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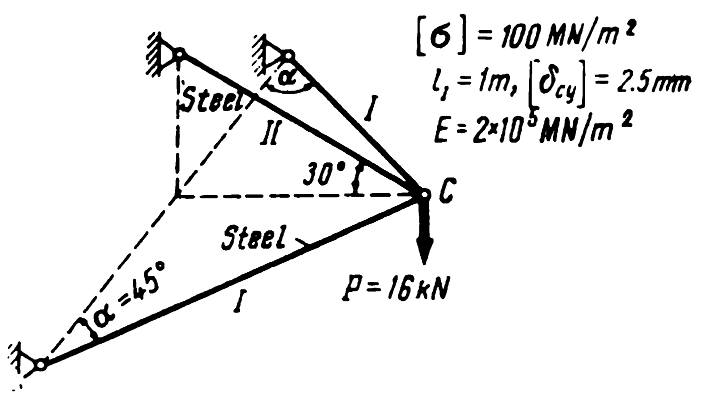

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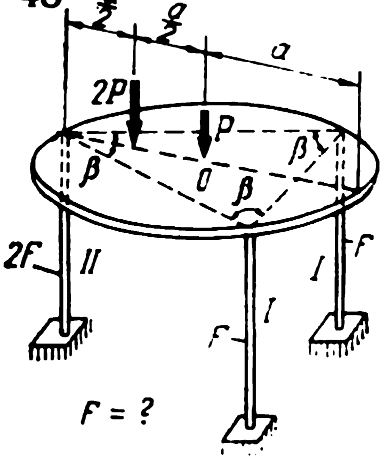

we obtain

$$\Delta r = \frac{pr^2}{E(R-r)} = \frac{20 \times 10^5 \times 10^2 \times 10^{-4}}{2 \times 10^{11} \times 0.1 \times 10^{-2}} = 0.01 \times 10^{-2} \text{ m} = 0.01 \text{ cm}$$

Problems 41 through 48. Find the cross-sectional areas F for the elastic elements of the given systems.

In problems where the conditions are specified by letters assume the allowable stress $[\sigma]$ to be equal for tension and compression for all

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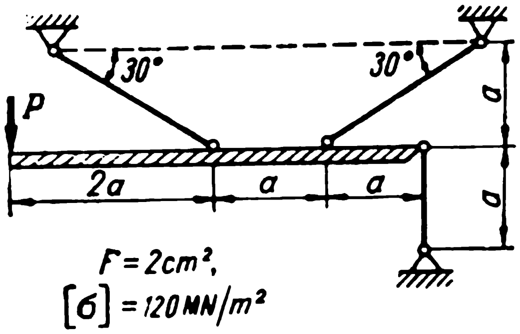

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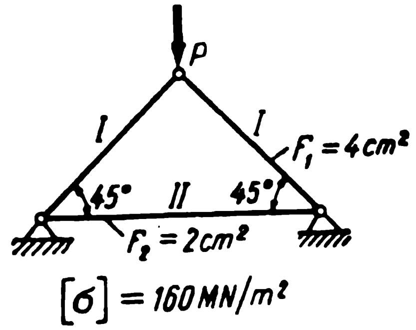

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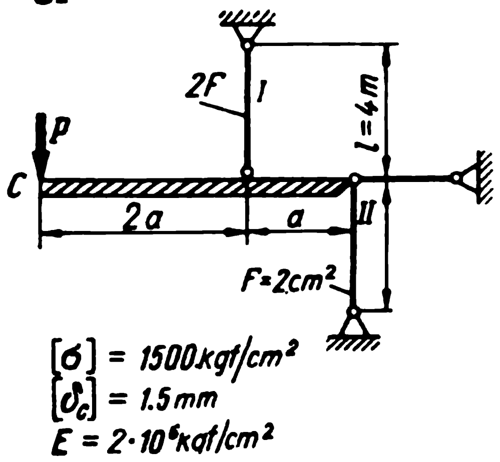

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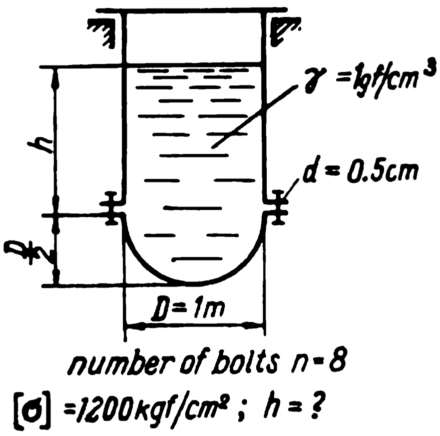

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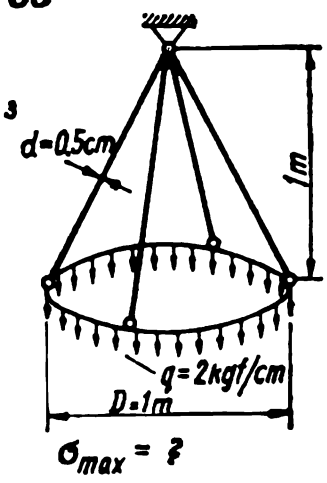


elastic elements of the system. If the elastic modulus E is not given, it is assumed to be known and equal for all the bars. In Problems 45 and 46 assume that $E = 2 \times 10^6$ kgf/cm^2 for steel.

Problems 49 through 53. Find the allowable force P or other quantities indicated in the conditions.

1.6.

Taking the Dead Weight into Account

For a prismatic bar subject to the action of its own weight and concentrated force P applied at the free end: the axial force at a cross section at a distance x from the free end is

$$N_x = P + \gamma F x \tag{14}$$

the normal stress at the same section is

$$\sigma_x = \frac{P}{F} + \gamma x \tag{15}$$

the required cross-sectional area is

$$F = \frac{P}{[\sigma] - \gamma l} \tag{16}$$

and the total elongation is

$$\Delta l=\frac{l}{EF}\left(P+\frac{Q}{2}\right) \tag{17}$$

where γ is the weight of unit volume of the bar material, l is the length of the bar and $Q = \gamma Fl$ is the weight of the bar.

For a bar of uniform strength, i.e. for one having equal normal stresses at each cross section, the cross-sectional area is found by the formula

$$F_x=\frac{P}{[\sigma]}\times e^{\frac{\gamma}{[\sigma]}x} \tag{18}$$

where e is the base of natural logarithms.

The total elongation of a bar of uniform strength is

$$\Delta l=\frac{[\sigma]\,l}{E} \tag{19}$$

For a stepped bar, the required cross-sectional area of an arbitrary i-th step will be

$$F_i=\frac{P\,[\sigma]^{i-1}}{([\sigma]-\gamma l_1)\,([\sigma]-\gamma l_2)\,([\sigma]-\gamma l_3)\,\ldots\,([\sigma]-\gamma l_i)} \tag{20}$$

and the total elongation

$$\Delta l=\frac{[\sigma]}{E}\sum l_i\left(1-\frac{\gamma l_i}{2\,[\sigma]}\right) \tag{21}$$

where l_1, l_2, l_3, ... are the lengths of the respective steps of the bar.

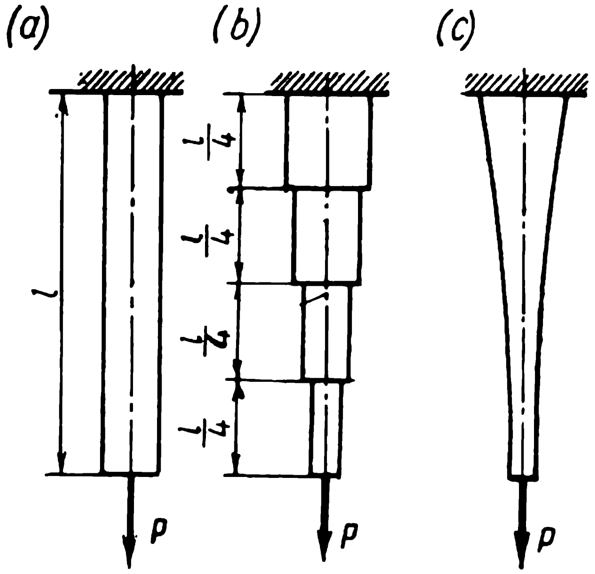


Fig. 7

Example 7. Given: $P = 16$ tnf; $\gamma = 8$ gf/cm³; $[\sigma]=1600$ kgf/cm²; $E = 2 \times 10^6$ kgf/cm² and $l = 40$ m (Fig. 7).

Find the cross-sectional area F_{pr}, weight Q_{pr} and total elongation Δl_{pr} of the prismatic bar; the maximum cross-sectional area F_{st}, weight Q_{st} and total elongation Δl_{st} of the stepped bar with four steps of equal length, and the maximum area F_{us}, weight Q_{us} and total elongation Δl_{us} of the bar of uniform strength.

Solution. For the prismatic bar (Fig. 7*a*) the area is calculated by formula (16). Thus

$$F_{pr}=\frac{16\times 10^3}{16\times 10^2-8\times 10^{-3}\times 4\times 10^3}\cong 10.204\ \text{cm}^2$$

The weight is

$$Q_{pr}=\gamma F_{pr}l=8\times10^{-3}\times10.204\times4\times10^{3}\cong326.53\text{ kgf}$$

The total elongation is calculated by formula (17):

$$\Delta l_{pr}=\frac{4\times10^{3}}{2\times10^{6}\times10.204}\left(16\times10^{3}+\frac{327}{2}\right)\cong3.168\text{ cm}$$

For the stepped bar (Fig. 7*b*) the maximum area is calculated by formula (20). Thus

$$F_{st}=\frac{P[\sigma]^{3}}{\left([\sigma]-\frac{1}{4}\gamma l\right)^{4}}=\frac{P}{[\sigma]\left(1-\frac{\gamma l}{4[\sigma]}\right)^{4}}$$

$$=\frac{16\times10^{3}}{16\times10^{2}\left(1-\frac{8\times10^{-3}\times4\times10^{3}}{4\times16\times10^{2}}\right)^{4}}\cong10.203\text{ cm}^{2}$$

The weight is

$$Q_{st}=[\sigma]F_{st}-P=16\times10^{2}\times10.203-16\times10^{3}\cong324.8\text{ kgf}$$

The total elongation is calculated by formula (21)

$$\Delta l_{st}=\frac{[\sigma]l}{E}\left(1-\frac{\gamma l}{2\times4[\sigma]}\right)$$

$$=\frac{16\times10^{2}\times4\times10^{3}}{2\times10^{6}}\left(1-\frac{8\times10^{-3}\times4\times10^{3}}{2\times4\times16\times10^{2}}\right)\cong3.192\text{ cm}$$

For the bar of uniform strength (Fig. 7*c*) the maximum cross-sectional area is found by formula (18). Thus

$$F_{us}=\frac{P}{[\sigma]}\times e^{\frac{\gamma l}{[\sigma]}}=\frac{16\times10^{3}}{16\times10^{2}}\times e^{\frac{8\times10^{-3}\times4\times10^{3}}{16\times10^{2}}}$$

$$=10e^{0.02}\cong10.202\text{ cm}^{2}$$

The weight is

$$Q_{us}=[\sigma]F_{us}-P=16\times10^{2}\times10.202-16\times10^{3}\cong323.3\text{ kgf}$$

The total elongation is found by formula (19)

$$\Delta l_{us}=\frac{[\sigma]l}{E}=\frac{16\times10^{2}\times4\times10^{3}}{2\times10^{6}}=3.2\text{ cm}$$

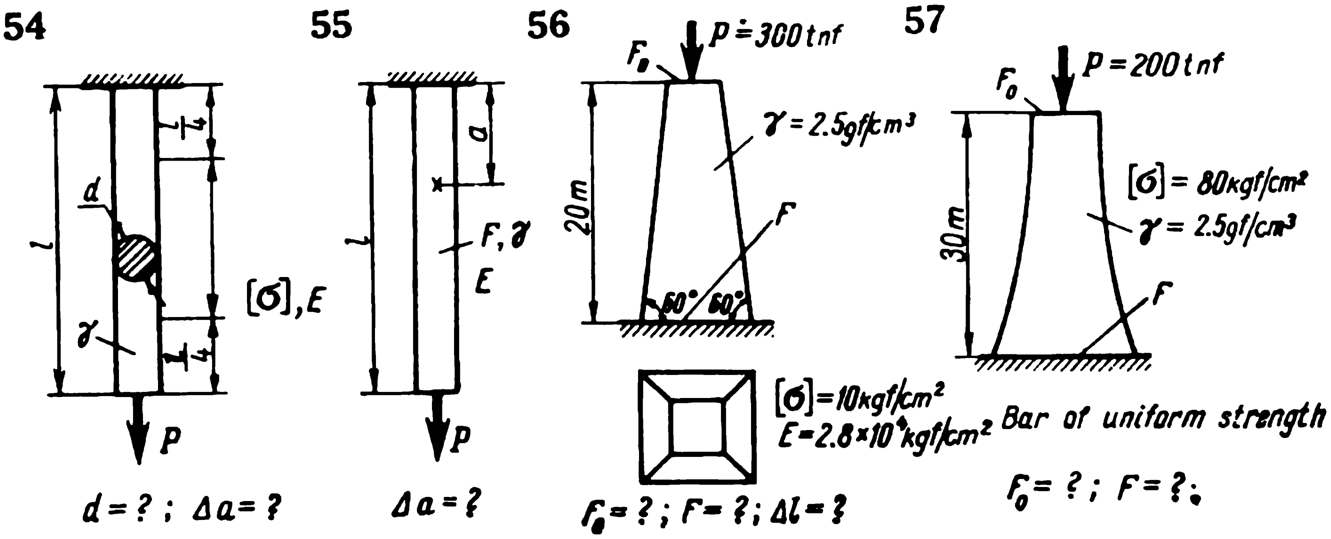
The obtained results show that for a 40-m steel bar the difference between a prismatic, stepped and uniform strength bar is insignificant.

As is evident from formula (16), the stress due to the dead weight of a prismatic bar reaches 5% of $[\sigma]$ for a bar length $l \geqslant \frac{0.05[\sigma]}{\gamma}$

In the case of a steel bar and assuming that $[\sigma] = 1600$ kgf/cm^2 and $\gamma = 8$ gf/cm^3,

$$l \geqslant \frac{0.05 \times 16 \times 10^2}{8 \times 10^{-3}} = 10^4 \text{ cm} = 100 \text{ m}$$

Problems 54 through 57. Find the sought-for quantities in the following problems.



1.7.

Statically Indeterminate Systems

Statically indeterminate systems are ones for which the equations of statics (equilibrium) are insufficient to determine the forces in all the elements.

The forces and stresses of statically indeterminate systems are calculated by simultaneous use of the equations of statics and those of combined displacements. The procedure is as follows.

First the equations of statics are written down, the degree of statical indeterminacy of the system is established and then the conditions of combined displacements are determined. These conditions are the geometrical relationships between the elongation of certain elements of the system.

The elongation of the elements is expressed in terms of the forces, according to Hooke's law, and they are substituted into the equations of combined displacements.

The axial forces in all the elements of the system are found by solving the equations of statics and of combined displacements simultaneously.

The same procedure is used to determine thermal stresses. Here the conditions of statics are established only for the forces, and the changes in the length of the heated or cooled elements are determined

by algebraically summing up the length increments due to forces and to temperature change. The total elongation due to a change in temperature is computed by the formula

$$\Delta l = l\alpha\Delta t \tag{22}$$

where l is the length of the bar, α is the mean coefficient of linear expansion of the material the bar is made of, and Δt is the change in temperature.

Assembly stresses are also determined from the equations of statics and of combined displacements. Here, the inevitable errors, within the stipulated accuracy, that occur in manufacturing the elements of the system should be taken into account when setting up the equations of combined displacements. Since the actual lengths of the manufactured elements differ only slightly from the design dimensions, usually the design and not the actual lengths are taken for determining the total elongation of the elements by Hooke's law.

In determining the maximum safe working force that develops the allowable stress, the stress in the most heavily loaded bar is assumed to be equal to the permissible value. The maximum safe working force is then determined on this basis.

Statically indeterminate systems are designed on the basis of their supporting power, using only the equations of statics. Under these conditions the axial forces are assumed to be equal to the product of the allowable stress by the cross-sectional area of all the elements in which the attainment of the yield point of the material by the stress leads to a change in the geometrical shape of the system. This method of design is based on replacing the actual stress-strain diagram by the idealized Prandtl diagram in which the yield step is assumed to be unlimited.

Example 8. Let

(a) $E_1 = E_2 = E_3 = E = 2 \times 10^6$ kgf/cm²; $[\sigma] = 1600$ kgf/cm²; $a = 0.4$ m; $b = 1.2$ m; $c = 0.4$ m; $\beta_1 = 45°$; $\beta_2 = 60°$; $\beta_3 = 30°$; $F_1 = 12$ cm²; $F_2 = 14$ cm² and $F_3 = 16$ cm² (Fig. 8).

(b) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 12.5 \times 10^{-6}$ and $\Delta t = 40°$;

(c) $\Delta_2 = 1.2$ mm which is the amount by which bars II are manufactured shorter than the design length.

Find: (a) P and P_{max}; (b) $\sigma_{I,\,II,\,III}$; (c) $\sigma_{I,\,II,\,III}$.

Solution. (a) *According to the allowable stress.* From the equation of statics it follows that the sums of the projections on the vertical axis of the forces and efforts exerted in the indicated joints (see Fig. 8*a*) are

$$2N_1 \sin\beta_1 = 2N_2 \sin\beta_2 \text{ and } 2N_2 \sin\beta_2 + 2N_3 \sin\beta_3 = P$$

From the condition of combined displacements, i.e. from the equality of displacements of the point of application of force P caused by the

elongation of bars *I* and *II* and contraction of bars *III* (Fig. 8*b*), we have

$$\delta_1 + \delta_2 = \delta_3$$

According to Hooke's law

$$\delta_1 = \frac{\Delta l_1}{\sin\beta_1} = \frac{N_1 l_1}{E_1F_1\sin\beta_1}; \quad \delta_2 = \frac{\Delta l_2}{\sin\beta_2}$$

$$= \frac{N_2 l_2}{E_2F_2\sin\beta_2}; \quad \delta_3 = \frac{\Delta l_3}{\sin\beta_3} = \frac{N_3 l_3}{E_3F_3\sin\beta_3}$$

From the geometry of the system

$$l_1 = \frac{a}{\sin\beta_1}; \quad l_2 = \frac{b}{\sin\beta_2}; \quad l_3 = \frac{c}{\sin\beta_3}$$

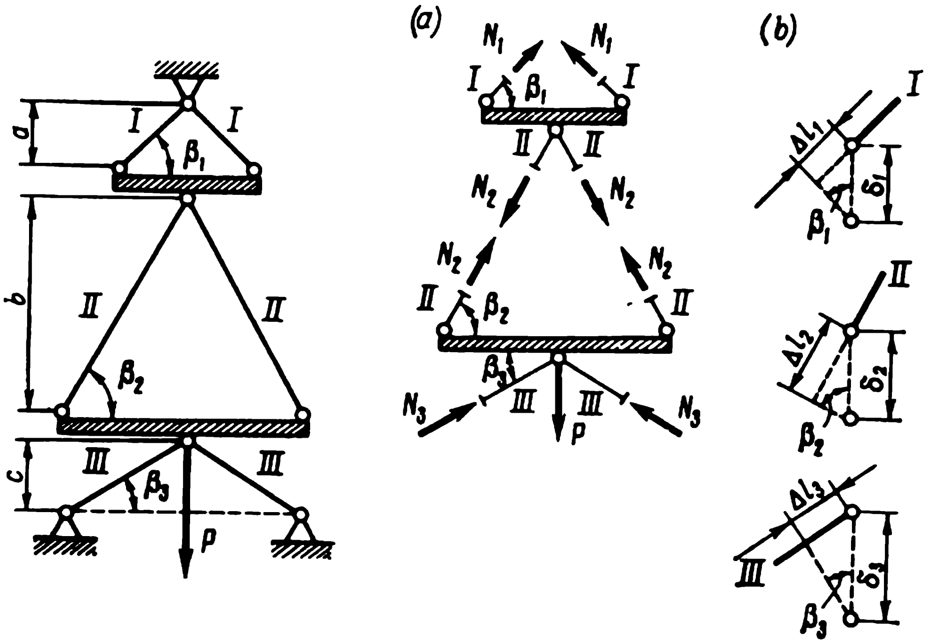


Fig. 8

Substituting these quantities into the equation of combined displacements we have

$$\frac{N_1 a}{E_1F_1\sin^2\beta_1} + \frac{N_2 b}{E_2F_2\sin^2\beta_2} = \frac{N_3 c}{E_3F_3\sin^2\beta}$$

For the given numerical values

$$\sin\beta_1 = \frac{\sqrt{2}}{2}; \quad \sin\beta_2 = \frac{\sqrt{3}}{2}; \quad \sin\beta_3 = \frac{1}{2};$$

$$\frac{a}{F_1\sin^2\beta_1} = \frac{40\times 4}{12\times 2} = \frac{20}{3}\ 1/\text{cm};$$

$$\frac{b}{F_2\sin^2\beta_2} = \frac{120\times 4}{14\times 3} = \frac{80}{7}\ 1/\text{cm}; \quad \frac{c}{F_3\sin^2\beta_3} = \frac{40\times 4}{16\times 1} = 10\ 1/\text{cm}$$

Substituting these quantities into the equations of statics and into the equations of combined displacements, we obtain the following system of three equations:

$$\left.\begin{aligned} \sqrt{2}N_1 &= \sqrt{3}N_2; \\ \sqrt{3}N_2 + N_3 &= P; \\ 14N_1 + 24N_2 &= 21N_3 \end{aligned}\right\}$$

Solving this system we obtain

$$N_1 = \sigma_I F_1 \cong 0.332P;\ N_2 = \sigma_{II} F_2 \cong 0.27P;$$
$$N_3 = \sigma_{III} F_3 \cong 0.53P$$

whence

$$\sigma_I = \frac{0.332}{12} P \cong 0.0276P; \quad \sigma_{II} = \frac{0.27}{14} P \cong 0.0193P;$$
$$\sigma_{III} = \frac{0.53}{16} P \cong 0.0331P$$

Since the maximum stress σ_{III} should not exceed $[\sigma]$, the allowable working force is

$$P \leqslant \frac{[\sigma]}{0.0331} = \frac{1600}{0.0331} \cong 48.340 \text{ kgf} \cong 48.3 \text{ tnf}$$

With this force the stresses in the bars of the system will have the following values:

$$\sigma_I = 48.3 \times 27.6 \cong 1334 \text{ kgf/cm}^2;$$
$$\sigma_{II} = 48.3 \times 19.3 \cong 932 \text{ kgf/cm}^2;$$
$$\sigma_{III} = 1600 \text{ kgf/cm}^2$$

(b) *According to the supporting power*. The system acquires a geometrically variable state, when the first and third bars begin "yielding". The equation of statics relating the forces exerted in these bars is of the form

$$2N_1 \sin\beta_1 + 2N_3 \sin\beta_3 = P$$

Assuming that $N_1 = [\sigma]\,F_1$ and $N_3 = [\sigma]\,F_3$ and substituting them into the equation of statics, we find the maximum force P_{max}:

$$P_{max} = 2\,[\sigma]\,(F_1 \sin\beta_1 + F_3 \sin\beta_3)$$
$$= 2 \times 1600 \left(12\frac{\sqrt{2}}{2} + 16\frac{1}{2}\right) \cong 52.750 \text{ kgf} = 52.75 \text{ tnf}$$

Thus, the load-carrying capacity of the system calculated on the basis of the supporting power exceeds that calculated on the basis of the allowable stress (assuming the same safety factor) by

$$\frac{P_{max} - P}{P} \times 100 = \frac{52.75 - 48.3}{48.3} \times 100 \cong 9.1\%$$

(c) *Determining the thermal stresses.*
From the equations of statics (Fig. 9*a*):

$$2N_1 \sin\beta_1 = 2N_2 \sin\beta_2;\ 2N_2 \sin\beta_2 = 2N_3 \sin\beta_3$$

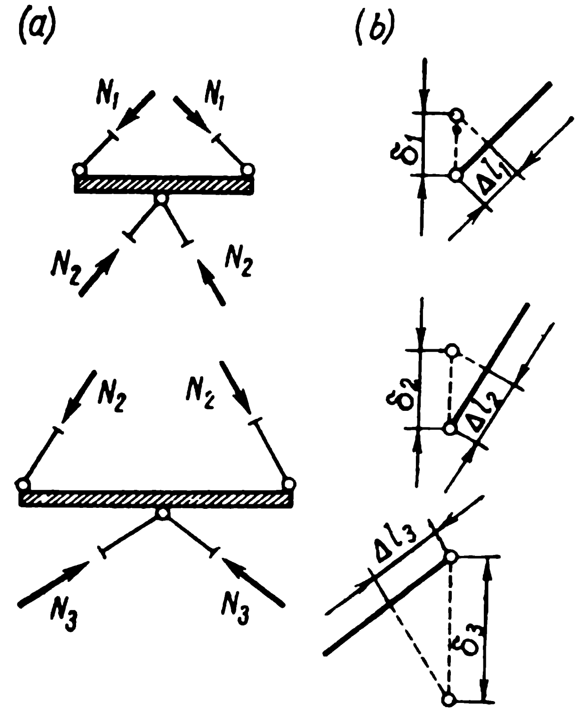


Fig. 9

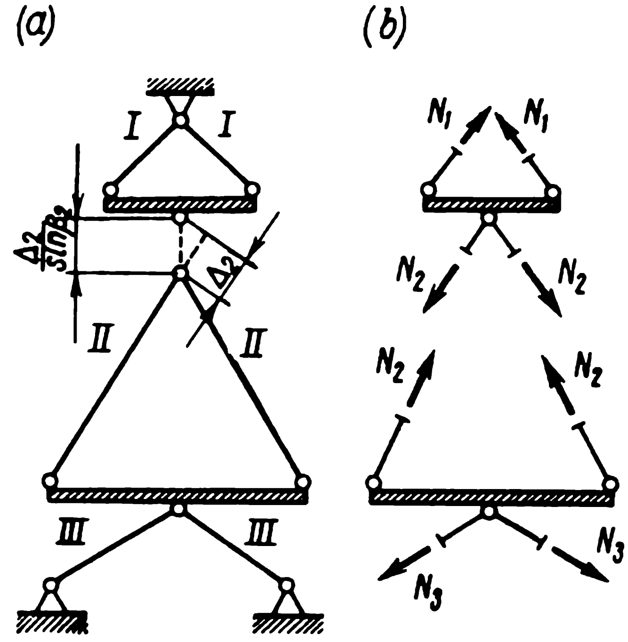


Fig. 10

From the equation of combined displacements (the condition of the constant height of the system) (Fig. 9*b*):

$$\delta_1 + \delta_2 + \delta_3 = 0$$

Since

$$\delta_1 = \frac{\Delta l_1}{\sin\beta_1};\quad \delta_2 = \frac{\Delta l_2}{\sin\beta_2};\quad \delta_3 = \frac{\Delta l_3}{\sin\beta_3};$$

$$\Delta l_1 = l_1\alpha_1\Delta t - \frac{N_1 l_1}{E_1F_1};\quad \Delta l_2 = l_2\alpha_2\Delta t - \frac{N_2 l_2}{E_2F_2};$$

$$\Delta l_3 = l_3\alpha_3\Delta t - \frac{N_3 l_3}{E_3F_3}$$

the equation of combined displacements acquires the following form:

$$\frac{a}{\sin^2\beta_1}\left(\alpha\Delta t - \frac{N_1}{EF_1}\right) + \frac{b}{\sin^2\beta_2}\left(\alpha\Delta t - \frac{N_2}{EF_2}\right)$$
$$+ \frac{c}{\sin^2\beta_3}\left(\alpha\Delta t - \frac{N_3}{EF_3}\right) = 0$$

Recalling that $N_1 = \sigma_I F_1$, $N_2 = \sigma_{II} F_2$ and $N_3 = \sigma_{III} F_3$ and taking the given numerical values into consideration, the equations of statics and that of combined displacements are reduced to the following system:

In mk(force)s (engineering) units

$$\left.\begin{aligned}6\sqrt{2}\,\sigma_I &= 7\sqrt{3}\,\sigma_{II};\\ 7\sqrt{3}\,\sigma_{II} &= 8\sigma_{III};\\ \sigma_I + 2\sigma_{II} + 2\sigma_{III} &= 5000\end{aligned}\right\}$$

Solving this system, we obtain $\sigma_I \cong 1105$ kgf/cm², $\sigma_{II} \cong 774$ kgf/cm² and $\sigma_{III} = 1172$ kgf/cm²

(d) *Determining the assembly stresses* (Fig. 10*a*).

The equations of statics (Fig. 10*b*) are

$$2N_1 \sin\beta_1 = 2N_2 \sin\beta_2; \quad 2N_2 \sin\beta_2 = 2N_3 \sin\beta_3$$

The equation of combined displacements is

$$\frac{\Delta l_1}{\sin\beta_1} + \frac{\Delta l_2}{\sin\beta_2} + \frac{\Delta l_3}{\sin\beta_3} = \frac{\Delta_2}{\sin\beta_2}$$

or

$$\frac{\sigma_I a}{\sin^2\beta_1} + \frac{\sigma_{II} b}{\sin^2\beta_2} + \frac{\sigma_{III} c}{\sin^2\beta_3} = E\frac{\Delta_2}{\sin^2\beta_2}$$

Substituting the given numerical values into the equations of statics and of combined displacements we obtain

$$\left.\begin{aligned}6\sqrt{2}\,\sigma_I &= 7\sqrt{3}\,\sigma_{II};\\ 7\sqrt{3}\,\sigma_{II} &= 8\sigma_{III};\\ \sigma_I + 2\sigma_{II} + 2\sigma_{III} &= 4000\end{aligned}\right\}$$

Solving these equations, we obtain: $\sigma_I \cong 886$ kgf/cm², $\sigma_{II} \cong 620$ kgf/cm² and $\sigma_{III} \cong 939$ kgf/cm².

Example 9. A cylindrical steel pipe *II* of inside radius $r_2 = 41.96$ mm and outside radius $R_2 = 43$ mm (Fig. 11) is fitted in the heated state on another cylindrical steel pipe *I* of inside radius $r_1 = 40$ mm and outside radius $R_1 = 42$ mm.

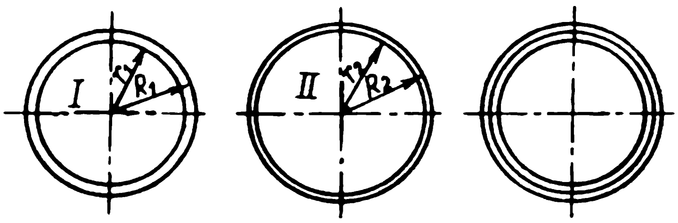


Fig. 11

Determine the stresses developed in the walls of pipe *I* (σ_I) and pipe *II* (σ_{II}) when the outer pipe cools to the temperature of the inner pipe if Young's modulus for the steel equals $E = 2 \times 10^6$ kgf/cm².

Solution. Instead of pipes let us consider rings of unit length (see Example 6). In cooling, ring *II* will exert a uniform external radial pressure p on ring *I*, while the latter, resisting deformation, will exert

a similar pressure on ring II, but from inside. The pressure p is determined from the condition that the sum of changes in the outside radius of ring I (ΔR_1) and in the inside radius of ring II (Δr_2) must be equal to the initial difference $R_1 - r_2$.

Since (see Example 6)

$$\Delta R_1 = \frac{p}{E} \times \frac{R_1^2}{R_1 - r_1} \quad \text{and} \quad \Delta r_2 = \frac{p}{E} \times \frac{r_2^2}{R_2 - r_2}$$

the equation of combined displacements can be written in the following form:

$$\frac{p}{E}\left(\frac{R_1^2}{R_1 - r_1} + \frac{r_2^2}{R_2 - r_2}\right) = R_1 - r_2$$

whence:

$$p = \frac{E(R_1 - r_2)}{\frac{R_1^2}{R_1 - r_1} + \frac{r_2^2}{R_2 - r_2}} = \frac{2 \times 10^6 \times 0.004}{\frac{4.2^2}{0.2} + \frac{4.196^2}{0.104}} \cong 31 \text{ kgf/cm}^2$$

The normal stresses σ_I and σ_{II} are determined by the formulas (see Example 6):

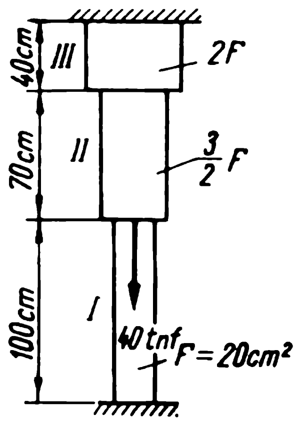
$$\sigma_I = \frac{pr_1}{R_1 - r_1} = \frac{31 \times 4.2}{0.2} = 651 \text{ kgf/cm}^2;$$

$$\sigma_{II} = \frac{pr_2}{R_2 - r_2} = \frac{31 \times 4.196}{0.104} \cong 1250 \text{ kgf/cm}^2$$

Problems 58 through 70. Determine the normal stresses in the elastic elements of the systems, due to the action of the indicated forces.

If the modulus of elasticity E is not specified, assume it to be equal for all elastic elements of the system.

58



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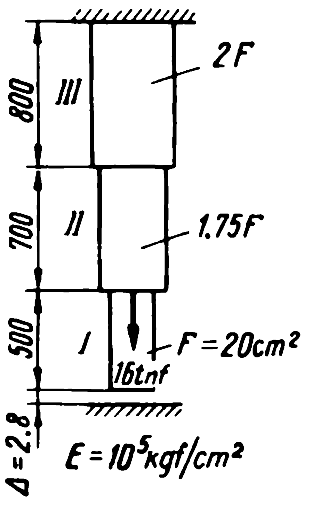

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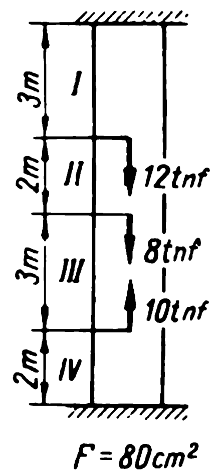

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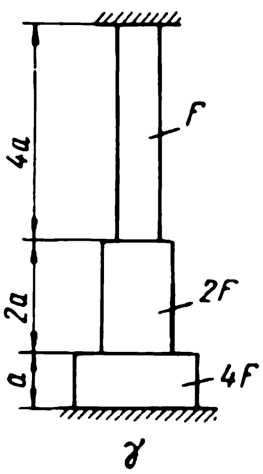

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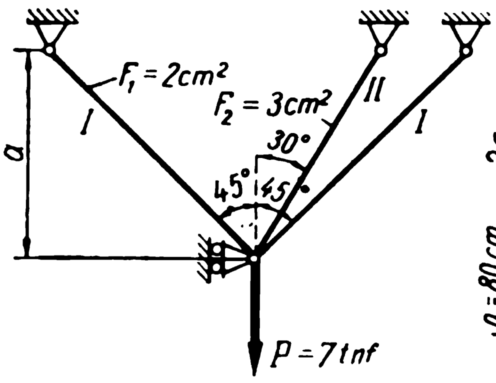

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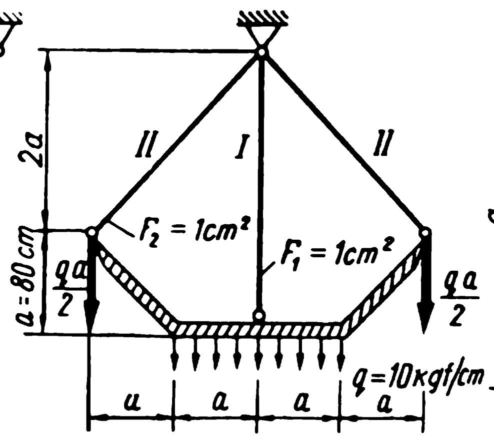

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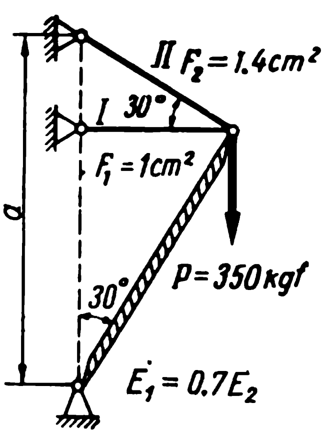

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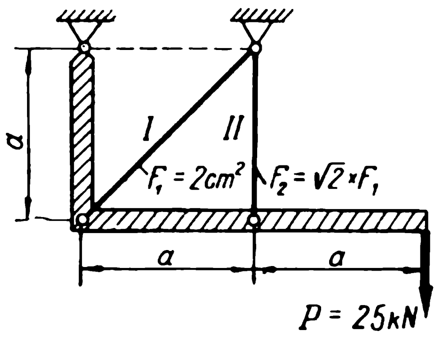

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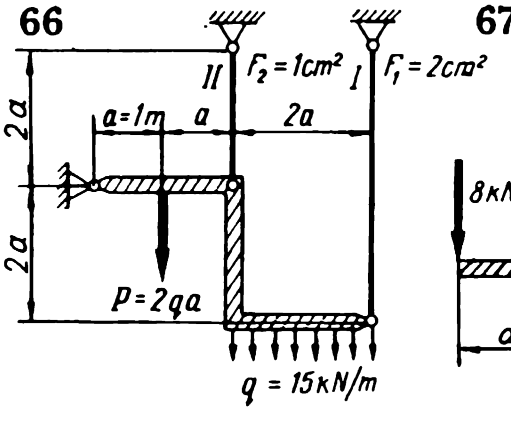

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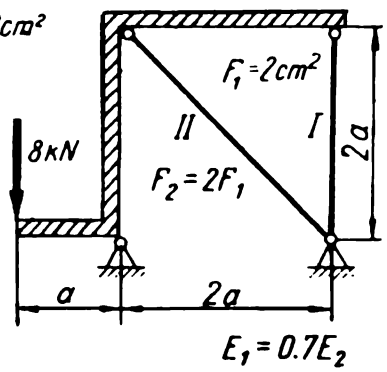

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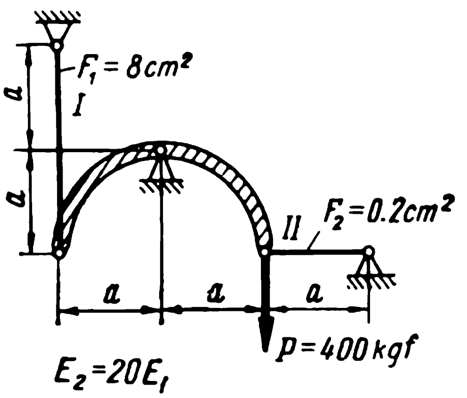

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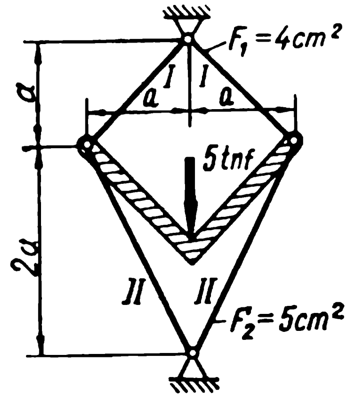

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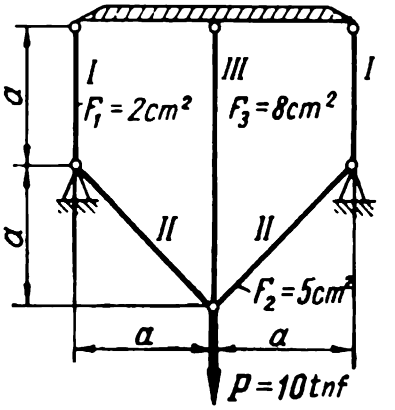
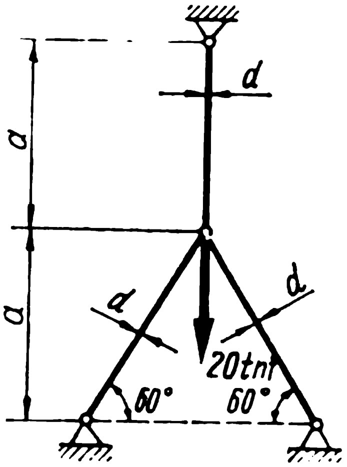


Problems 71 through 73. Determine the dimensions of the cross sections for the elements of the system.

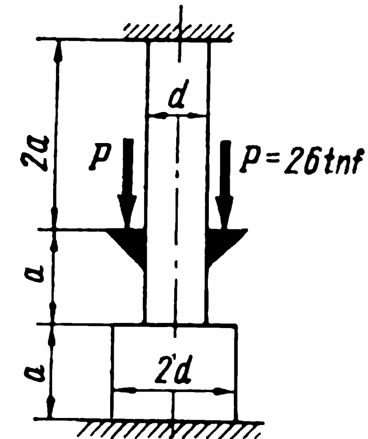
Problems 74 through 79. Find the assembly stresses if Δ is the linear manufacturing error in an elastic element of the system. For bars assume that $E = 2 \times 10^6$ kgf/cm². In Problems 78. and 79, $E = 2 \times 10^5$ MN/m².

71



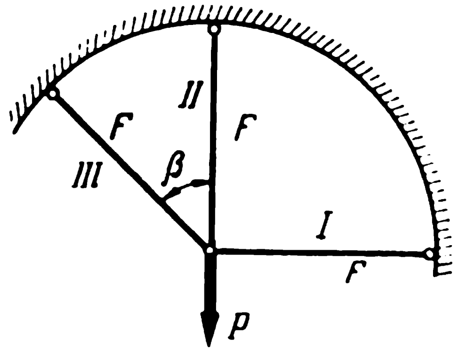
$[\sigma] = 1000 \text{kgf/cm}^2$
$d = ?$

72



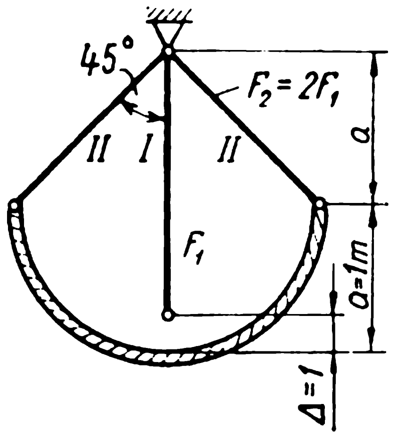
$[\sigma_t] = 20 \text{kgf/cm}^2$;
$[\sigma_c] = 40 \text{kgf/cm}^2$
$d = ?$

73



$[\sigma]$; $N_I = (N_I)_{max}$
$\beta = ?$; $F = ?$

74



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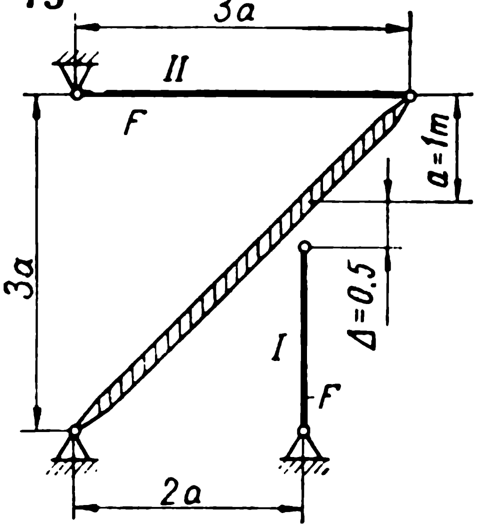

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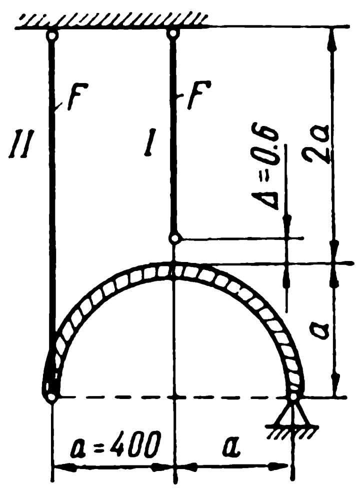

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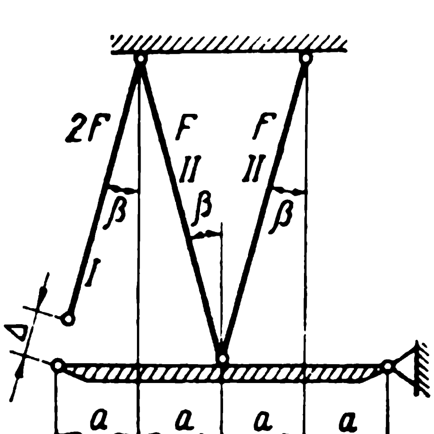

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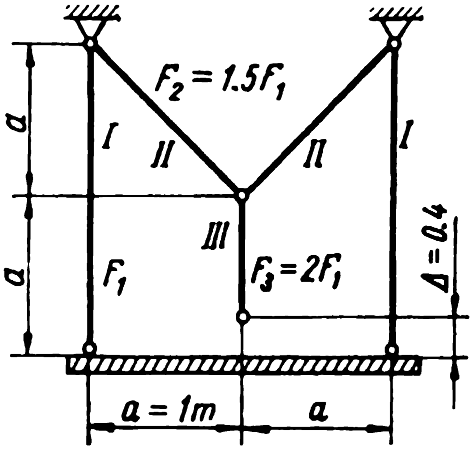

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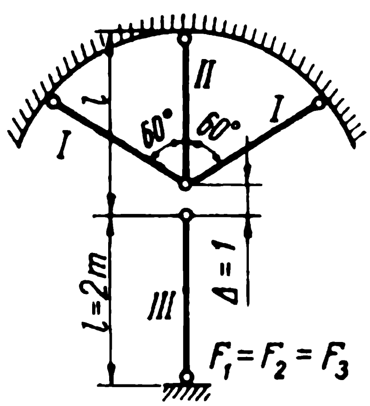


Problems 80 through 87. Determine the thermal stresses.

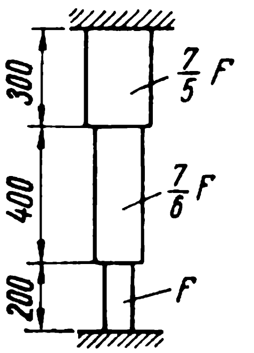
Notation: Δt = change in temperature (deg C) of the entire system; Δt_i = change in temperature of the i-th element of the system; st = = steel; cp = copper.

For steel: $E = 2 \times 10^6$ kgf/cm² and $\alpha = 125 \times 10^{-7}$;

for copper: $E = 1 \times 10^6$ kgf/cm² and $\alpha = 165 \times 10^{-7}$.

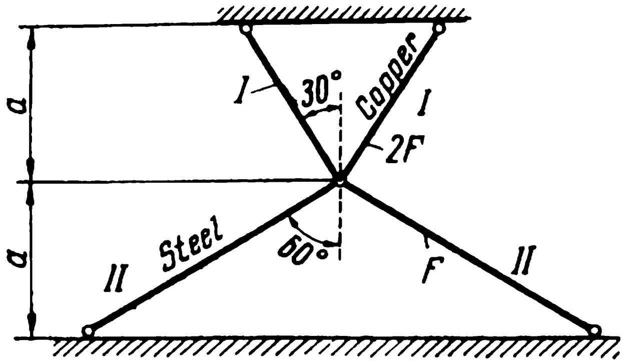
In Problems 84, 86 and 87, for steel assume that $E = 2 \times 10^5$ MN/m²

80



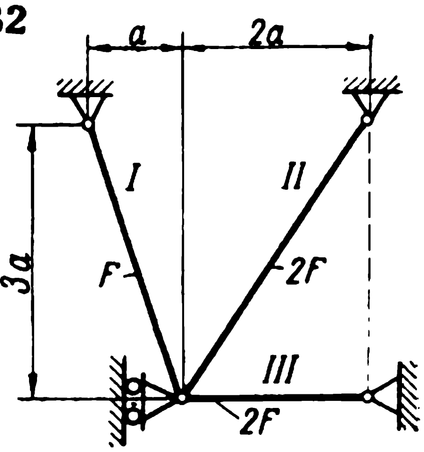
Steel
$\Delta t = 36°$

81



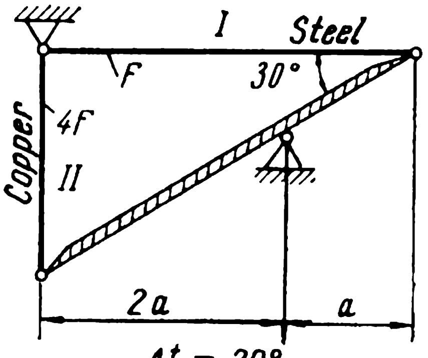
$\Delta t = 10°$

82



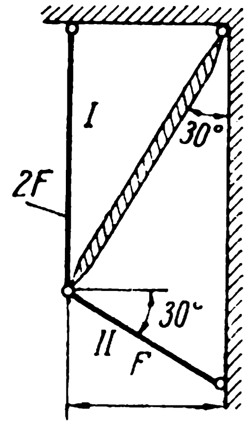
Steel; $\Delta t = 40°$

83



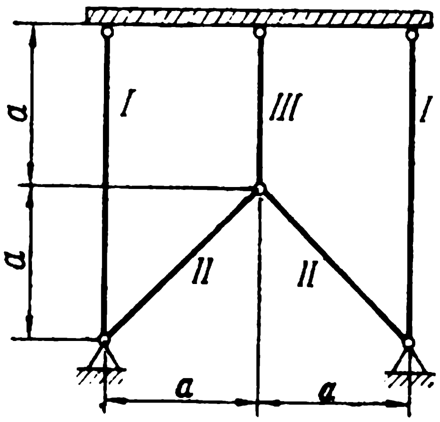
$\Delta t = 20°$

84



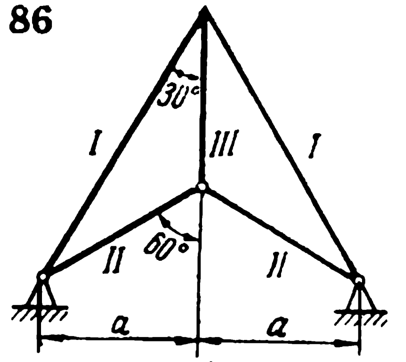
Steel ; $\Delta t = 14°$

85



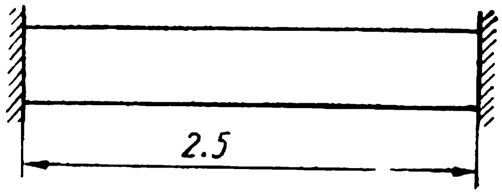
Steel; F ; $\Delta t_1 = 70°$

86



Steel ; F ; $\Delta t_3 = 17.3°$

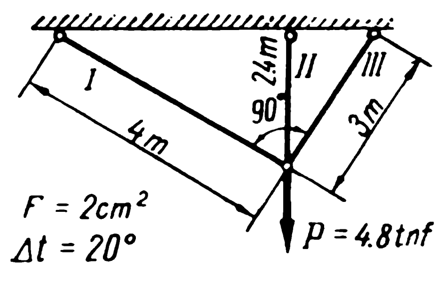
87



Steel; $F = 22$ cm²; $\Delta t = 80°$
Elastic walls with a displacement of 0.25mm each from a force of 10tnf

Problems 88 through 93. Determine the stresses due to the action of the forces and those due to the change in temperature.

88



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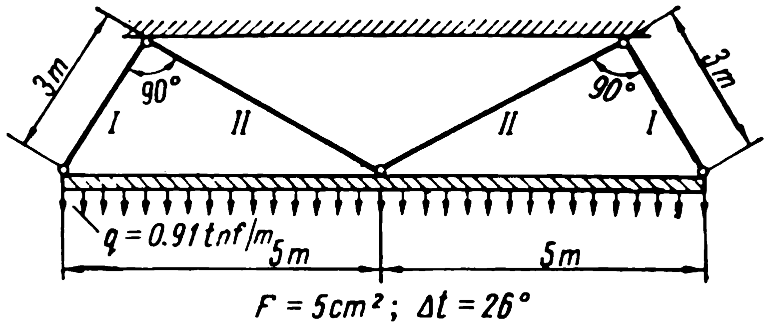

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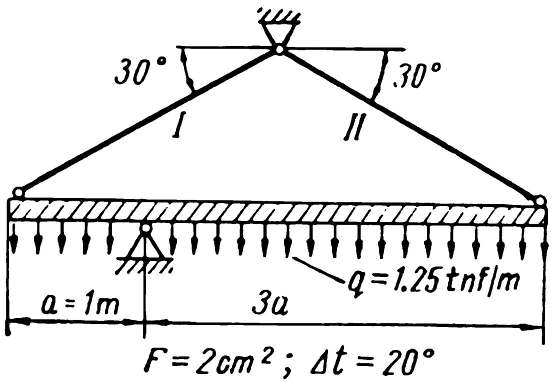

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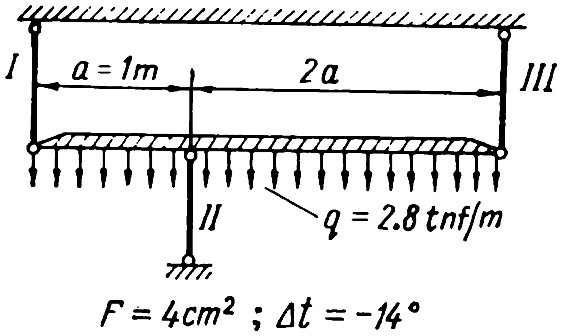

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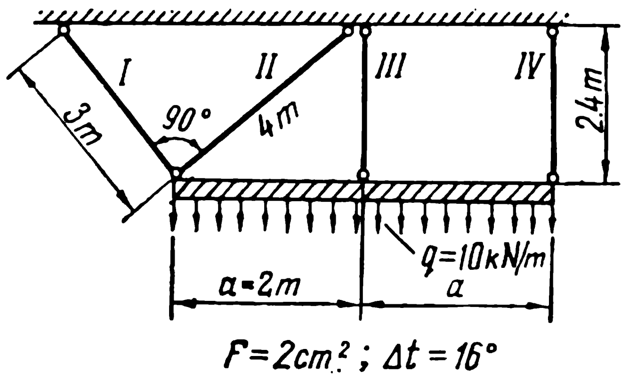

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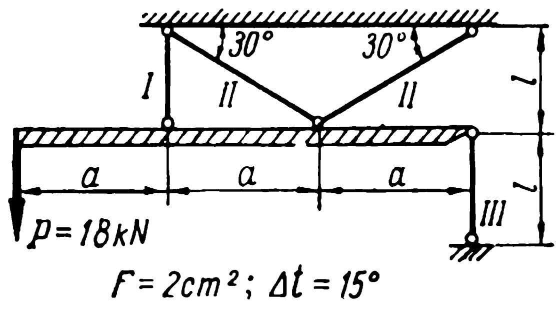


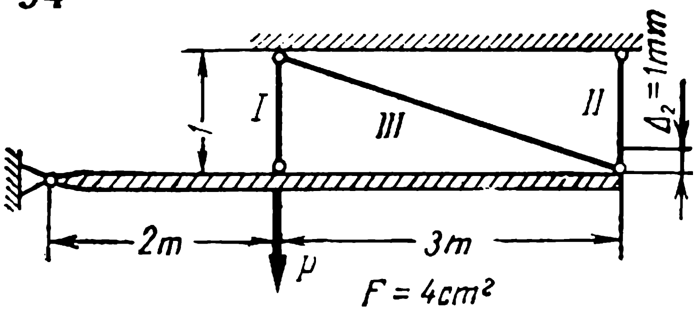
Notation: σ_{i_p} and σ_{i_t} = stresses in the i-th bar due to the action of the forces and to the temperature change, respectively.

For the bars assume that $E = 2 \times 10^6$ kgf/cm² and $\alpha = 12 \times 10^{-6}$. In Problems 92 and 93 assume that $E = 2 \times 10^5$ MN/m².

Problems 94 through 101. Determine the allowable forces calculated on the basis of the permissible stress $[P]$ and of the supporting power $[P']$, and also find the assembly σ_{i_a} and thermal $\sigma_{i_{th}}$ stresses for the conditions specified in the diagrams.

For the bars assume that $[\sigma] = 1600$ kgf/cm², $E = 2 \times 10^6$ kgf/cm² and $\alpha = 12 \times 10^{-6}$. In Problems 99 and 101 assume that $[\sigma] = 160$ MN/m² and $E = 2 \times 10^5$ MN/m².

94



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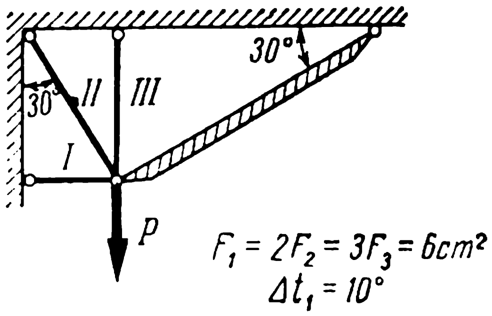

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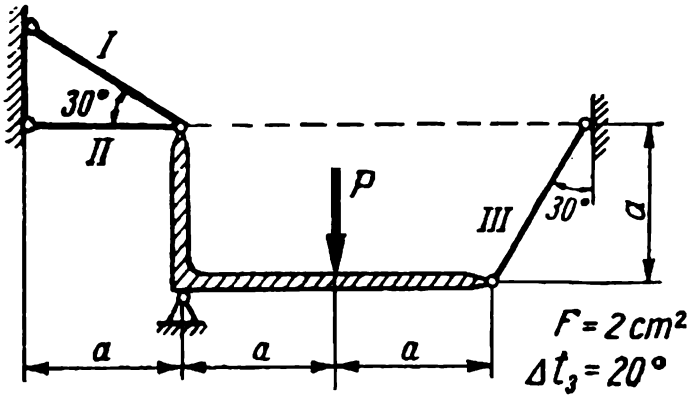

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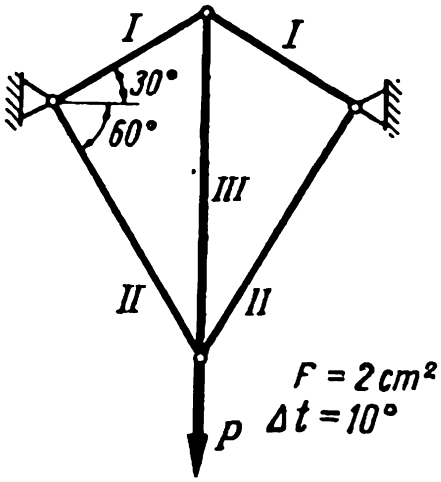

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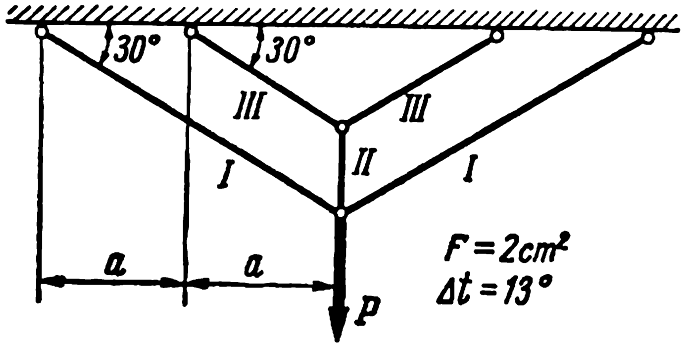

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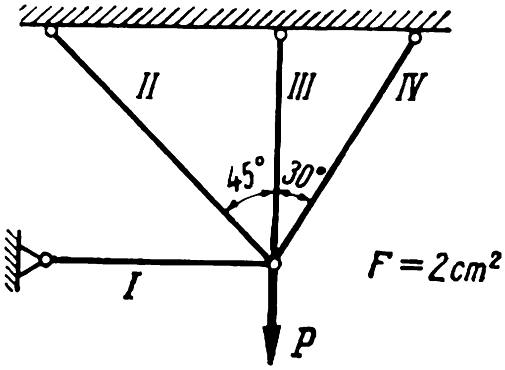

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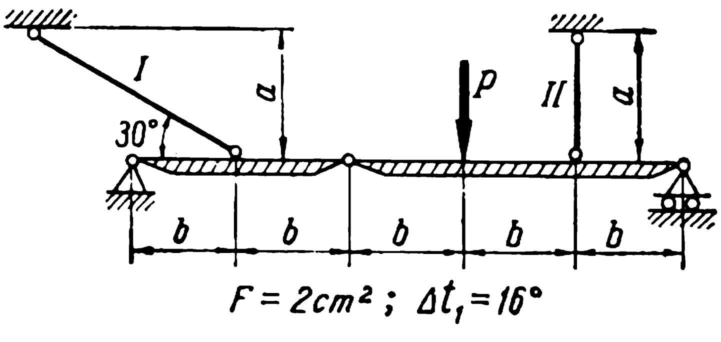

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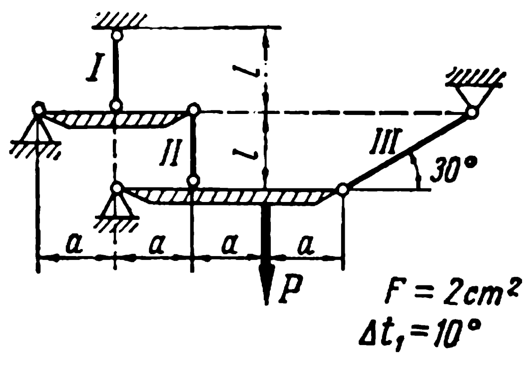
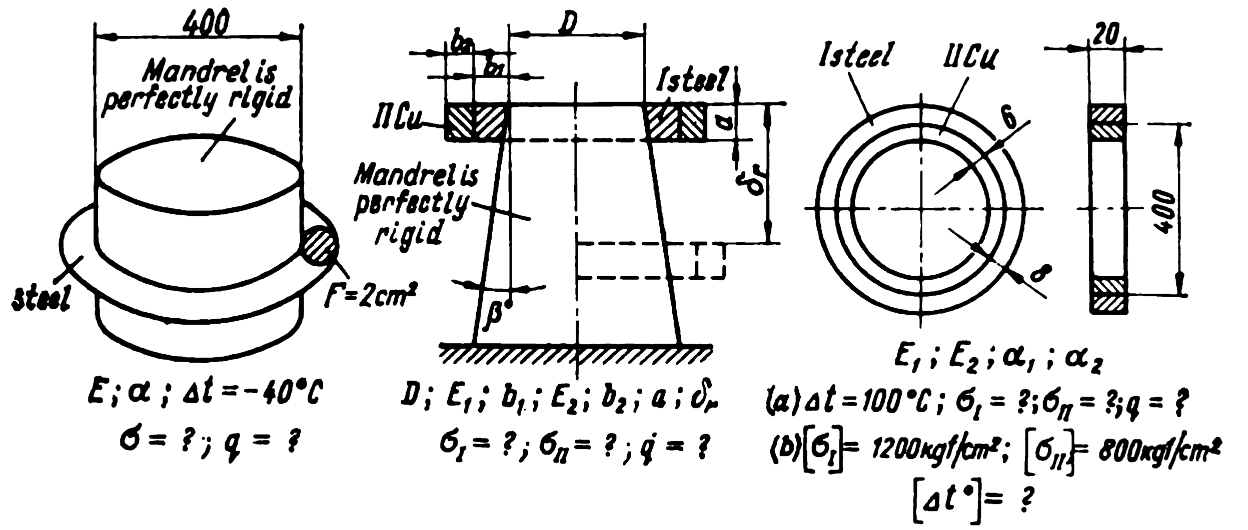


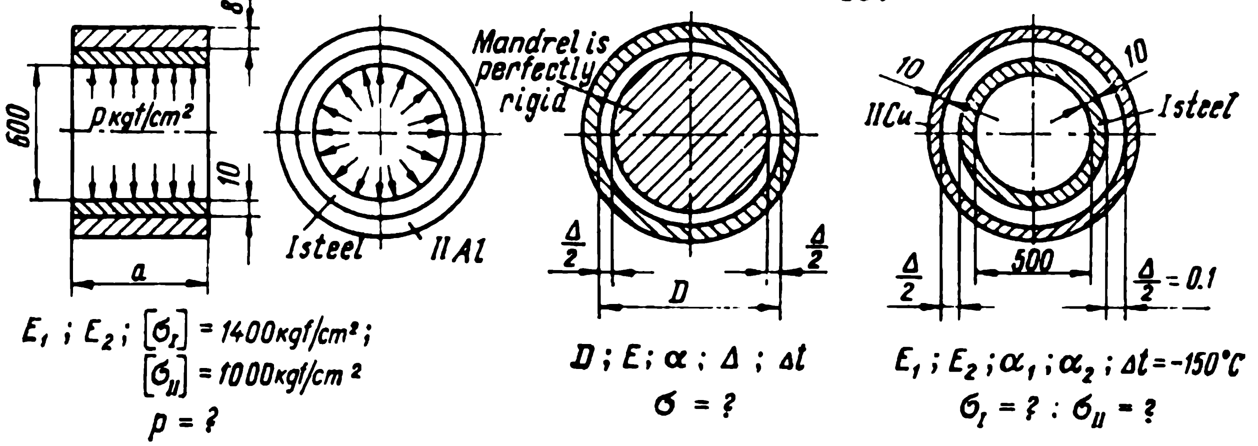
Problems 102 through 107. Determine the quantities indicated in the conditions of the problems on thin-walled cylindrical rings and pipes.

102 **103** **104**



Notation: p = pressure; q = intensity of the distributed load between the mandrel and the ring, or between the rings (kgf/cm).

105 **106** **107**



For steel (st): $E = 2 \times 10^6$ kgf/cm²; $\alpha = 125 \times 10^{-7}$;
for copper (Cu): $E = 1 \times 10^6$ kgf/cm²; $\alpha = 165 \times 10^{-7}$;
for aluminium (Al): $E = 0.7 \times 10^6$ kgf/cm².
In Problems 106 and 107 determine σ provided that clearance Δ is eliminated by lowering the temperature of the system.

CHAPTER 2. STATES OF STRESS AND STRENGTH THEORIES

2.1.

Linear, Planar and Volumetric States of Stress

When a solid is in a volumetric state of stress, three principal nonzero stresses $\sigma_1 > \sigma_2 > \sigma_3$ (Fig. 12*a*) are produced along the faces of an elementary cube taken near to the point being considered in the solid. The areas they act upon (which are void of shearing stresses) are called the *principal areas of stress*. The axes (*I*, *II*, *III*) perpendicular to these areas are called the *principal axes of stress*.

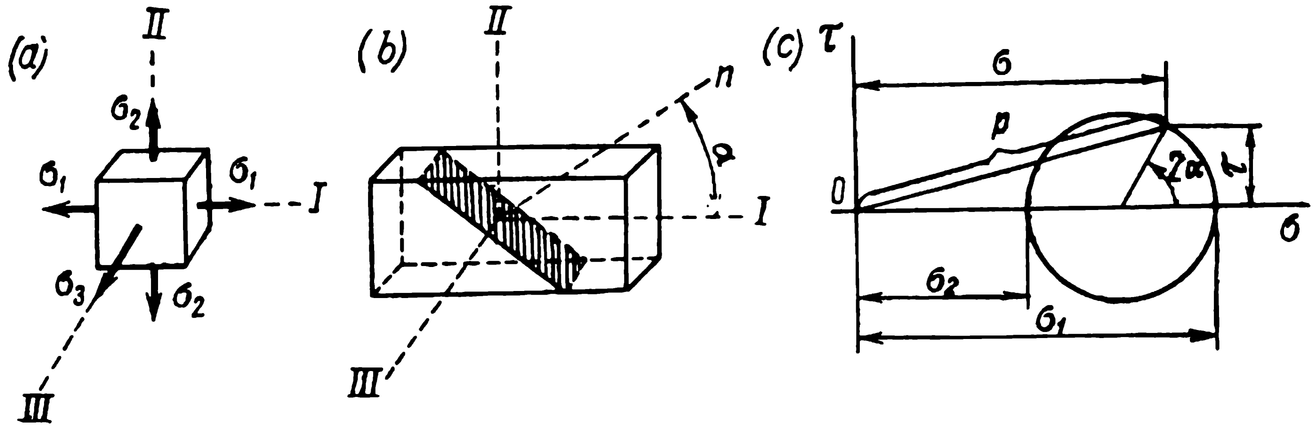


Fig. 12

For inclined sectional planes the normal σ, shearing τ and resultant p stresses are determined from the following formulas:

for planes parallel to axis *III* (Fig. 12*c*):

$$\left.\begin{aligned} \sigma &= \sigma_1 \cos^2\alpha + \sigma_2 \sin^2\alpha; \\ \tau &= \frac{\sigma_1-\sigma_2}{2}\sin 2\alpha; \\ p &= \sqrt{\sigma^2+\tau^2} = \sqrt{\sigma_1^2\cos^2\alpha + \sigma_2^2\sin^2\alpha} \end{aligned}\right\} \qquad (24)$$

for planes parallel to axis *II* (Fig. 13*a*):

$$\left.\begin{aligned} \sigma &= \sigma_1 \cos^2\alpha + \sigma_3 \sin^2\alpha; \\ \tau &= \frac{\sigma_1-\sigma_3}{2}\sin 2\alpha; \\ p &= \sqrt{\sigma_1^2\cos^2\alpha + \sigma_3^2\sin^2\alpha} \end{aligned}\right\} \qquad (25)$$

for planes parallel to axis *I* (Fig. 14*a*):

$$\left.\begin{aligned} \sigma &= \sigma_2 \cos^2\beta + \sigma_3 \sin^2\beta; \\ \tau &= \frac{\sigma_2 - \sigma_3}{2} \sin 2\beta; \\ p &= \sqrt{\sigma_2^2 \cos^2\beta + \sigma_3^2 \sin^2\beta} \end{aligned}\right\} \tag{26}$$

These stresses can be found graphically, using the circle diagrams in Figs. 12*c*; 13*b* and 14*b*. The general circle diagram of stresses is obtained by superposing the three diagrams as shown in Fig. 14*c*.

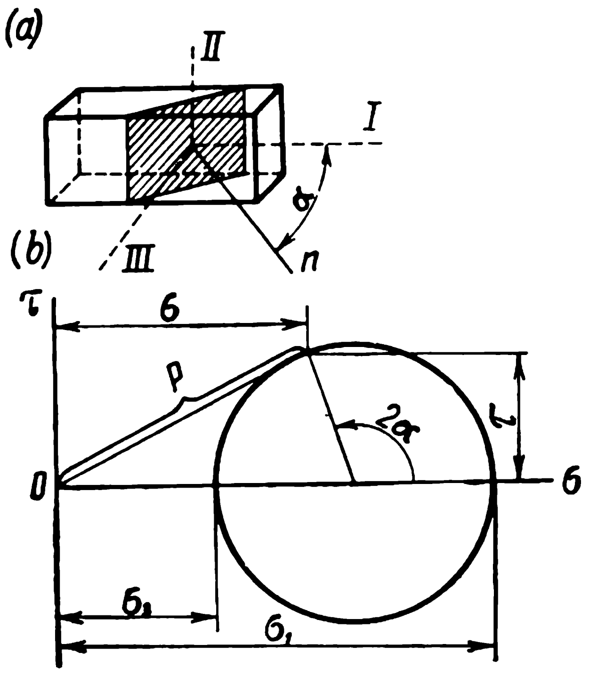


Fig. 13

The extremal shearing stresses are equal to:

$$\left.\begin{aligned} \tau_1 &= \pm \frac{\sigma_2 - \sigma_3}{2}; \\ \tau_2 &= \pm \frac{\sigma_1 - \sigma_3}{2}; \\ \tau_3 &= \pm \frac{\sigma_1 - \sigma_2}{2}; \end{aligned}\right\} \tag{27}$$

of which τ_2 is the maximum (in magnitude).

These stresses occur in planes inclined at 45° to the direction of the principal stresses: τ_1—in two mutually perpendicular planes parallel to axis *I* (Fig. 15*a*); τ_2—in two mutually perpendicular planes parallel to axis *II* (Fig. 15*b*) and τ_3—in two mutually perpendicular planes parallel to axis *III* (Fig. 15*c*).

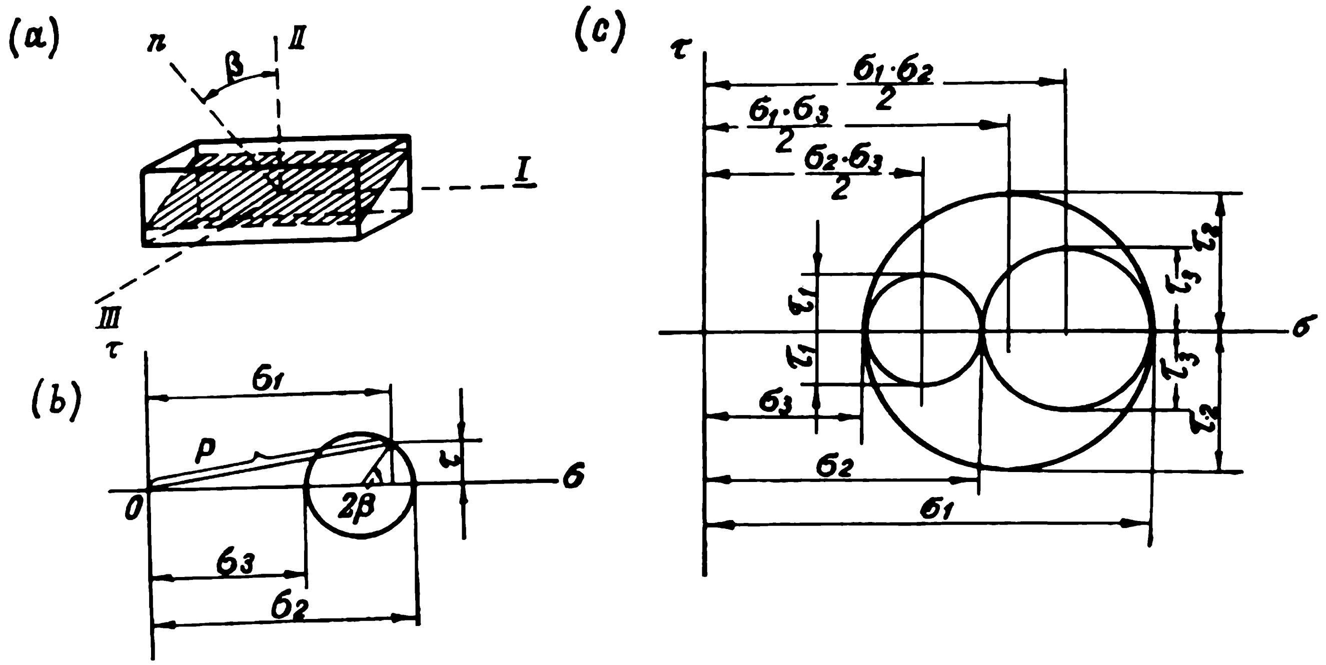


Fig. 14

The octahedral normal (σ_0), shearing (τ_0), and resultant (p_0) stresses, acting on an area equally inclined to the three principal axes of stress (Fig. 16), are determined from the following formulas:

$$\left.\begin{aligned}\sigma_0 &= \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3);\\ \tau_0 &= \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2};\\ p_0 &= \sqrt{\frac{1}{3}(\sigma_1^2+\sigma_2^2+\sigma_3^2)}\end{aligned}\right\} \quad (28)$$

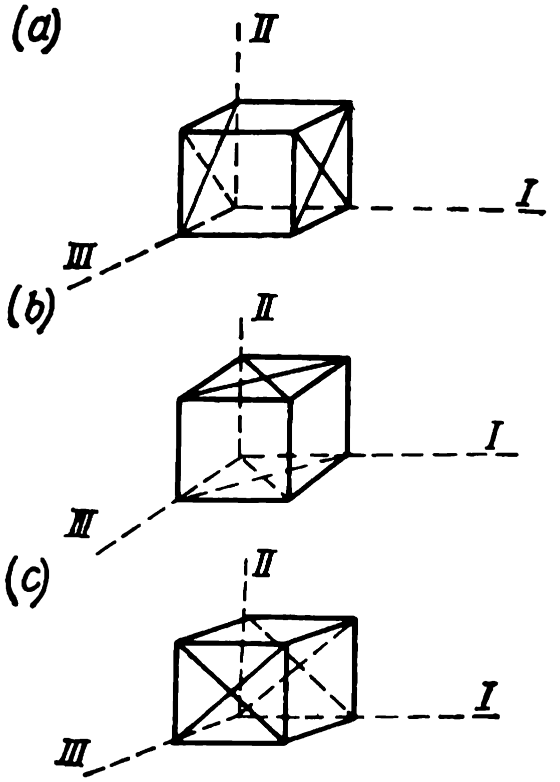


Fig. 15

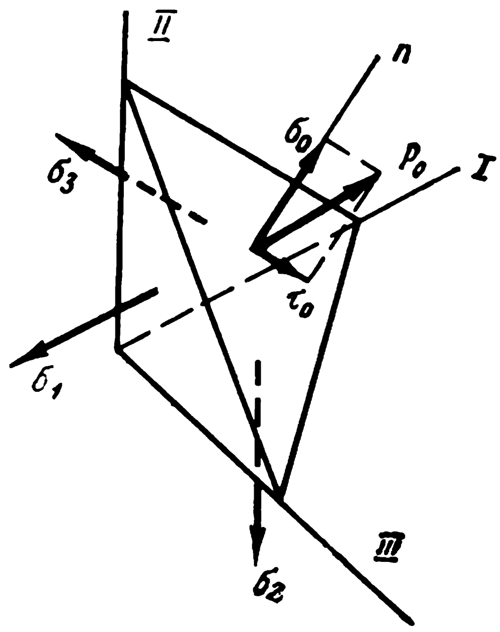


Fig. 16

The principal linear deformations ε_1, ε_2, ε_3 (unit elongations occurring along the directions of action of the principal stresses) have the following values:

$$\left.\begin{aligned}\varepsilon_1 &= \frac{1}{E}[\sigma_1-\mu(\sigma_2+\sigma_3)];\\ \varepsilon_2 &= \frac{1}{E}[\sigma_2-\mu(\sigma_3+\sigma_1)];\\ \varepsilon_3 &= \frac{1}{E}[\sigma_3-\mu(\sigma_1+\sigma_2)]\end{aligned}\right\} \quad (29)$$

The unit volume change is

$$\frac{\Delta V}{V} = \varepsilon_1+\varepsilon_2+\varepsilon_3 = \frac{1-2\mu}{E}(\sigma_1+\sigma_2+\sigma_3) \quad (30)$$

The quantity $3 \times \frac{1-2\mu}{E} = \chi$ is called the *coefficient of compressibility of a material*, and the inverse quantity $\frac{E}{3(1-2\mu)} = K$ is called the *bulk modulus of a material*.

The specific strain energy is

$$u = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3) = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad (31)$$

The specific strain energy due to distortion is

$$u_{dist} = \frac{1+\mu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \quad (32)$$

The specific strain energy due to change of volume is

$$u_{vol} = \frac{1-2\mu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 \quad (33)$$

All the formulas relating to the volumetric state of stress can also be applied to the planar state of stress, if one principal stress is equated to zero, as well as to the linear state of stress, if two principal stresses are equated to zero.

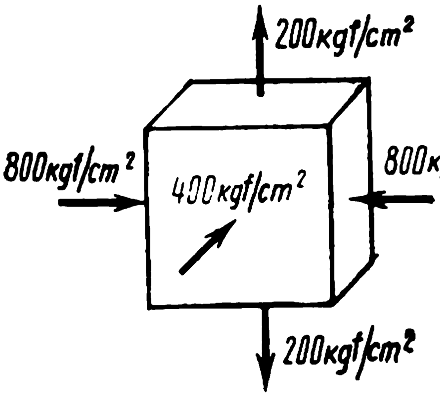


Fig. 17

Example 10. Let the state of stress be as shown in Fig. 17 with

$$E = 2 \times 10^6 \text{ kgf/cm}^2 \text{ and } \mu = 0.3$$

Determine the following quantities analytically and graphically: $\tau_{1,2,3}$; σ' and τ' in a plane parallel to axis *I* at $\beta = 30°$; σ'' and τ'' in a plane parallel to axis *II* at $\alpha = 60°$; σ''' and τ''' in a plane parallel to axis *III* at $\alpha = 30°$; p_0, σ_0, τ_0; $\varepsilon_{1,2,3}$; $\frac{\Delta V}{V}$; u, u_{dist} and u_{vol}.

Solution. Since the principal stresses are: $\sigma_1 = 200$ kgf/cm², $\sigma_2 = -400$ kgf/cm² and $\sigma_3 = -800$ kgf/cm², then, from formula (27), the extremal shearing stresses will be

$$\tau_1 = \pm\frac{-400+800}{2} = \pm 200 \text{ kgf/cm}^2;$$

$$\tau_2 = \pm\frac{200+800}{2} = \pm 500 \text{ kgf/cm}^2;$$

$$\tau_3 = \pm\frac{200+400}{2} = \pm 300 \text{ kgf/cm}^2$$

From formulas (26), in a plane parallel to axis I at

$$\beta = 30^\circ$$

$$\sigma' = -400\cos^2 30^\circ - 800\sin^2 30^\circ = -500 \text{ kgf/cm}^2;$$

$$\tau' = \frac{-400+800}{2}\sin 60^\circ \cong 173 \text{ kgf/cm}^2$$

From formulas (25), in a plane parallel to axis II at

$$\alpha = 60^\circ$$

$$\sigma'' = 200\cos^2 60^\circ - 800\sin^2 60^\circ = -550 \text{ kgf/cm}^2;$$

$$\tau'' = \frac{200+800}{2}\sin 120^\circ \cong 433 \text{ kgf/cm}^2$$

From formulas (24), in a plane parallel to axis III at $\alpha = 30^\circ$

$$\sigma''' = 200\cos^2 30^\circ - 400\sin^2 30^\circ = 50 \text{ kgf/cm}^2;$$

$$\tau''' = \frac{200+400}{2}\sin 60^\circ = 260 \text{ kgf/cm}^2$$

The graphical determination of all the stresses is shown in the circle diagram of Fig. 18.

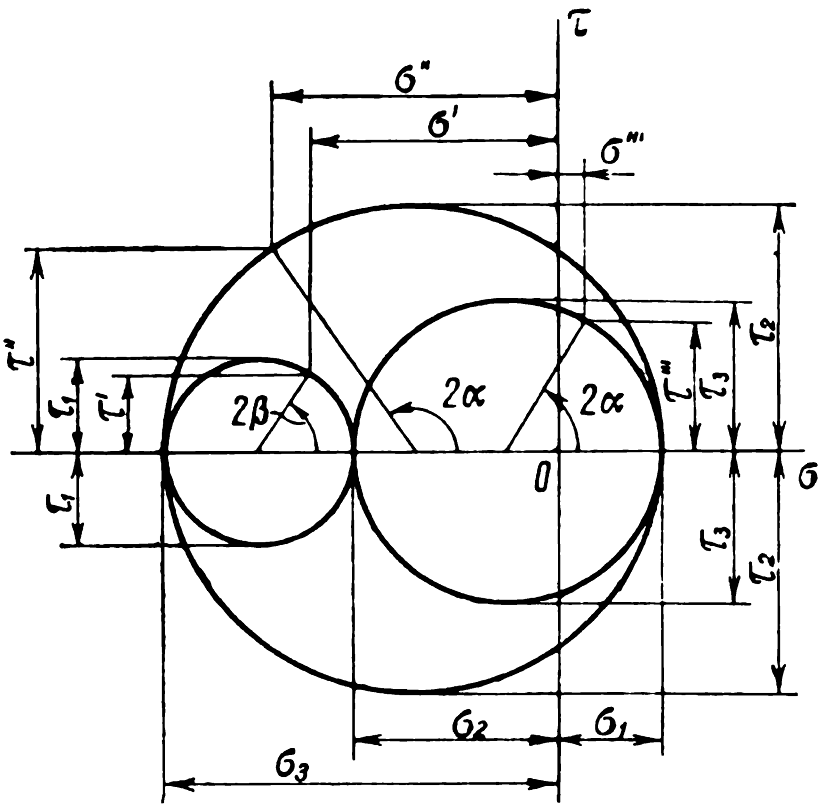


Fig. 18

The octahedral stresses are determined by formulas (28):

$$p_0 = 100\sqrt{\frac{1}{3}(4+16+64)} \cong 529 \text{ kgf/cm}^2;$$

$$\sigma_0 = \frac{1}{3}(200-400-800) \cong -333 \text{ kgf/cm}^2;$$

$$\tau_0 = \frac{100}{3}\sqrt{9+4+100} \cong 354 \text{ kgf/cm}^2$$

The principal linear deformations are found by formulas (29):

$$\varepsilon_1 = \frac{100}{2\times 10^6}(2+0.3\times 1.2) = 2.8\times 10^{-4};$$

$$\varepsilon_2 = \frac{100}{2\times 10^6}(-4+0.3\times 6) = -1.1\times 10^{-4};$$

$$\varepsilon_3 = \frac{100}{2\times 10^6}(-8+0.3\times 2) = -3.7\times 10^{-4}$$

The unit volume change is determined by formula (30):

$$\frac{\Delta V}{V} = 10^{-4}(2.8-1.1-3.7) = -2\times 10^{-4}$$

The specific strain energy and the strain energy due to change of volume are found by formulas (31) and (33):

$$u = 100\times 10^{-4}\left(\frac{2\times 2.8}{2}+\frac{4\times 1.1}{2}+\frac{8\times 3.7}{2}\right)$$
$$= 19.8\times 10^{-2} \text{ kgf-cm/cm}^3;$$

$$u_{vol} = \frac{1-0.6}{6\times 2\times 10^6}\times 10^4(2-4-8)^2 \cong 3.3\times 10^{-2} \text{ kgf-cm/cm}^3$$

The specific strain energy due to distortion is

$$u_{dist} = u - u_{vol} = (19.8-3.3)\times 10^{-2} = 16.5\times 10^{-2} \text{ kgf-cm/cm}^3$$

Problems 108 through 122. Determine the following quantities analytically and by means of circle diagrams:

1. The extremal shearing stresses τ_1, τ_2 and τ_3.
2. The normal σ_α and shearing τ_α stresses:

(a) in a plane parallel to axis *I*, where the normal (to the plane) makes an angle $\beta = 30°$ with axis *II*;

(b) in a plane parallel to axis *II*, where the normal (to the plane) makes an angle $\alpha = 60°$ with axis *I*;

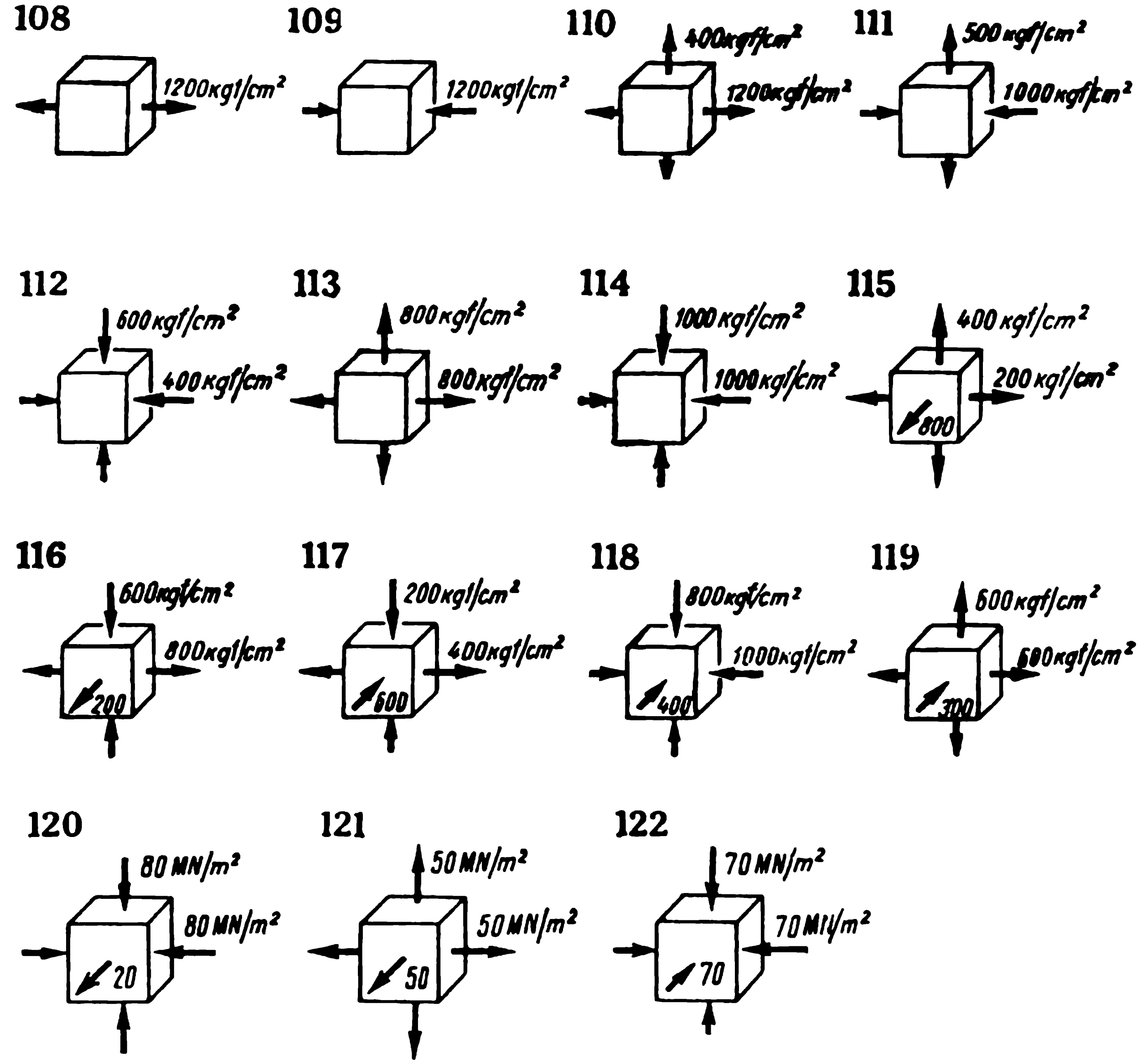
(c) in a plane parallel to axis *III*, where the normal (to the plane) makes an angle $\alpha = 30°$ with axis *I*.

3. The octahedral resultant p_0, normal σ_0 and shearing τ_0 stresses.

Problems 108′ through 122′. Determine the principal linear deformations ε_1, ε_2 and ε_3; unit volume change $\frac{\Delta V}{V}$; specific strain energy u and its parts due to distortion u_{dist} and change of volume u_{vol}.

Analyse the states of stress indicated in Problems 108 through 122.

Assume that $E = 2\times 10^6$ kgf/cm^2 and $\mu = 0.3$. In Problems 120, 121 and 122 assume that $E = 2\times 10^5$ MN/m^2.



2.2.

Strength Theories and Equivalent Stresses

The various strength theories propose criteria which determine the strength of an element of a material subject to complex stress conditions. According to these criteria, equivalent, or reduced, stresses (σ_{eq}) are established, i.e. stresses due to uniaxial tension of an element of a material which makes its stressed state equal to a given complex state of stress.

Irrespective of the strength theory adopted, the condition for the mechanical strength of an element of a material in any state of stress has the form

$$\sigma_{eq} \leqslant [\sigma_t] \tag{34}$$

For the volumetric state of stress of an element, the equivalent stresses have the following values:

according to the maximum normal stress theory

$$\sigma_{eq_I}=\sigma_1 \quad \text{at} \quad \sigma_1>0^{*} \tag{35}$$

according to the maximum linear strain theory

$$\sigma_{eq_{II}}=\sigma_1-\mu(\sigma_2+\sigma_3) \tag{36}$$

according to the maximum shearing stress theory

$$\sigma_{eq_{III}}=\sigma_1-\sigma_3 \tag{37}$$

according to the theory of specific strain energy due to distortion

$$\sigma_{eq_{IV}}=\sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]} \tag{38}$$

according to the theory of limiting states of stress

$$\sigma_{eq_V}=\sigma_1-\nu\sigma_3 \tag{39}$$

where

$$\nu=\frac{[\sigma_t]}{[\sigma_c]} \tag{40}$$

Example 11. For the volumetric state of stress (see Fig. 19), $\sigma_1 = 200$ kgf/cm², $\sigma_2 = -400$ kgf/cm², $\sigma_3 = -800$ kgf/cm² and $\mu = 0.3$.

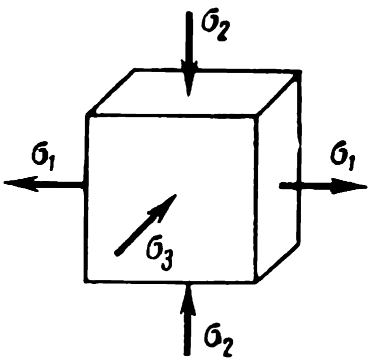


Fig. 19

Determine the equivalent stresses according to the above strength theories.

In determining the equivalent stress according to the theory of limiting states of stress assume that $\nu = 0.25$.

Solution. $\sigma_{eq_I} = 200$ kgf/cm², $\sigma_{eq_{II}} = 200 + 0.3 \quad (400 + 800) = 560$ kgf/cm², $\sigma_{eq_{III}} = 200 + 800 = 1000$ kgf/cm².

$$\sigma_{eq_{IV}}=100\sqrt{\frac{1}{2}[(2+4)^2+(-4+8)^2+(-8-2)^2]}\cong 872 \text{ kgf/cm}^2;$$

$$\sigma_{eq_V}=200+0.25\times 800=400 \text{ kgf/cm}^2$$

Problems 108″ through 122″. Determine the values of equivalent stresses using the strength theories.

Analyze the states of stress indicated in Problems 108 through 122, assuming that $\mu = 0.3$. For the 5th theory (that of limiting states of stress) assume that $\nu = 0.5$.

---

* Sometimes, when $|\sigma_3|>\sigma_1$, the strength according to the 1st theory is computed from the formulas:

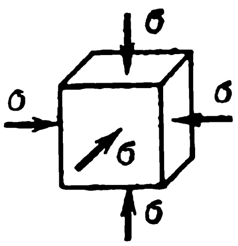
$$\sigma_1 \leqslant [\sigma_t] \text{ and } [\sigma_3] \leqslant [\sigma_c]$$

Problems 123 through 127. Determine the quantities indicated in the conditions of the problems.

Notation: $\varkappa$ = compressibility factor of the material; K = bulk modulus; p = intensity of the distributed load. Friction is to be neglected in all cases.

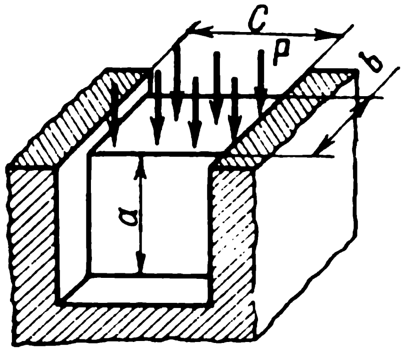
In Problem 123 assume for steel that $E = 2 \times 10^6$ kgf/cm² and $\mu = 0.28$; for copper that $E = 1 \times 10^6$ kgf/cm² and $\mu = 0.34$; and for aluminium that $E = 0.7 \times 10^6$ kgf/cm² and $\mu = 0.33$.

123



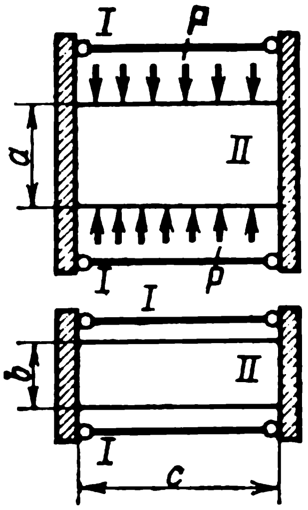
(a) steel
(b) copper
(c) aluminium
$\varkappa = ?$; $K = ?$

124



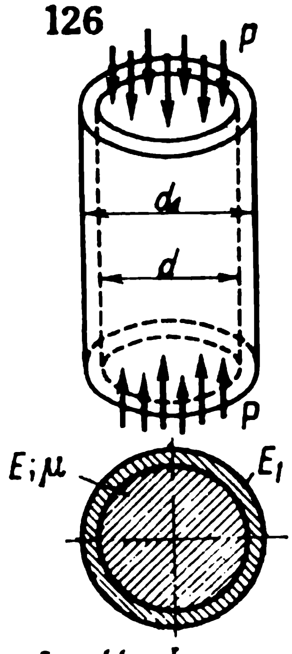
p; a; b; c; μ; E
$\sigma_{1,2,3} = ?$; $\Delta a = ?$; $\Delta b = ?$

125



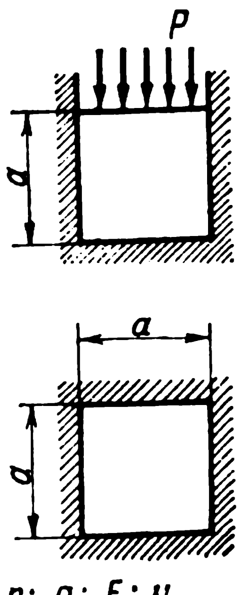
E_1; F_1; E_2; μ_2
$\sigma_I = ?$; $(\sigma_{1,2,3})_{II} = ?$ $\left(\frac{\Delta V}{V}\right)_{II} = ?$

126



In the bar
$\sigma_1 = ?$; $\sigma_2 = ?$; $\sigma_3 = ?$

127



p; a; E; μ
$\sigma_{1,2,3} = ?$ $\Delta a = ?$
$U = ?$

CHAPTER 3. THIN-WALLED VESSELS

This treatment will be confined to vessels having the form of a thin-walled solid of revolution and subject to a continuous internal pressure of intensity p, not necessarily uniform but symmetrically distributed with reference to the axis of revolution (Fig. 20). If the thickness of the wall is small in comparison with the radii of curvature and there are no discontinuities such as sharp bends in the meridional curves, the stresses can be calculated with sufficient accuracy by neglecting the bending of the wall of the vessel, i.e. by using the so-called membrane theory of calculation.

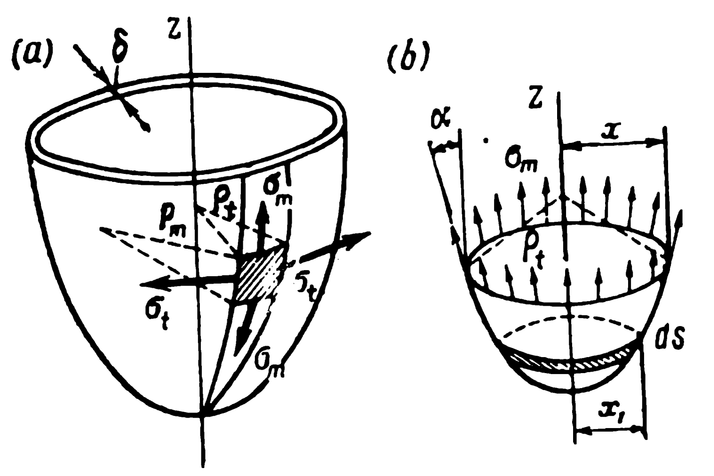


Fig. 20

According to this theory, from the condition of equilibrium of an element isolated from the wall of the vessel near the point being considered by two meridional sections and two sections perpendicular to them (see Fig. 20*a*), we obtain an equation (Laplace's equation) for determining the circumferential (σ_t) and meridional (σ_m) normal stresses

$$\frac{\sigma_t}{\rho_t}+\frac{\sigma_m}{\rho_m}=\frac{p}{\delta} \tag{41}$$

where ρ_t = circumferential radius of curvature of the wall at the level of the point being considered

ρ_m = meridional radius of curvature of the wall at the level of the point being considered

p = intensity of the internal pressure which is a function of coordinate z only

δ = thickness of the wall of the vessel.

From the condition of equilibrium of the portion of the vessel cut away by the sections taken perpendicular to the meridians at the level of the point being considered (Fig. 20*b*) we obtain one more equation

$$\sigma_m \delta x \cos\alpha = Z \tag{42}$$

where: x = radius of the circular cross section at the level being considered

α = angle between the z-axis and a line tangent to the meridian at the same level

Z = sum of the projections on the z-axis of the forces acting on the cut-away portion of the vessel (Z is referred to an arc equal to the radius),

$$Z = \int_0^x p x_1 \, dx_1 \tag{43}$$

Here x_1 is the running radius of the circular cross section of the vessel.

Solving equations (41) and (42) simultaneously, we obtain the following values for the stresses σ_t and σ_m:

$$\left.\begin{aligned} \sigma_t &= \frac{p\rho_t}{\delta} - \frac{Z}{\delta\rho_m \cos^2\alpha}; \\ \sigma_m &= \frac{Z}{\delta\rho_t \cos^2\alpha} \end{aligned}\right\} \tag{44}$$

In particular cases:

1. For a vessel with a straight generatrix $\rho_m = \infty$ and $\rho_t = \rho$. Then:

$$\left.\begin{aligned} \sigma_t &= \frac{p\rho}{\delta}; \\ \sigma_m &= \frac{Z}{\delta\rho \cos^2\alpha} \end{aligned}\right\} \tag{45}$$

2. For a spherical vessel $\rho_t = \rho_m = \rho$. Then:

$$\left.\begin{aligned} \sigma_t &= \frac{p\rho}{\delta} - \frac{Z}{\delta\rho \cos^2\alpha}; \\ \sigma_m &= \frac{Z}{\delta\rho \cos^2\alpha} \end{aligned}\right\} \tag{46}$$

(a) For p = const (pressure of a gas or vapour):

$$Z = \frac{px^2}{2} = \frac{1}{2} p\rho^2 \cos^2\alpha \tag{47}$$

$$\left.\begin{aligned} \sigma_t &= \frac{p\rho_t}{2\delta}\left(2 - \frac{\rho_t}{\rho_m}\right); \\ \sigma_m &= \frac{p\rho_t}{2\delta} \end{aligned}\right\} \tag{48}$$

$$\text{At } \rho_m = \infty \text{ and } \rho_t = \rho: \ \sigma_t = 2\sigma_m = \frac{p\rho}{\delta}. \tag{49}$$

$$\text{At } \rho_t = \rho_m = \rho: \ \sigma_t = \sigma_m = \frac{p\rho}{2\delta}. \tag{50}$$

(b) For $p = \gamma (h - z)$ (liquid pressure, Fig. 21)

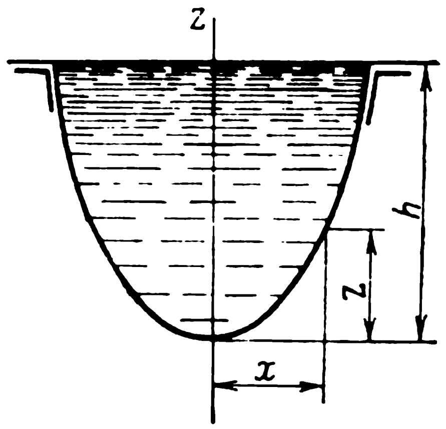


Fig. 21

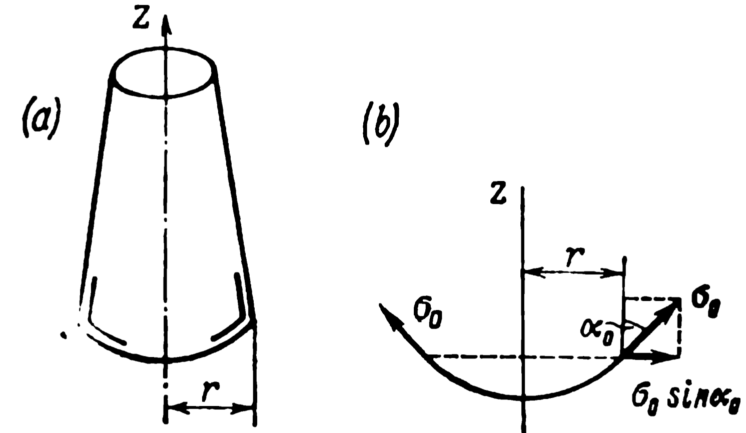


Fig. 22

where: γ = weight of unit volume of the liquid
h = height of the liquid level in the vessel
z = running ordinate.

$$Z = \gamma \left(\frac{h\rho_t^2 \cos^2 \alpha}{2} - Z_1 \right) \tag{51}$$

The quantity

$$Z_1 = \int^x zx\, dx \tag{52}$$

is easily determined, if the equation of the generatrix of the vessel $z = z(x)$ is known.

The third principal normal stress $\sigma_r = -p$ is developed on the internal surface of the vessel's walls. In most cases it is very small compared with σ_t and σ_m, and may be neglected in structural design.

If the walls of the vessel have a sharp bend (Fig. 22*a*), then edge forces occur in the section at the corner which may lead to considerable overstresses. These are not taken into account by the membrane theory. To reduce the effect of these forces the section at the joint is often strengthened by a reinforcing ring.

If the meridional normal stresses in the section at the joint are $\sigma_m = \sigma_0$ (Fig. 22*b*), then the linear thrust force is

$$q_0 = \sigma_0 \delta \sin \alpha_0 \tag{53}$$

The required area F of a reinforcing ring of radius r can be found by the formula

$$F = \frac{q_0 r}{[\sigma]} = \frac{\sigma_0 \delta r \sin \alpha_0}{[\sigma]} \tag{54}$$

Example 12. Let $\gamma = 1.2$ gf/cm³; $h_1 = 4$m; $r = 1$ m; $\alpha = 60°$ and $[\sigma] = 1000$ kgf/cm² (Fig. 23). Find δ and F.

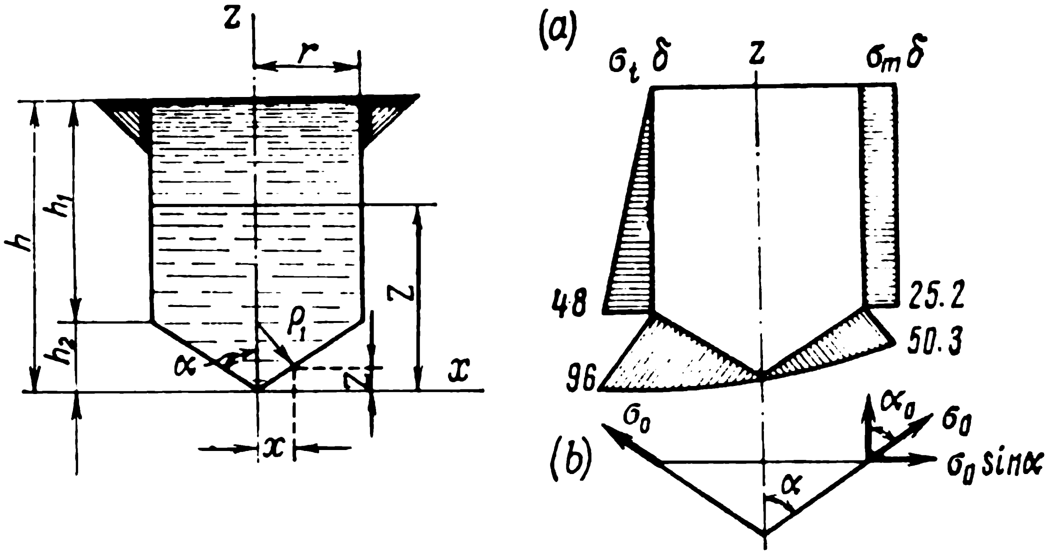


Fig. 23

Solution. For the conical part of the vessel we can write

$$(0 \leqslant z \leqslant h_2)\ \rho_m = \infty; \quad \rho_t = \rho = \frac{x}{\cos \alpha} :$$

$z = x \cot \alpha$; $p = \gamma (h - z) = \gamma (h - x \cot \alpha)$
From formula (52)

$$Z_1 = \cot \alpha \int_0^x x^2\, dx = \frac{x^3}{3} = \cot \alpha$$

According to formula (51) the weight of the liquid contained in the cut-away part of the vessel

$$Z = \gamma \left(\frac{hx^2}{2} - \frac{x^3}{3} \cot \alpha \right) = \gamma x^2 \left(\frac{h}{2} - \frac{x}{3} \cot \alpha \right)$$

According to formula (45), the circumferential σ_t and meridional σ_m normal stresses at points of the wall of the conical part of the vessel, at any arbitrary level specified by coordinate x, are

$$\sigma_t = \frac{\gamma (h - x \cot \alpha) x}{\delta \cos \alpha} = \frac{\gamma x}{\delta} \left(\frac{h}{\cos \alpha} - \frac{x}{\sin \alpha} \right);$$

$$\sigma_m = \frac{\gamma x^2 \left(\frac{h}{2} - \frac{x}{3} \cot \alpha \right)}{\delta x \cos \alpha} = \frac{\gamma x}{\delta} \left(\frac{h}{2 \cos \alpha} - \frac{x}{3 \sin \alpha} \right)$$

Since

$$h = h_1 + r\cot\alpha \quad \text{and} \quad \frac{h}{\cos\alpha} = \frac{h_1}{\cos\alpha} + \frac{r}{\sin\alpha}$$

then

$$\sigma_{t_{x=0}} = 0; \quad \sigma_{m_{x=0}} = 0; \quad \sigma_{t_{x=r}} = \frac{\gamma r}{\delta}\left(\frac{h}{\cos\alpha} - \frac{r}{\sin\alpha}\right) = \frac{\gamma r h_1}{\delta\cos\alpha};$$

$$\sigma_{m_{x=r}} = \frac{\gamma r}{2\delta}\left(\frac{h}{\cos\alpha} + \frac{r}{3\sin\alpha}\right)$$

For the given numerical values

$$\sigma_{t_{x=r}} = \frac{1.2\times 10^{-3}\times 10^2\times 4\times 10^2\times 2}{\delta} = \frac{96}{\delta}\ \text{kgf/cm}^2;$$

$$\sigma_{m_{x=r}} = \frac{1.2\times 10^{-3}\times 10^2}{2\delta}\left(\frac{4\times 10^2\times 2}{1} + \frac{10^2\times 2}{3\sqrt{3}}\right) \cong \frac{50.3}{\delta}\ \text{kgf/cm}^2$$

For the cylindrical part of the vessel $(h_2 \leqslant z \leqslant h)$ $\rho_m = \infty$; $\rho_t = \rho = r$; $p = \gamma\,(h - z)$; $\alpha = 0$ and Z is equal to Z of the conical part at $x = r$, i.e. $Z = Z_{x=r} = \gamma r^2\left(\frac{h}{2} - \frac{r}{3}\cot\alpha\right)$.

The circumferential σ_t and meridional σ_m normal stresses at points of the wall of the cylindrical part of the vessel are found by formula (43). Thus

$$\sigma_t = \frac{\gamma r}{\delta}(h - z);$$

$$\sigma_m = \frac{\gamma r}{2\delta}\left(h - \frac{2}{3}r\cot\alpha\right) = \frac{\gamma r}{2\delta}\left(h_1 + \frac{h_2}{3}\right) = \frac{\gamma r}{2\delta}\left(h_1 + \frac{r}{3}\cot\alpha\right)$$

$$\sigma_{t_{z=h_2}} = \frac{\gamma r}{\delta}h_1, \quad \sigma_{t_{z=h}} = 0; \quad \sigma_m = \text{const}$$

For the given numerical values

$$\sigma_{t_{z=h_2}} = \frac{1.2\times 10^{-3}\times 10^2\times 4\times 10^2}{\delta} = \frac{48}{\delta}\ \text{kgf/cm}^2;$$

$$\sigma_m = \frac{1.2\times 10^{-3}\times 10^2}{2\delta}\left(4\times 10^2 + \frac{10^2}{3}\times 0.577\right) \cong \frac{25.2}{\delta}\ \text{kgf/cm}^2$$

Diagrams of the stresses σ_t and σ_m are shown in Fig. 23*a*.

The upper section of the conical part of the vessel, where $\sigma_t = \frac{96}{\delta}$ and $\sigma_m = \frac{50.3}{\delta}$ is considered to be the dangerous one.

Using the 3rd strength theory, we can determine the wall thickness δ of the vessel. Assuming that $\sigma_r = 0$, we obtain the following design

equation:

$$\sigma_t = \frac{96}{\delta} = [\sigma]$$

whence

$$\delta = \frac{96}{[\sigma]} = \frac{96}{1000} = 0.096 \text{ cm} = 0.96 \text{ mm}$$

which we round off to $\delta = 1$ mm.

At the joint of the conical and cylindrical parts (Fig. 23*b*)

$$\sigma_0 = \sigma_m = 503 \text{ kgf/cm}^2$$

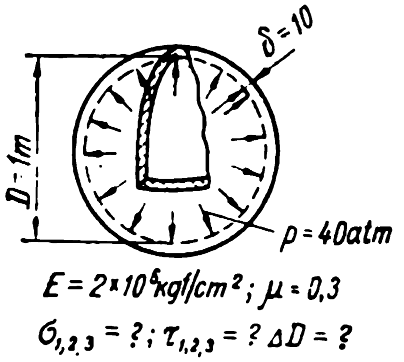
Since $\alpha_0 = \alpha = 60^\circ$, from formula (52) the area F of the reinforcing ring, according to formula (52), should be

$$F = \frac{503 \times 0.1 \times 10^2 \times 0.87}{10^3} \cong 4.4 \text{ cm}^2$$

The dimensions δ and F thus determined should be checked, taking into consideration the design features and stability of the reinforcing ring.

Problems 128 through 133. Determine the quantities indicated in the conditions of the problems.

128



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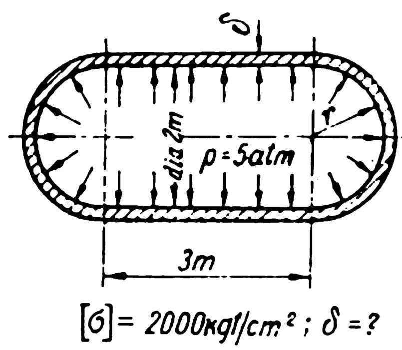

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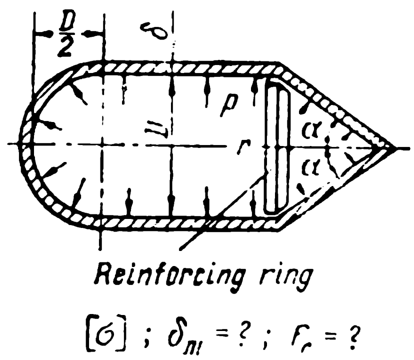

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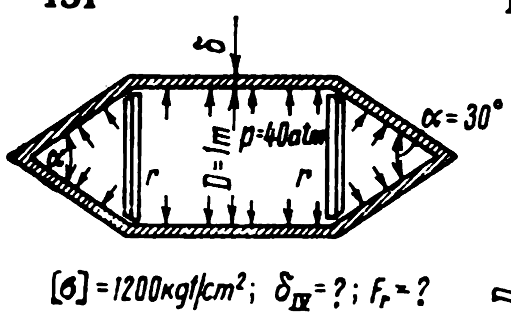

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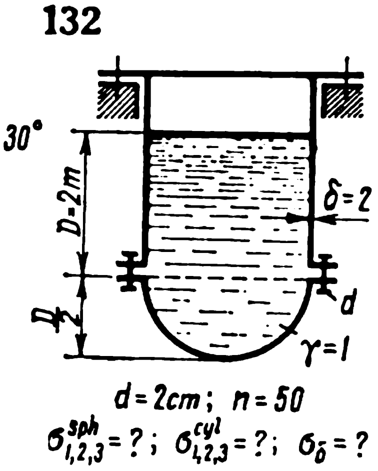

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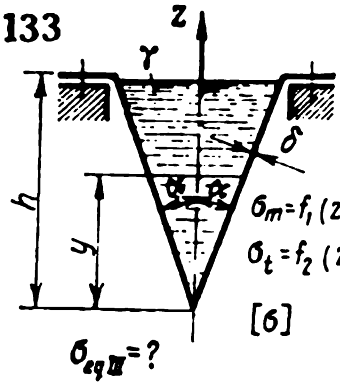

Notation: p = internal pressure of gas
γ = specific weight of liquid

δ = thickness of the wall (δ_{III} according to the 3rd and δ_{IV} according to the 4th strength theory)
F_r = area of the reinforcing ring
d = diameter of one bolt
n = number of bolts
$\sigma_{1,2,3}^{sph}$ = principal stresses at the dangerous point of the spherical portion of the vessel
$\sigma_{1,2,3}^{cyl}$ = same in the cylindrical portion.

All the necessary calculations should be carried out in accordance with the membrane theory.

In Problem 133 investigate and plot the graphs of the changes in the meridional (σ_m) and circumferential (σ_t) stresses, and also the equivalent stress σ_{eqIII} calculated according to the 3rd strength theory as a function of the coordinate z.

Write the equations for the strength condition.

CHAPTER 4. SHEAR

Pure shear is a state of stress in which the faces of an isolated element are acted upon only by shearing stresses. Such a state of stress is characterized by linear displacement of two parallel faces of the

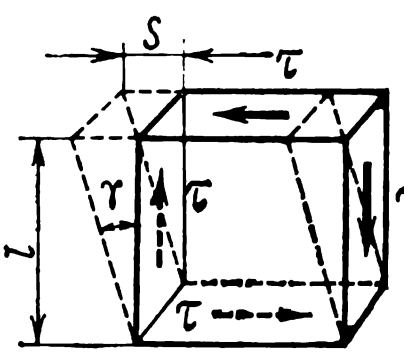

Fig. 24

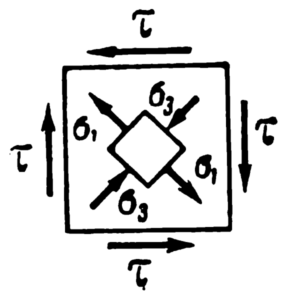

Fig. 25

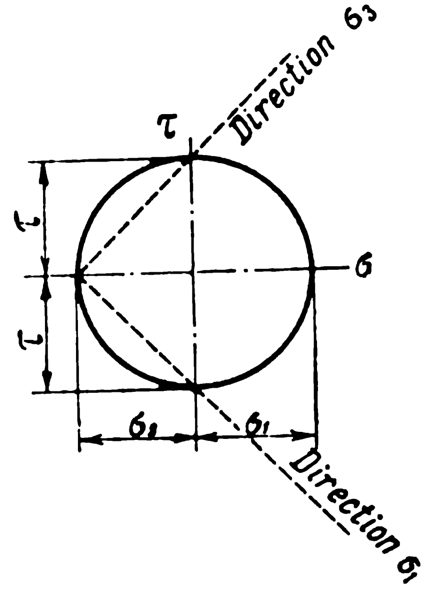

Fig. 26

element relative to each other. *Total shear* is the magnitude of linear displacement s (Fig. 24).

The ratio of the total shear s between two adjacent faces to the distance l between these faces

$$\frac{s}{l} = \tan\gamma \cong \gamma \tag{55}$$

gives the unit *shearing deformation* or the *angle of shear.*

The angles of shear γ_1, γ_2, γ_3 (i.e. the angular amounts of change in the right angles between the planes of action of the extremal shearing stresses τ_1, τ_2, τ_3, which are equal in magnitude but opposite in sign) are determined according to Hooke's law:

$$\left.\begin{aligned} \gamma_1 &= \varepsilon_2 - \varepsilon_3 = \frac{\tau_1}{G}; \\ \gamma_2 &= \varepsilon_1 - \varepsilon_3 = \frac{\tau_2}{G}; \\ \gamma_3 &= \varepsilon_1 - \varepsilon_2 = \frac{\tau_3}{G} \end{aligned}\right\} \tag{56}$$

where

$$G = \frac{E}{2(1+\mu)} \tag{57}$$

is the modulus of elasticity in shear (also called the shear modulus or the rigidity modulus).

The unit shearing deformation γ_0 which occurs under the action of an octahedral shearing stress τ_0 is called the *octahedral shear*,

$$\gamma_0 = \frac{\tau_0}{G} = \frac{2}{3}\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2} \tag{58}$$

In a state of pure shearing stress (Fig. 25) the principal stresses in planes at an angle of 45° are

$$\sigma_1 = -\sigma_3 = \tau \tag{59}$$

the principal linear strains are

$$\left.\begin{aligned} \varepsilon_1 &= -\varepsilon_3 = \varepsilon = \frac{1+\mu}{E}\tau; \\ \varepsilon_2 &= 0 \end{aligned}\right\} \tag{60}$$

and the principal shearing strain (angle of shear) is

$$\gamma = 2\varepsilon \tag{61}$$

The centre of Mohr's circle in the circle diagram of the state of stress is located at the origin of coordinates (Fig. 26).

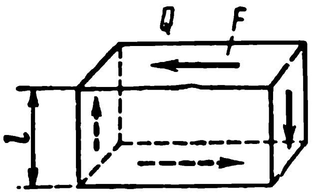

Fig. 27

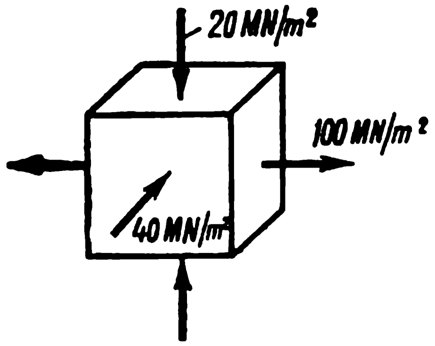

Fig. 28

If the shearing stresses are considered to be uniformly distributed over the area F of their action (Fig. 27), then the shearing force is

$$Q = \tau F \tag{62}$$

On the basis of formulas (55) and (56), we can write Hooke's law for shear in the following way:

$$\Delta s = \frac{Ql}{GF} \tag{63}$$

The strain energy in shear is determined from the formula

$$U = \frac{Q^2 l}{2GF} = \frac{\Delta s^2 GF}{2} = \frac{Q \Delta s}{2} \tag{64}$$

The quantity

$$u = \frac{\tau\gamma}{2} = \frac{\tau^2}{2G} \tag{65}$$

is called the elastic strain energy per unit volume in shear.

Example 13. Determine $\gamma_{1,2,3}$ and γ_0 for a given state of stress (Fig. 28), if $E = 2 \times 10^5$ MN/m² and $\mu = 0.25$.

Solution. The principal stresses in the given volumetric state of stress are:

$$\sigma_1 = 100\ \text{MN/m}^2;\quad \sigma_2 = -20\ \text{MN/m}^2;\quad \sigma_3 = -40\ \text{MN/m}^2$$

Then, according to formulas (27), the extremal shearing stresses are:

$$\tau_1 = \frac{-20+40}{2} = 10\ \text{MN/m}^2;$$

$$\tau_2 = \frac{100+40}{2} = 70\ \text{MN/m}^2;$$

$$\tau_3 = \frac{100+20}{2} = 60\ \text{MN/m}^2$$

The modulus of elasticity in shear is found by formula (57):

$$G = \frac{2 \times 10^5}{2(1+0.25)} = 8 \times 10^4\ \text{MN/m}^2$$

The principal angles of shear (shearing strains) are found from expressions (56):

$$\gamma_1 = \frac{10}{8 \times 10^4} = 1.25 \times 10^{-4};$$

$$\gamma_2 = \frac{70}{8 \times 10^4} = 8.75 \times 10^{-4};$$

$$\gamma_3 = \frac{60}{8 \times 10^4} = 7.5 \times 10^{-4}$$

According to formula (58), the octahedral shear is

$$\gamma_0 = \frac{2}{3} \times 10^{-4} \sqrt{1.25^2 + 8.75^2 + 7.5^2} \cong 7.72 \times 10^{-4}$$

Problems 134 through 140. Determine the quantities indicated in the conditions of the problems.

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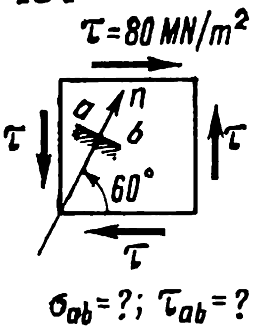

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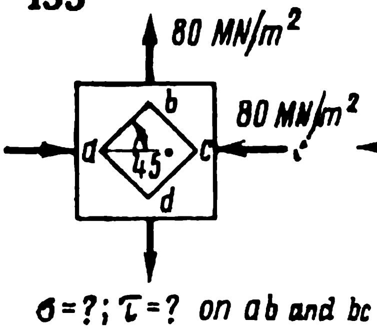

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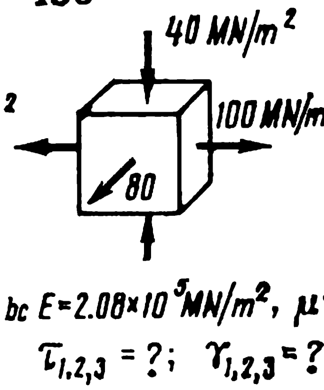

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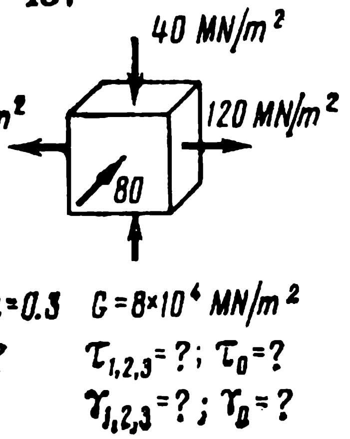

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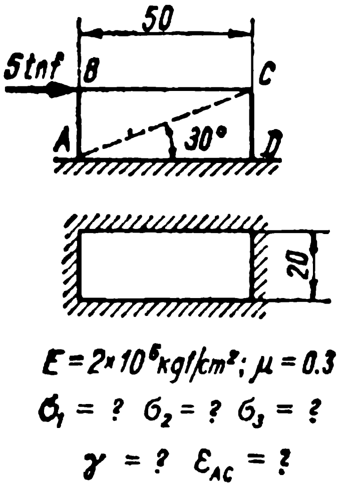

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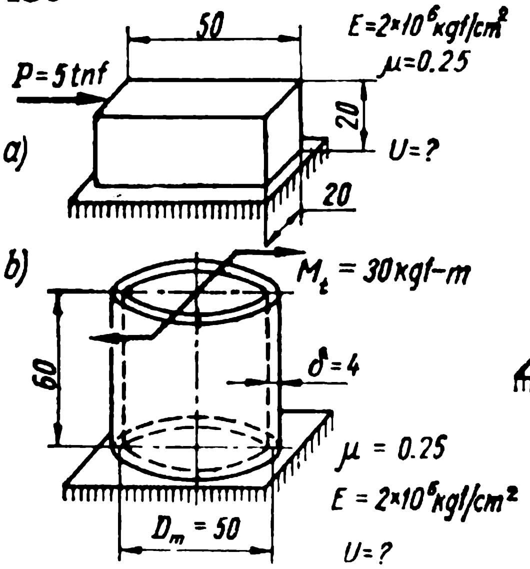

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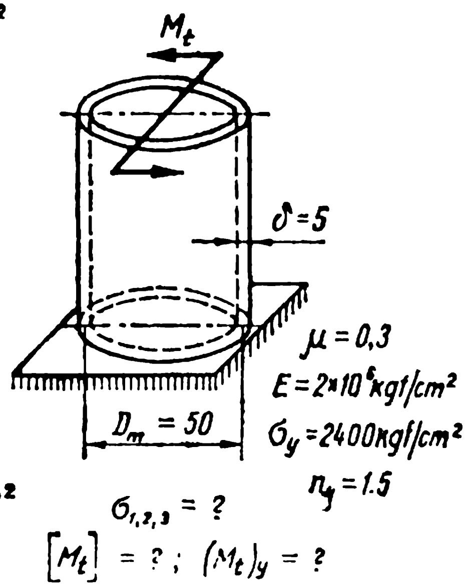

Notation: σ_y = yield point
n_y = factor of safety based on yield point
$(M_t)_y$ = value of the moment corresponding to the yield condition of the material.

Bending is to be neglected and tangential stresses are assumed to be uniformly distributed throughout the thickness of the tubes.

CHAPTER 5. DESIGN OF VARIOUS JOINTS

In various joints (secured by means of bolts, pins, studs, keys, cotters, rivets, welds, mortises, etc.) the elements are subject to tension, compression, shear and bearing stresses.

Under real conditions tension and compression usually occur in the pure form, whereas pure shear is practically nonexistent. Shear is usually accompanied by bending, or by tension or compression. Therefore, in addition to shearing stresses, the shear planes are also subject to normal stresses. But, since the normal stresses are low compared to the shearing stresses, engineering design takes only shear into account. In the case of wooden elements, shear is often called splitting. Shearing stresses are usually assumed to be uniformly distributed over the area F_{sh} of the corresponding sections. Errors introduced by this assumption are compensated for by the value of the allowable shearing stress $[\tau]$.

What we have called bearing action is a compressive force over a relatively small area. In other words, it is a localized compression due to elements pressing against each other. Since we do not know the law governing the distribution of pressures over the surface acted on, for simplicity in design calculations it is conventionally assumed that the pressures are uniformly distributed not over this surface, but over an area (F_{br}) which is the projection of the surface on a plane perpendicular to the direction of the bearing force P_{br}. This error is also taken into account by the value of the allowable bearing stress $[\sigma_{br}]$.

To ensure a saving of the material joints should be designed on the basis of equal strength of their elements using the following design formulas:

(1) for tension or compression

$$F \geqslant \frac{N}{[\sigma]} \tag{66}$$

(2) for shear

$$F_{sh} \geqslant \frac{Q}{[\tau]} \tag{67}$$

(3) for bearing action

$$F_{br} \geqslant \frac{P_{br}}{[\sigma_{br}]} \tag{68}$$

The areas in the above formulas are "net" areas, i.e. ones allowing for possible weakening due to various holes, circular grooves, recesses, etc.

If calculations are carried out both on the basis of permissible shear and of permissible bearing stresses, the larger of the two calculated areas is to be used in the design.

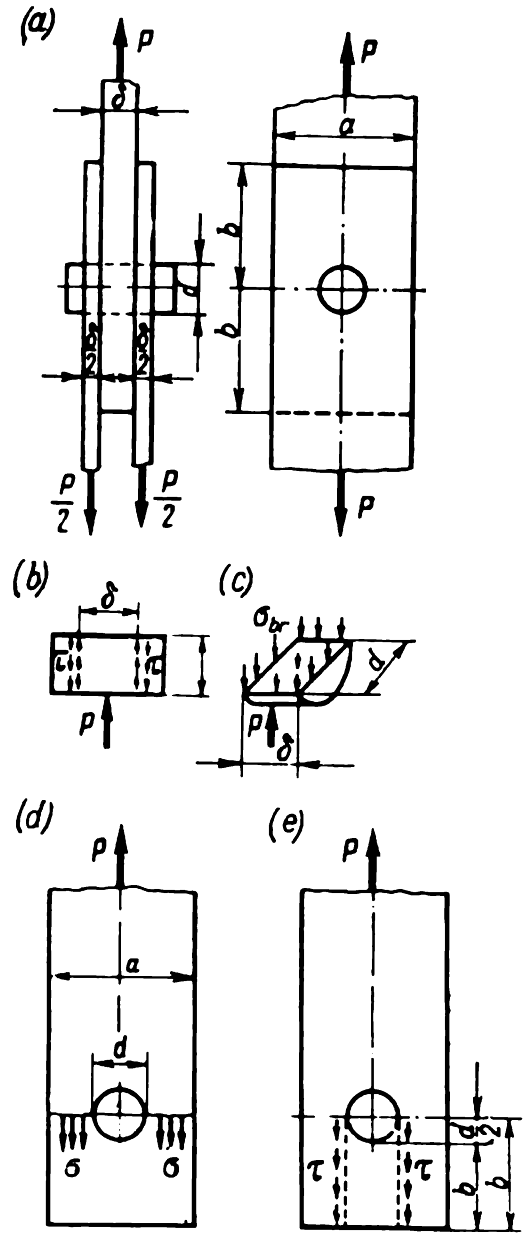

Fig. 29

Practically it is not always possible to observe the condition of equal strength of the elements of a joint. Usually, certain additional design considerations come into effect which are dealt with in study courses on the design of the components of machinery, and metal and wooden structures.

Example 14. Let $P = 4$ tnf; $[\sigma] = 1600$ kgf/cm^2; $[\tau] = 1200$ kgf/cm^2; $[\sigma_{br}] = 3200$ kgf/cm^2 (Fig. 29*a*).

Find d, δ, a and b.

Solution. 1. Find the diameter d of the pin on the basis of the shear strength (Fig. 29*b*):

$$2\,\frac{\pi d^2}{4} \geqslant \frac{P}{[\tau]}; \quad d \geqslant \sqrt{\frac{2P}{\pi[\tau]}} = \sqrt{\frac{2\times 4\times 10^3}{\pi\times 12\times 10^2}} \cong 1.46 \text{ cm}$$

2. Determine the thickness δ of the plate on the basis of the bearing strength (Fig. 29*c*):

$$\delta d \geqslant \frac{P}{[\sigma_{br}]}; \quad \delta \geqslant \frac{P}{d[\sigma_{br}]} = \frac{4\times 10^3}{1.46\times 32\times 10^2} \cong 0.86 \text{ cm}$$

3. Determine the width a of the strip on the basis of the tensile strength (Fig. 29*d*):

$$(a-d)\,\delta \geqslant \frac{P}{[\sigma]}; \quad a = \frac{P}{\delta[\sigma]} + d = \frac{4\times 10^3}{0.86\times 16\times 10^2} + 1.46 \cong 2.92 \text{ cm}$$

4. Determine the length b of the strip from the pin to the end on the basis of the shear strength (Fig. 29*e*):

$$2b'\delta \geqslant \frac{P}{[\tau]}; \quad b' = \frac{P}{2\delta[\tau]} = \frac{4\times 10^3}{2\times 0.86\times 12\times 10^2} \cong 1.94 \text{ cm}$$

Hence, $b = b' + \frac{d}{2} = 1.94 + 0.73 = 2.67$ cm

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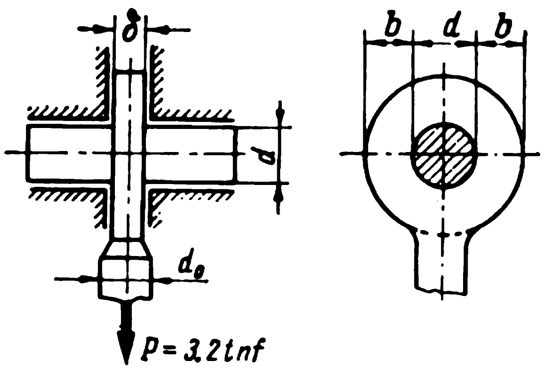

$[\sigma] = 1600 \text{ kgf/cm}^2$; $[\tau] = 0.7[\sigma]$; $[\sigma_{br}] = 2[\sigma]$

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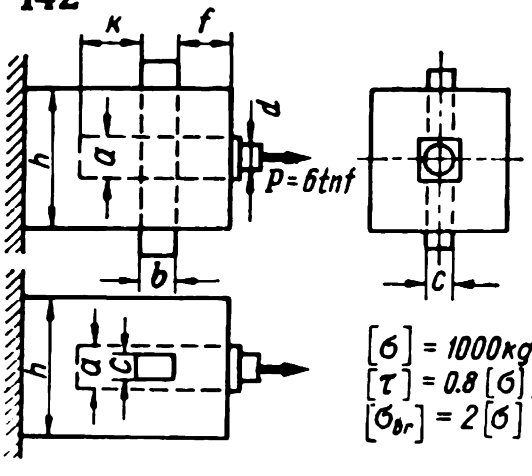
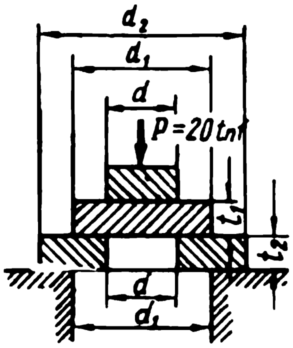


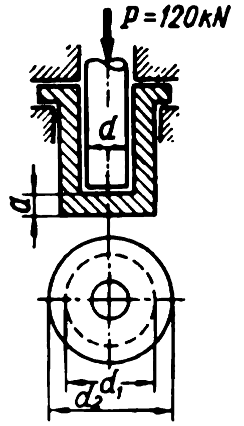
$[\sigma] = 1000 \text{ kgf/cm}^2$;
$[\tau] = 0.8[\sigma]$;
$[\sigma_{br}] = 2[\sigma]$

143



$[\tau] = 800 \text{ kgf/cm}^2$;
$[\sigma] = 1600 \text{ kgf/cm}^2$

144



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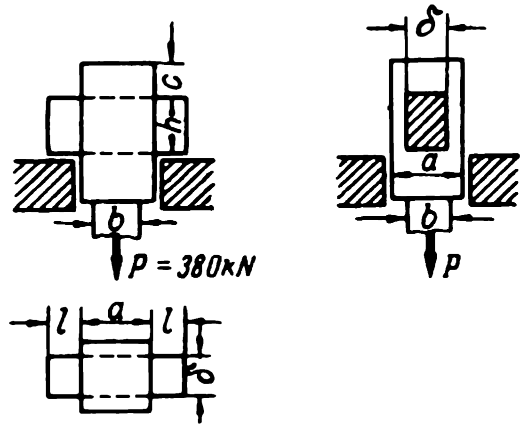

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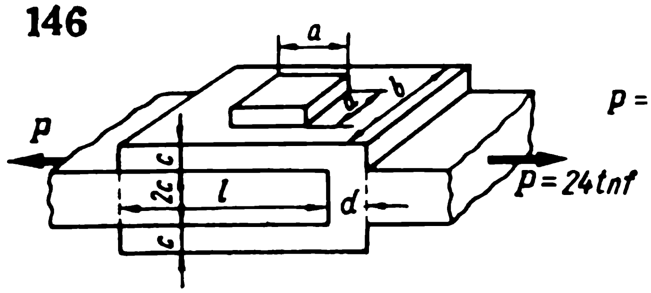

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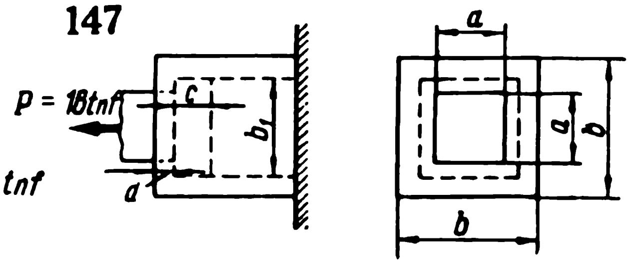

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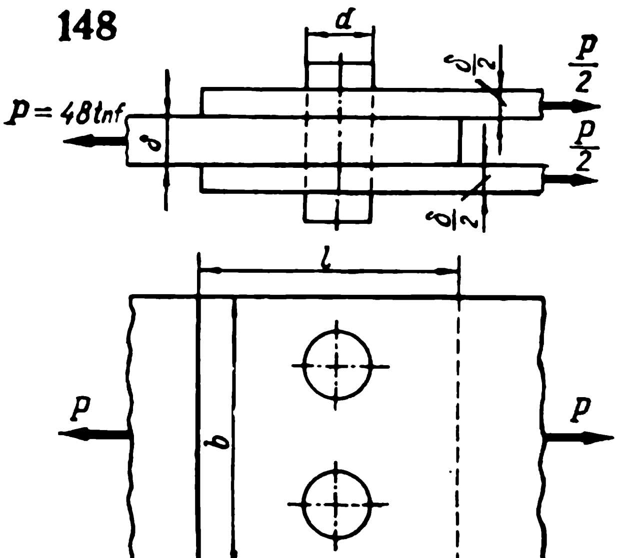

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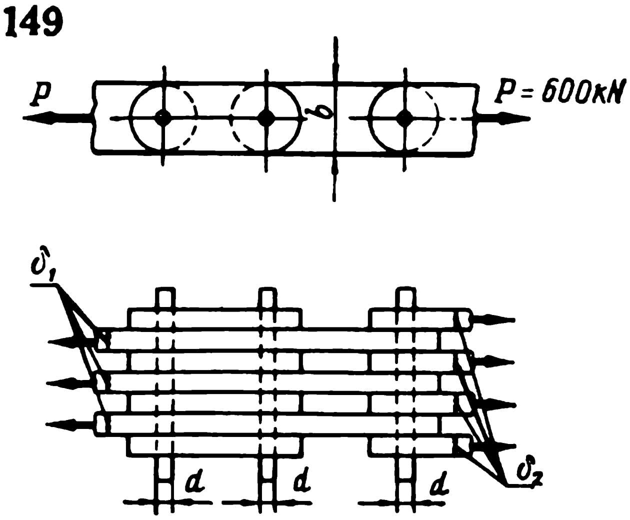


Problems 141 through 149. Determine all the dimensions of the elements of joints illustrated in the accompanying figures.

Design calculations should be carried out proceeding from the condition of equal strength of the elements.

In Problems 146, 147 and 148 assume that $[\sigma] = 1600$ kgf/cm^2; $[\tau] = 1200$ kgf/cm^2; $[\sigma_{br}] = 3200$ kgf/cm^2. In Problems 144, 145 and 149 assume that $[\sigma] = 160$ MN/m^2; $[\tau] = 120$ MN/m^2; $[\sigma_{br}] =$ $= 320$ MN/m^2.

CHAPTER 6. GEOMETRICAL PROPERTIES OF PLANE FIGURES

6.1.

Static Moments of a Section

The term *static moment of the area of a figure with respect to axes z and y* (Fig. 30) is applied to definite integrals of the form

$$\left.\begin{aligned} S_z &= \int_F y\,dF; \\ S_y &= \int_F z\,dF \end{aligned}\right\} \tag{69}$$

where F is the area of the figure; dF is its element; and z and y are the coordinates of this element.

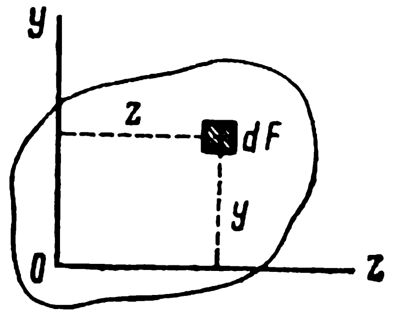


Fig. 30

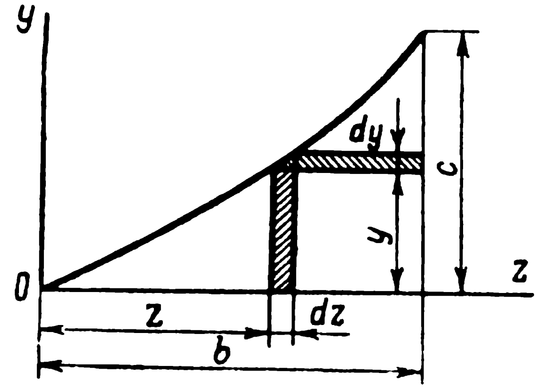


Fig. 31

The coordinates z_c and y_c of the centre of gravity of the figure are found from the equations

$$\left.\begin{aligned} z_c &= \frac{S_y}{F}; \\ y_c &= \frac{S_z}{F} \end{aligned}\right\} \tag{70}$$

The static moments of the area of the figure with respect to the centroidal axes are equal to zero.

Example 15. Find the coordinates of the centre of gravity of the area of a figure bounded by the straight lines b and c, and the parabola $y = az^n$ (Fig. 31).

Solution. Let the elemental area of the figure be $dF = y\,dz = az^n\,dz$. Then the area of the figure is

$$F = \int_F dF = a\int_0^b z^n\,dz = \frac{ab^{n+1}}{n+1} = \frac{bc}{n+1}$$

The static moment of area F with respect to axis y is found by formula (69)

$$S = \int_F z\,dF = a\int_0^b z^{n+1}\,dz = \frac{ab^{n+2}}{n+2} = \frac{b^2c}{n+2}$$

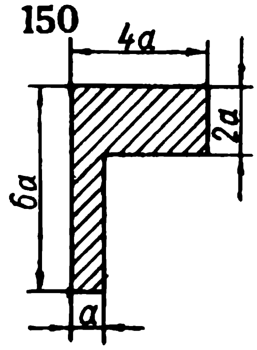
To find the static moment of the area of this figure with respect to axis z we consider an element of the area. Thus

$$dF = (b - z)\,dy = an\,(b - z)\,z^{n-1}\,dz$$

Then, from formula (69)

$$S_z = \int_F y\,dF = a^2n\int_0^b z^{2n-1}(b-z)\,dz = a^2nb^{2n+1}\left(\frac{1}{2n} - \frac{1}{2n+1}\right)$$
$$= \frac{a^2b^{2n+1}}{2(2n+1)} = \frac{bc^2}{2(2n+1)}$$

150



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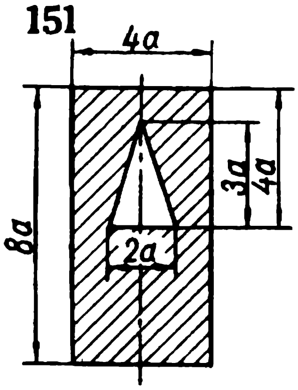

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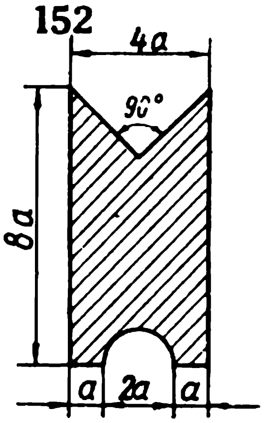

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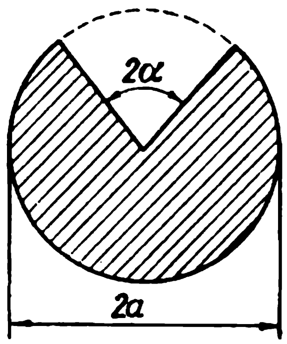

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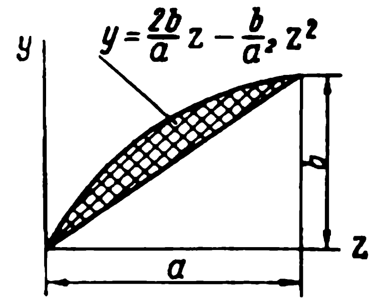

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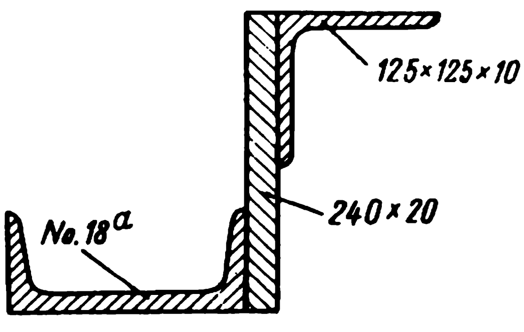

From equations (70) for the coordinates of the centre of gravity of the figure we obtain

$$z_c = \frac{S_y}{F} = \frac{b^2c\,(n+1)}{(n+2)\,bc} = \frac{n+1}{n+2}\,b;$$

$$y_c = \frac{S_z}{F} = \frac{bc^2}{2\,(2n+1)} \times \frac{n+1}{bc} = \frac{n+1}{2\,(2n+1)}\,c$$

Problems 150 through 155. Determine the position of the centre of gravity of the illustrated figures.

6.2.

Moments of Inertia of the Area of a Section

Definite integrals of the form

$$\left.\begin{aligned} I_z &= \int_F y^2\,dF; \\ I_y &= \int_F z^2\,dF \end{aligned}\right\} \qquad (71)$$

are called the *axial, linear* or *equatorial moments of inertia of the area of a figure* (Fig. 32) *with respect to axis z and axis y,*

$$I_{yz} = \int_F yz\,dF \qquad (72)$$

is the product of inertia of the area with respect to two mutually perpendicular axes z and y, and

$$I_p = \int_F \rho^2\,dF = I_y + I_z \qquad (73)$$

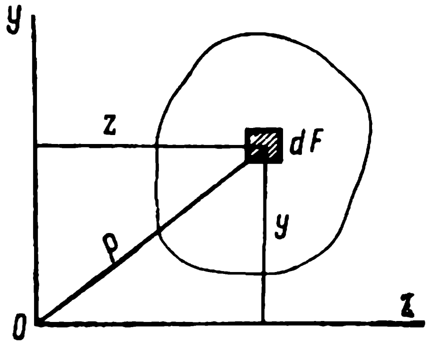

Fig. 32

is the polar moment of inertia of the area with respect to the origin of coordinates O.

The moments of inertia with respect to parallel axes, one pair of which $(z_0O_0y_0)$ are centroidal axes, are given by (Fig. 33):

$$\left.\begin{aligned} I_z &= I_{z_0} + a^2F; \\ I_y &= I_{y_0} + b^2F; \\ I_{yz} &= I_{y_0z_0} + abF; \\ I_p &= I_{p_0} + (a^2 + b^2)\,F \end{aligned}\right\} \qquad (74)$$

The moments of inertia with respect to axes rotated through an angle α (Fig. 34) are:

$$\left.\begin{aligned} I_u &= \frac{I_z+I_y}{2} + \frac{I_z-I_y}{2}\cos 2\alpha - I_{yz}\sin 2\alpha; \\ I_v &= \frac{I_z+I_y}{2} - \frac{I_z-I_y}{2}\cos 2\alpha + I_{yz}\sin 2\alpha; \\ I_{uv} &= \frac{I_z-I_y}{2}\sin 2\alpha + I_{yz}\cos 2\alpha; \\ I_{p_{uv}} &= I_{p_{yz}} \end{aligned}\right\} \quad (75)$$

The principal axes of inertia of a plane figure, i.e. two mutually perpendicular axes with respect to which the product of inertia of the

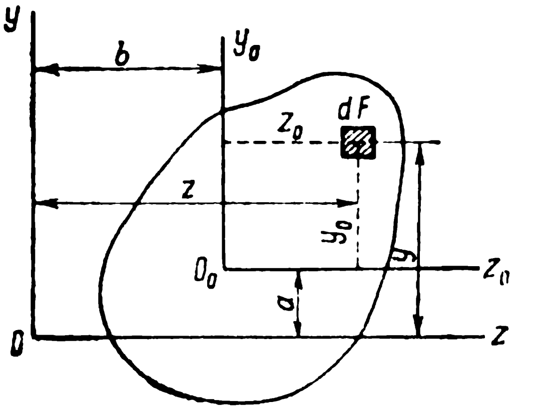

Fig. 33

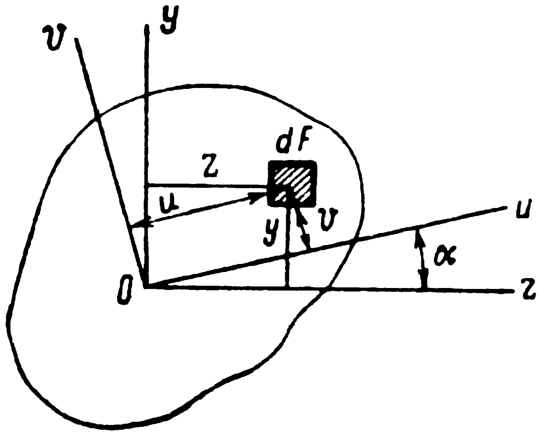

Fig. 34

area of the figure equals zero, occupy a position specified by the equation

$$\tan 2\alpha = \frac{2I_{yz}}{I_y - I_z} \quad (76)$$

The principal moments of inertia, i.e. the axial moments of inertia calculated with respect to the principal axes of inertia, have the following extremal values:

$$I_{\substack{\max \\ \min}} = \frac{I_z+I_y}{2} \pm \frac{1}{2}\sqrt{(I_z-I_y)^2 + 4I_{yz}^2} \quad (77)$$

If $I_{yz} < 0$, the principal axis with respect to which the moment of inertia is maximal passes through quadrants *I* and *III*.

If $I_{yz} > 0$, the principal axis with respect to which the moment of inertia is maximal passes through quadrants *II* and *IV*.

Principal axes passing through the centre of gravity of the area of a figure are called the *principal centroidal axes*, and the moments of inertia with respect to these axes are called the *principal centroidal moments of inertia*.

The positive values of the quantities

$$i_z=\sqrt{\frac{I_z}{F}};\quad i_y=\sqrt{\frac{I_y}{F}};\quad i_u=\sqrt{\frac{I_u}{F}};\ \text{etc.} \tag{78}$$

are called the *radii of inertia of a plane figure with respect to the corresponding axis.*

An ellipse plotted to the equation

$$\frac{z^2}{i_y^2}+\frac{y^2}{i_z^2}=1 \tag{79}$$

is also called the *ellipse of the inertia* of the figure. Here axes y and z are the principal axes of inertia of the figure. The ellipse of inertia is usually plotted on the principal centroidal axes of the plane figure.

An axis of symmetry of a plane figure is a principal axis of inertia of this figure. Any axis perpendicular to an axis of symmetry is a second principal axis of inertia of the figure for the point of intersection of the axes.

If a plane figure has at least two axes of symmetry which are not perpendicular to each other, all axes passing through the centre of gravity of this figure are its principal centroidal axes of inertia. The axial moments of inertia of the area of the figure, calculated with respect to these axes, are equal to each other.

Example 16. Find the principal centroidal moments of inertia and the polar moment of inertia, and plot the centroidal ellipse of inertia for a regular n-sided polygon with sides of length a (Fig. 35).

Solution. The central angle opposite side a is

$$\alpha=\frac{2\pi}{n}$$

The radius of the circumscribed circle is

$$R=\frac{a}{2\sin\frac{\alpha}{2}}$$

The radius of the inscribed circle is

$$r=R\cos\frac{\alpha}{2}$$

We consider one triangle with the apex angle α (Fig. 36a) and find its axial moments of inertia I'_u, I'_v and polar moment of inertia I'_p.

The area of an elemental strip of width dv is

$$dF=a_1\,dv=\frac{a}{r}v\,dv$$

The axial moment of inertia of the area of the triangle about axis u is equal to

$$I'_u=\int_F v^2\,dF=\frac{a}{r}\int_0^r v^3\,dv=\frac{ar^3}{4}$$

The area of an elemental strip of width du (Fig. 36*b*) is

$$dF = r_1\,du = \frac{r}{a}(a-2u)\,du$$

The axial moment of inertia of the area of the triangle about axis v is

$$I'_v = 2\int\limits_{\frac{F}{2}} u^2\,dF = 2\frac{r}{a}\int\limits_0^{\frac{a}{2}} u^2(a-2u)\,du = \frac{a^3r}{48}$$

The polar moment of inertia of the area of the triangle about point O is

$$I'_p = I'_u + I'_v = \frac{ar^3}{4} + \frac{a^3r}{48} = \frac{2R\sin\frac{\alpha}{2}R^3\cos^3\frac{\alpha}{2}}{4}$$

$$+\frac{8R^3\sin^3\frac{\alpha}{2}R\cos\frac{\alpha}{2}}{48} = \frac{R^4}{6}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(3\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}\right)$$

$$=\frac{R^4}{12}\sin\alpha\,(2+\cos\alpha)$$

Since all the triangles into which the n-sided polygon is divided are equal to one another and their apexes are at point O, the polar

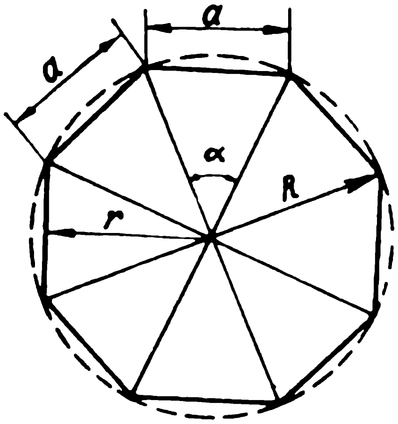

Fig. 35

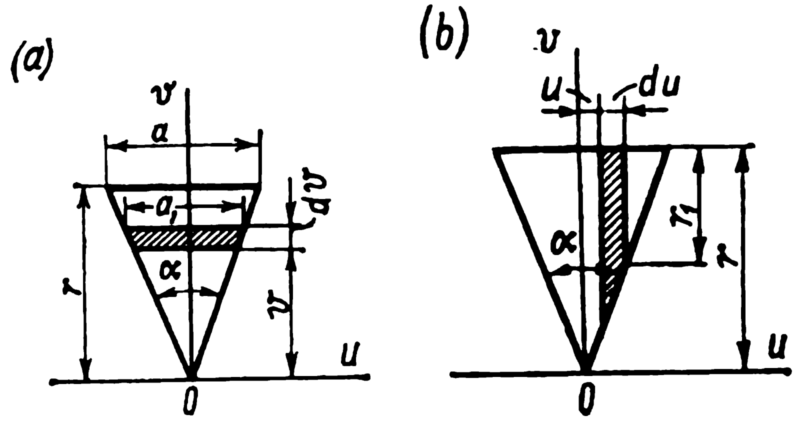

Fig. 36

moment of inertia of the area of the entire n-sided polygon about point O is

$$I_p = nI'_p = \frac{nR^4}{12}\sin\alpha\,(\cos\alpha+2) = \frac{\pi R^4}{6\alpha}\sin\alpha\,(\cos\alpha+2)$$

In a regular n-sided polygon there are at least two axes of symmetry which are not perpendicular to each other. Therefore all the centroidal

axes are principal axes of inertia and the moments of inertia of the area of the polygon about them are equal to one another and to I, hence

$$I=\frac{1}{2}I_p=\frac{nR^4}{24}\sin\alpha\,(\cos\alpha+2)=\frac{\pi R^4}{12\alpha}\sin\alpha\,(\cos\alpha+2)$$

Since the area of an n-sided polygon is equal to

$$F=\frac{nar}{2}=\frac{n}{2}2R^2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=\frac{n}{2}R^2\sin\alpha$$

the principal radii of inertia of the polygon can be expressed by

$$i=\sqrt{\frac{I}{F}}=\frac{R}{2\sqrt{3}}\sqrt{2+\cos\alpha}$$

and the ellipse of inertia is the circle described by this radius.

Example 17. Construct the centroidal ellipse of inertia for the plate, channel and angle section shown in Fig. 37.

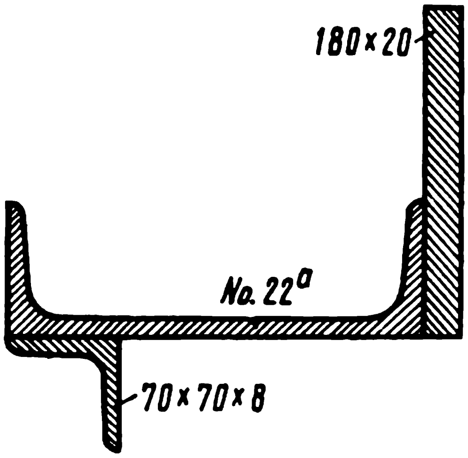

Fig. 37

Solution. The built-up section, drawn to a larger scale (Fig. 38), is referred to the system of coordinates zy whose axes are parallel to the sides of the component sections.

Each component of the section (angle *1*, channel *2* and plate *3*) are referred to the centroidal axes z_iy_i which are parallel to the axes zy.

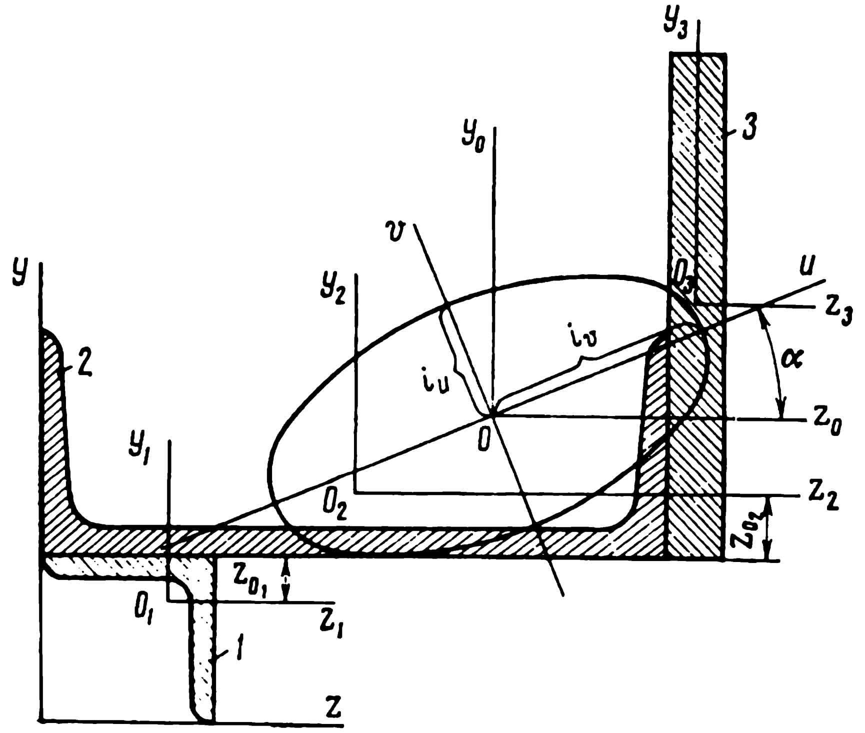

Fig. 38

First we find the coordinates of the centre of gravity of the plate. The coordinates of the centre of gravity of the angle $z_{0_1} = 2.02$ cm and of the channel $z_{0_2} = 2.46$ cm are taken from the data for rolled steel shapes listed in handbooks.

Subsequent calculations are preferably made in the tabulated form. All the required quantities for the plate are readily computed and those for the angle and channel are obtained from the tables for standard rolled steel shapes. Thus, for example, for an equal angle of size $70 \times 70 \times 8$ mm, the area F_1 in the table is 10.7 cm², for the No. 22 channel $F_2 = 28.8$ cm² and for the plate $F_3 = 18 \times 2 = 36$ cm². The coordinates of the centres of gravity of each of the components with respect to the axes yz are:

for the angle $z_1 = y_1 = 7 - 2.02 = 4.98$ cm;
for the channel $z_2 = 11$ cm; $y_2 = 7 + 2.46 = 9.46$ cm;
for the plate $z_3 = 22 + 1 = 23$ cm; $y_3 = 7 + 9 = 16$ cm.

1. *Determining the Coordinates of the Centre of Gravity of the Built-Up Section with Respect to the zy Axes.*

Component No.	Area F_i	Coordinates of the centres of gravity with respect to the zy axes		Static moments with respect to the z and y axes	
		z_i	y_i	$S_{z_i} = F_i y_i$	$S_{y_i} = F_i z_i$
	cm²	cm		cm³	
1	10.7	4.98	4.98	53.4	53.4
2	28.8	11	9.46	272.4	316.8
3	36	23	16	576	828
	75.5 area of the whole section			901.8 for the area of the whole section	1198.2

The coordinates of the centre of gravity of the section can be found by the equation

$$z_c = \frac{S_y}{F} = \frac{1198.2}{75.5} \cong 15.87 \text{ cm};$$

$$y_c = \frac{S_z}{F} = \frac{901.8}{75.5} \cong 11.94 \text{ cm}$$

Next we refer the section to the centroidal axes z_0y_0 which are parallel to the axes zy.

Then we find the coordinates z_{0_i} and y_{0_i} of the centres of gravity of the components of the section with respect to the axes z_0y_0. Thus: for the angle

$$z_{01} = 4.98 - 15.87 = -10.89 \text{ cm};$$
$$y_{01} = 4.98 - 11.94 = -6.96 \text{ cm}$$

for the channel

$$z_{02} = 11 - 15.87 = -4.87 \text{ cm};$$
$$y_{02} = 9.46 - 11.94 = -2.48 \text{ cm}$$

for the plate

$$z_{03} = 23 - 15.87 = 7.13 \text{ cm};$$
$$y_{03} = 16 - 11.94 = 4.06 \text{ cm}$$

Now we find the moments of inertia of the components of the section with respect to their centroidal axes z_iy_i.

For the angle (Fig. 39) they are taken from the tables for standard rolled steel shapes:

$$I_{z_1} = I_{y_1} = 48.2 \text{ cm}^4$$

Using the formulas for axes rotated through an angle we obtain

$$I_{z_1y_1} = \frac{I_{u_1} - I_{v_1}}{2} \sin 2(-45^\circ)$$
$$= \frac{76.4 - 20}{2}(-1) = -28.2 \text{ cm}^4$$

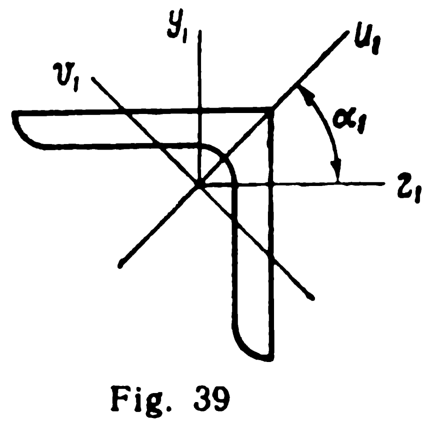

Fig. 39

The axes u_1v_1 are the principal axes of inertia for the angle because axis u_1 is the axis of symmetry, hence $I_{u_1v_1} = 0$. The values $I_{u_1} = 76.4$ cm^4 and $I_{v_1} = 20$ cm^4 are given in the table of standard shapes.

For the channel the values taken from the table are $I_{z_2} = 187$ cm^4; $I_{y_2} = 2330$ cm^4; $I_{z_2y_2} = 0$, because for the channel the axes z_2 and y_2 are the principal axes of inertia.

The calculation for the plate is as follows:

$$I_{z_3} = \frac{2 \times 18^3}{12} = 972 \text{ cm}^4; \quad I_{y_3} = \frac{18 \times 2^3}{12} = 12 \text{ cm}^4;$$

$$I_{z_3y_3} = 0$$

2. *Moments of Inertia of the Section with Respect to the Axes* z_0y_0

Component	F_i	z_{0i}	y_{0i}	I_{z_i}	I_{y_i}	$I_{z_iy_i}$	$F_iz_{0i}^2$	$F_iy_{0i}^2$	$F_iz_{0i}y_{0i}$	$I_{z_{0i}}=I_{z_i}+F_iy_{0i}^2$	$I_{y_{0i}}=I_{y_i}+F_iz_{0i}^2$	$I_{z_{0i}y_{0i}}=I_{z_iy_i}+F_iz_{0i}y_{0i}$
	cm^2	cm		cm^4								
1	10.7	−10.89	−6.96	48.2	48.2	−28.2	1269	518.3	811	566.5	1317.2	782.8
2	28.8	−4.87	−2.48	187	2330	0	683	177.1	347.8	364.1	3013	347.8
3	36	7.13	4.06	972	12	0	1830	593.3	1042	1565.3	1842	1042
										2496	6172	2173
										for the area of the whole section		

3. *Position of the Principal Centroidal Axes of Inertia of the Section* (*uv*).

According to equation (76)

$$\tan 2\alpha=\frac{2I_{z_0y_0}}{I_{y_0}-I_{z_0}}=\frac{2\times 2174}{6172-2496}=1.182$$

From tables of trigonometric functions we find that $2\alpha = 49°46'$, hence $\alpha = 24°53'$.

Since angle $\alpha > 0$, it is plotted counterclockwise from axis z. Then we draw the principal centroidal axes of inertia u and v of the section.

Since $I_{z_0y_0} > 0$, the axis with the maximum moment of inertia, i.e. axis v, passes through quadrants *II* and *IV*.

4. *Principal Centroidal Moments of Inertia of the Section.* According to equation (77)

$$I_{\substack{\max\\ \min}}=I_{\substack{v\\ u}}=\frac{I_{z_0}+I_{y_0}}{2}\pm\frac{1}{2}\sqrt{(I_{z_0}-I_{y_0})^2+4I_{z_0y_0}^2}$$

$$=\frac{2496+6172}{2}\pm\frac{1}{2}\sqrt{(2496-6172)^2+4\times 2173^2}=4334\pm 2846 \quad cm^4$$

Thus

$$I_{\max}=I_v=7180\ cm^4;\qquad I_{\min}=I_u=1488\ cm^4$$

5. *Principal Radii of Inertia of the Section and the Construction of the Centroidal Ellipse of Inertia.*

According to formulas (78), the principal radii of inertia are

$$i_{\max}=i_v=\sqrt{\frac{I_v}{F}}=\sqrt{\frac{7180}{75.5}}\cong 9.75\ cm;$$

$$i_{\min}=i_u=\sqrt{\frac{I_u}{F}}=\sqrt{\frac{1488}{75.5}}\cong 4.44\ cm$$

Since the equation of the ellipse of inertia is of the form $\frac{u^2}{i_v^2} + \frac{v^2}{i_u^2} = 1$, its semi-axes are the radii of inertia i_u on axis v and i_v on axis u. We plot the obtained values for i_u and i_v and construct the ellipse of inertia on them (see Fig. 38).

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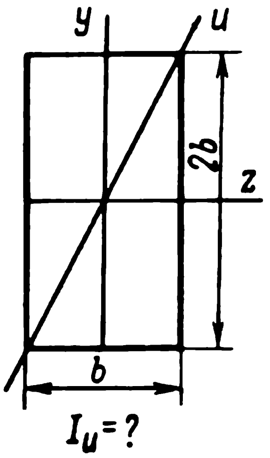

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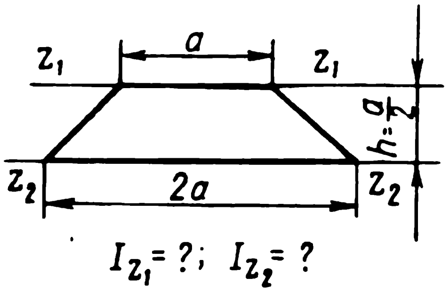

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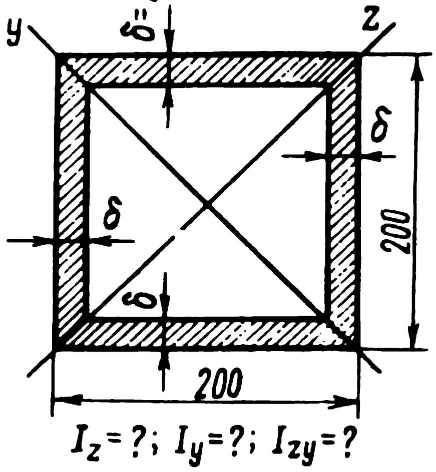

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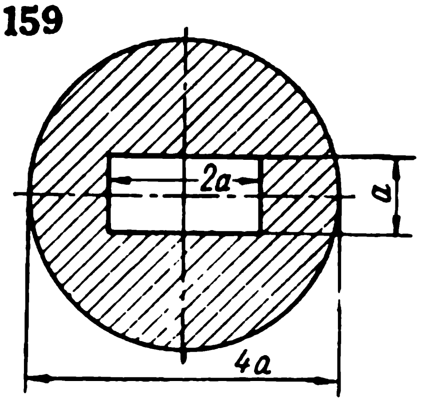

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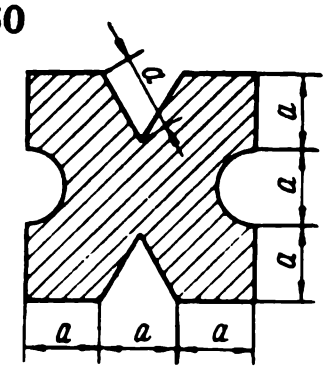

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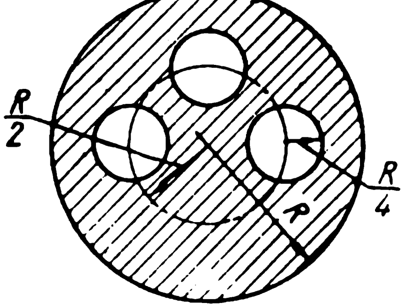

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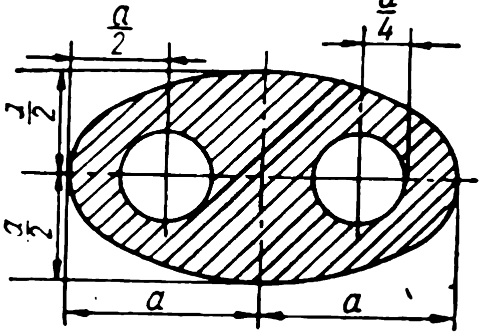

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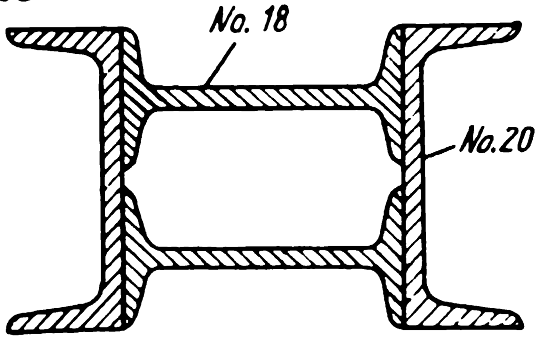

Problems 156 through 158. Find the moments of inertia of the figures about the axes shown in the problems.

Problems 159 through 163. Find the principal centroidal moments of inertia.

CHAPTER 7. TORSION

7.1.

Torque

Torque is calculated by the method of sections. The torque M_t in a cross section of a rod is equal to the algebraic sum of moments of all the external couples (concentrated couples M and those distributed along the length of the rod with the intensity m) acting about the geometric axis of the rod, and to one side of the cross section being considered.

The general formula for the torque in an arbitrary cross section of a rod is

$$M_t = \sum M + \sum \int m \, dx \tag{80}$$

Integration is carried out along the length of each part which is subject to a distributed moment, and summation is carried out over all the parts located to one side of the cross section under consideration.

The following sign convention is used. If an observer views the cross section from the side of the outward normal and sees the moment acting in the counterclockwise direction the moment is considered to be positive; otherwise it is negative.

The following dependences exist between the torque M of a couple given in kgf-cm, rotational speed n (rpm) of the couple and the power N:

$$M = 71{,}620 \frac{N}{n}; \tag{81a}$$

$$M = 97{,}360 \frac{N}{n} \tag{81b}$$

In formula (81a) the power is expressed in hp (metric horse power); in formula (81b), in kW (kilowatts).

In the International System of Units (SI) the ratio between torque M in Newton-metres (N-m), the angular speed of rotation ω in 1/sec and power N in watts is given by the formula

$$M = \frac{N}{\omega} \tag{81c}$$

Since torque is proportional to power, a diagram of the distribution of power over the length of a shaft, rotating at constant speed and,

transmitting power to operative units, can be plotted instead of a torque diagram.

Example 18. Let M be the torque in Fig. 40.

Plot the diagram for M_t.

Solution. Cutting the rod at cross sections in each part, we can set up the following equations in accordance with formula (80) and taking

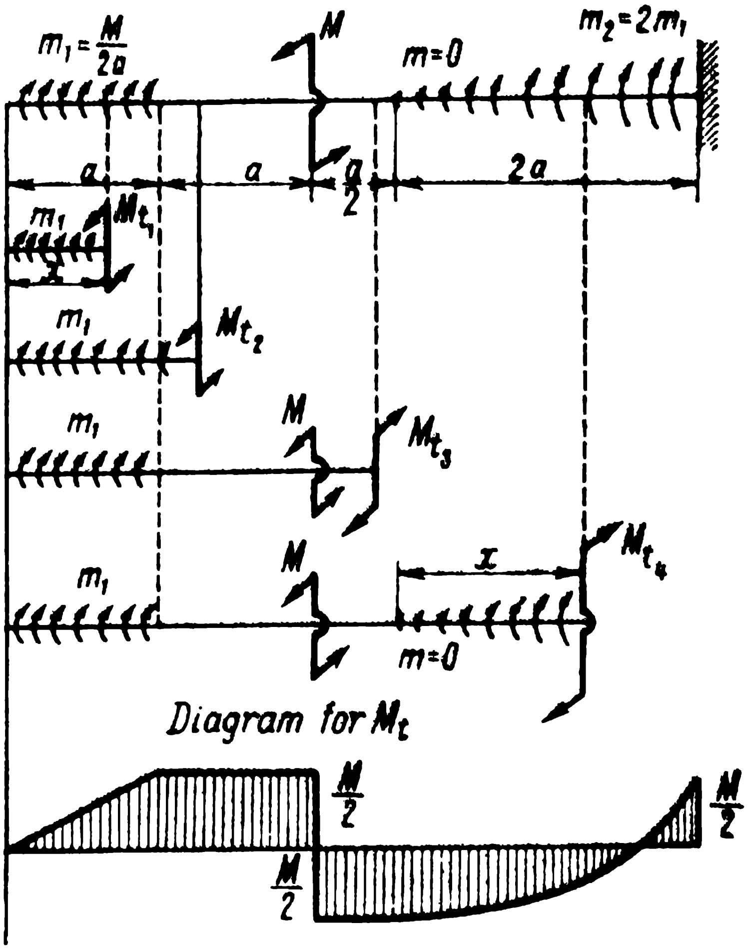

Fig. 40

the sign convention rule into account:

$$M_{t_1}=m_1x=\frac{M}{2a}x;\quad M_{t_1}\Big|_{x=0}=0;\quad M_{t_1}\Big|_{x=a}=\frac{M}{2};$$

$$M_{t_2}=m_1a=\frac{M}{2};\quad M_{t_3}=m_1a-M=\frac{M}{2}-M=-\frac{M}{2};$$

$$M_{t_4}=m_1a-M+\int_0^x\frac{2m_1}{2a}x\,dx=\frac{M}{2}-M+\frac{M}{2a^2}\int_0^x x\,dx$$

$$=-\frac{M}{2}+\frac{Mx^2}{4a^2};\quad M_{t_4}\Big|_{x=0}=-\frac{M}{2};\quad M_{t_4}\Big|_{x=2a}=\frac{M}{2}$$

The diagram for M_t is illustrated in Fig. 40.

Problems 164 through 167. Plot the diagrams of the torques.

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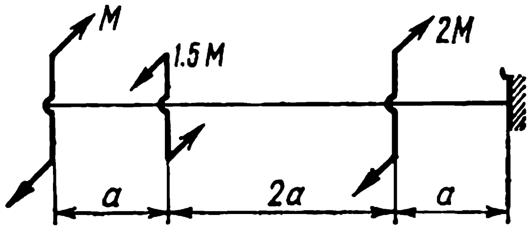

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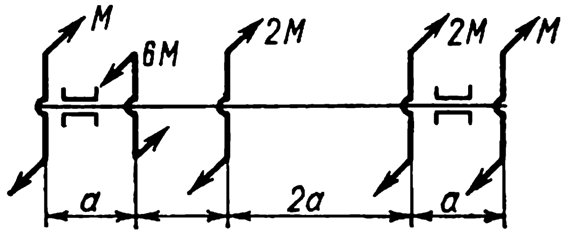

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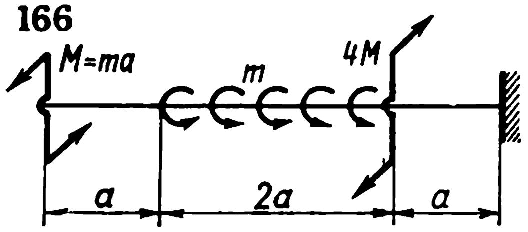

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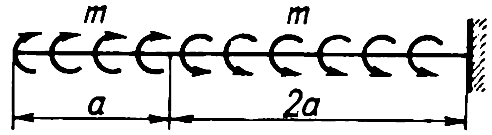

7.2.

Shearing Stresses, Angle of Twist and Elastic Strain Energy

For a round cylindrical rod of diameter $d = 2r$, the shearing stresses τ, at an arbitrary point at the distance ρ from the centre in a cross section, are found by the formula

$$\tau = \frac{M_t \rho}{I_p} \tag{82}$$

in which $I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \cong 0.1d^4$ is the polar moment of inertia of a circular cross section.

The maximum shearing stresses, at points most remote from the centre, are

$$\tau_{max} = \frac{M_t}{W_p} \tag{83}$$

in which $W_p = \frac{I_p}{r} = \frac{\pi r^3}{2} = \frac{\pi d^3}{16} \cong 0.2d^3$ is the polar section modulus of a circular cross section.

For rods of noncircular cross section the maximum shearing stresses are found by the formula

$$\tau_{max} = \frac{M_t}{W_t} \tag{84}$$

in which W_t = section modulus for torsion which is given for various cross sections in handbooks and textbooks on the strength of materials.

The angle of twist φ over a length l in which the torque M_t is constant is found by the formula for Hooke's law:

$$\varphi = \frac{M_t l}{G I_t} \tag{85}$$

in which I_t = "moment of inertia" of a cross section of a rod in torsion. This moment is equal to I_p for a rod of circular cross section; for other sections it is given in textbooks on the strength of materials and in handbooks.

If the rod has several portions over which M_t varies in accordance with some law, the total angle of twist (angle through which the end sections of the rod turn with respect to each other) is found from the expression

$$\varphi = \sum \int \frac{M_t\,dx}{GI_t} \tag{86}$$

Integration is carried out over the length of each portion; summation, over all the portions of the rod.

For shafts the angles of twist can be conveniently calculated for both sides of the cross section at which the drive pulley of the shaft is located.

The general formula for calculating the amount of elastic strain energy accumulated in a rod in torsion is of the form

$$U = \sum \int \frac{M_t^2\,dx}{2GI_t} \tag{87}$$

Here integration and summation are performed in the same way as for determining the angle of twist.

Example 19. The quantities M, a, d and G being known (Fig. 41), plot diagrams for M_t and φ; find $\tau_{\max_{I,\,II}}$ and U.

Solution. The torque over portion I is $M_{t_I} = -M$.

The torque in an arbitrary cross section of portion II is

$$M_{t_{II}} = -M + 2M - m\,(2a - x_2) = M - \frac{2M}{a}\,(2a - x_2)$$

$$= M\left(\frac{2x_2}{a} - 3\right);$$

$$M_{t_{II}\atop x_2=0} = -3M; \quad M_{t_{II}\atop x_2=2a} = M; \quad \tau_{\max_I} = \frac{M}{W_t}$$

For a square section with the side c, $W_t \cong 0.208c^3$.

Since $c = \frac{a}{\sqrt{2}}$; $W_t \cong 0.208\,\frac{d^3}{2\sqrt{2}} \cong 0.0736d^3$.

Therefore

$$\tau_{\max_I} = \frac{M}{0.0736d^3} \cong 13.6\,\frac{M}{d^3}; \quad \tau_{\max_{II}} = \frac{3M}{W_p} = \frac{3M}{\pi d^3} \times 16 \cong 15.3\,\frac{M}{d^3}$$

The left-hand section is fixed and cannot turn, hence the angle of twist should be calculated from the left-hand end of the rod.

For an arbitrary section of portion *II*

$$\varphi_{II_{x_2}}=\frac{M}{GI_p}\int_0^{x_2}\left(\frac{2x_2}{a}-3\right)dx_2=\frac{M}{GI_p}\left(\frac{x_2^2}{a}-3x_2\right);\quad \varphi_{II_{x_2=0}}=0;$$

$$\varphi_{II_{x_2=2a}}=\frac{M}{GI_p}(4a-6a)=-\frac{2Ma}{GI_p}=-\frac{2Ma}{G\pi d^4}\times 32\cong-20.37\frac{Ma}{Gd^4}$$

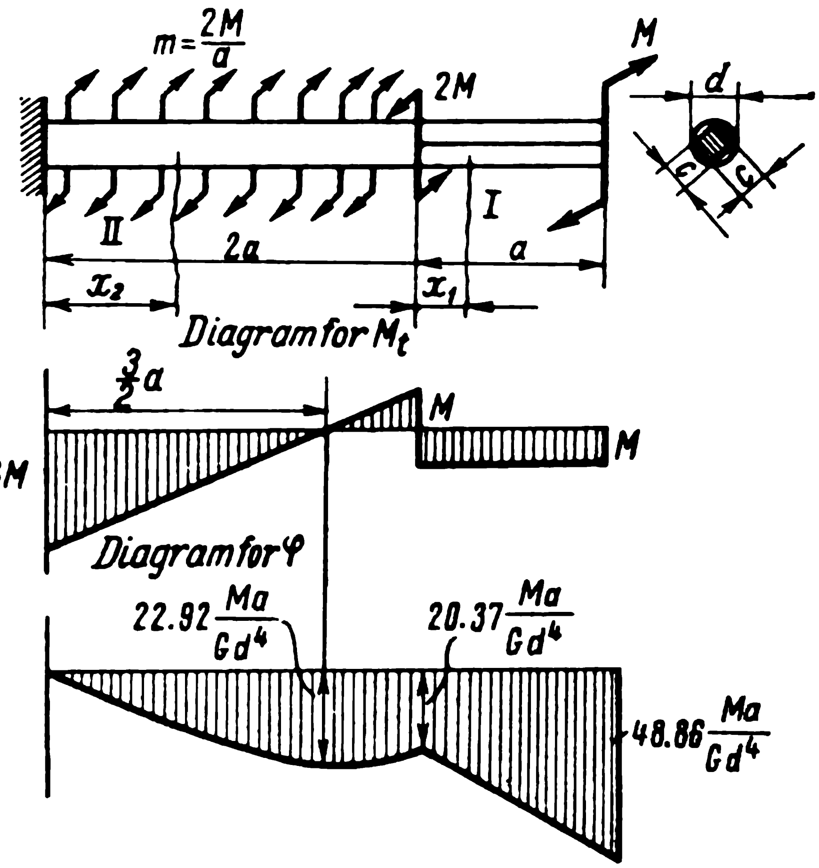

Fig. 41

Next we find the extremum of angle φ_{II}:

$$\frac{d\varphi_{II_{x_2}}}{dx_2}=\frac{M}{GI_p}\left(2\frac{x_2}{a}-3\right)=0;\quad x_2=\frac{3}{2}a;$$

$$\varphi_{II_{x_2=\frac{3}{2}a}}=\frac{M}{GI_p}\left(\frac{9a}{4}-\frac{9a}{2}\right)=-\frac{9}{4}\frac{Ma}{GI_p}$$

$$=-\frac{9}{4}\frac{Ma}{G\pi d^4}\times 32\cong-22.92\frac{Ma}{Gd^4}$$

For an arbitrary section of portion *I*

$$\varphi_{I_{x_1}}=\varphi_{II_{x_2=2a}}-\frac{Mx_1}{GI_t};\quad \varphi_{I_{x_1=0}}=\varphi_{II_{x_2=2a}};$$

$$\varphi_{I_{x_1=a}}=-\frac{2Ma}{GI_p}-\frac{Ma}{GI_t}$$

Since

$$I_t \cong 0.1404c^4 = 0.1404\frac{d^4}{4} \cong 0.0351d^4$$

and

$$\frac{1}{I_t} \cong \frac{1}{0.0351d^4} \cong \frac{28.49}{d^4}$$

then

$$\varphi_{I_{x_1=a}} = -(20.37+28.49)\frac{Ma}{Gd^4} = -48.86\frac{Ma}{Gd^4}$$

The amount of elastic strain energy in the twisted rod is obtained by formula (87):

$$U = U_I + U_{II} = \frac{M^2a}{2GI_t} + \frac{M^2}{2GI_{t_0}}\int_0^{2a}\left(\frac{2x_2}{a}-3\right)dx$$

$$= \frac{M^2a}{Gd^4}\left[14.25+5.1\left(\frac{32}{3}-24+18\right)\right] \cong 38\frac{M^2a}{Gd^4}$$

Problems 168 through 173. Plot the diagrams of the torque M_t and angle of twist φ and find the maximum shearing stresses τ_{max} and elastic strain energy U accumulated in the volume of the bar. Problem 170 should be solved in the International System of Units (SI).

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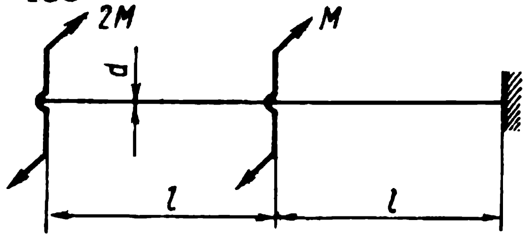

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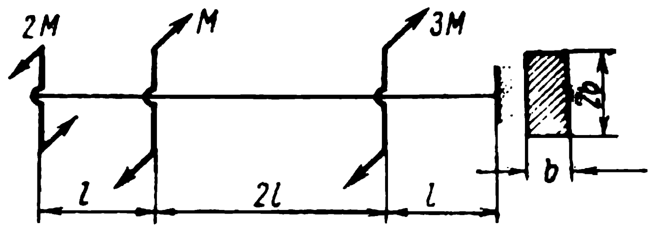

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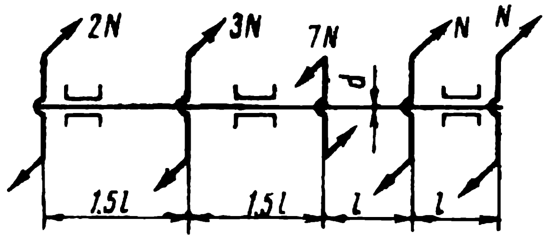

ω

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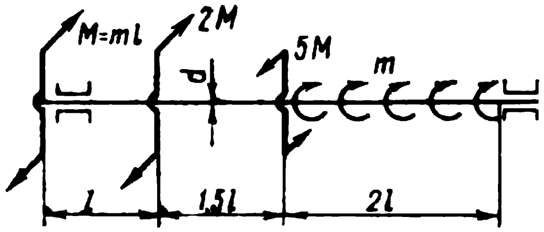

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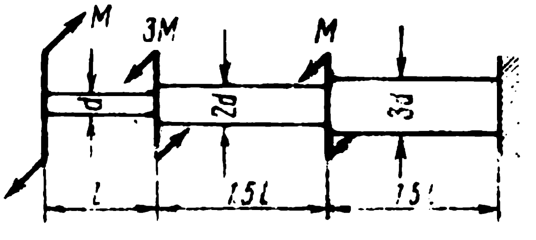

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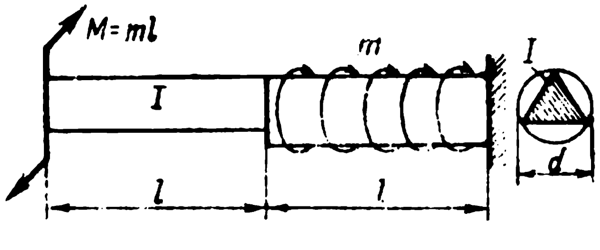

7.3.

Strength and Rigidity (Stiffness)

For a rod of constant cross section in torsion the dimensions of the cross section are calculated and assigned by means of the design formula

$$W_t \geqslant \frac{\max M_t}{[\tau]} \tag{88}$$

in which $\max M_t$ is the maximum torque (absolute value).

To satisfy an additional condition that the maximum angle of twist φ_{max} must not exceed the allowable value $[\varphi]$, the cross section is checked for stiffness by the use of the expression

$$I_t \geqslant \frac{\max M_t l_r}{G[\varphi]} \tag{89}$$

in which l_r is the rated length for which the allowable angle of twist is given.

If the allowable angle of twist is specified in degrees per metre of length $[\varphi°]$, it should be converted into radians $\left([\varphi] = [\varphi°]\frac{\pi}{180}\right)$ and in formula (89) it should be assumed that $l_r = 100$ cm.

If a shaft of constant section for transmitting power is to be designed, one of the expressions (81) should be substituted into formulas (88) and (89) for $\max M_t$, and the maximum power value from the power diagram should be used as the rated (design) power N_r.

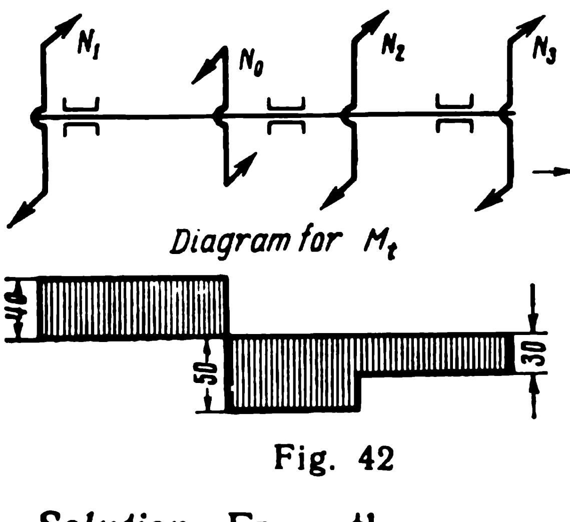

Fig. 42

For bars of solid circular and annular cross section $W_t = W_p$ and $I_t = I_p$.

Example 20. Let $N_1 = 40$ hp, $N_2 = 20$ hp, $N_3 = 30$ hp, $n = 1000$ rpm; $\alpha = \frac{d}{D} = 0.6$; $[\tau] = 450$ kgf/cm²; $[\varphi°] = 2$ deg/m and $G = 8 \times 10^5$ kgf/cm² (Fig. 42).

Find D and d.

Solution. From the power diagram of Fig. 42 the rated power $N_r = 50$ hp.

Since the polar section modulus of an annular cross section is

$$W_p = \frac{\pi D^3}{16}(1-\alpha^4) \text{ in which } \alpha = \frac{d}{D}$$

design formula (88), taking relation (81a) into account, can be written as

$$\frac{\pi D^3}{16}(1-\alpha^4) \geqslant \frac{71{,}620}{n[\tau]} N_r$$

Whence

$$D=\sqrt[3]{\frac{16}{\pi}\,\frac{71{,}620N_r}{n\,[\tau]\,(1-\alpha^4)}}\cong 71.4\sqrt[3]{\frac{50}{10^3\times 450\times 0.87}}\cong 3.64\ \text{cm}$$

Since the permissible angle of twist is given in degrees per metre of length, formula (89) can be put in the following form:

$$I_p=\frac{\pi D^4}{32}(1-\alpha^4)\geqslant\frac{71{,}620N_r\times 100\times 180}{nG\,[\varphi^\circ]\,\pi}$$

Whence

$$D=\sqrt[4]{\frac{71{,}620\times 100\times 180\times 32N_r}{\pi^2 nG\,[\varphi^\circ]\,(1-\alpha^4)}}\cong 253.4\sqrt[4]{\frac{N_r}{nG\,[\varphi^\circ]\,(1-\alpha^4)}}$$

$$=253.4\sqrt[4]{\frac{50}{10^3\times 8\times 10^5\times 2\times 0.87}}\cong 3.49\ \text{cm}$$

D should be taken equal to 3.64 cm, then $d=0.6\times 3.64\cong$ $\cong 2.18$ cm.

Example 21. Let $d=4$ cm; $\omega=80$ rad/sec; $G=8\times 10^4$ MN/m², $[\tau]=60$ MN/m² and $[\varphi]=2\times 10^{-2}$ rad/m (Fig. 43).

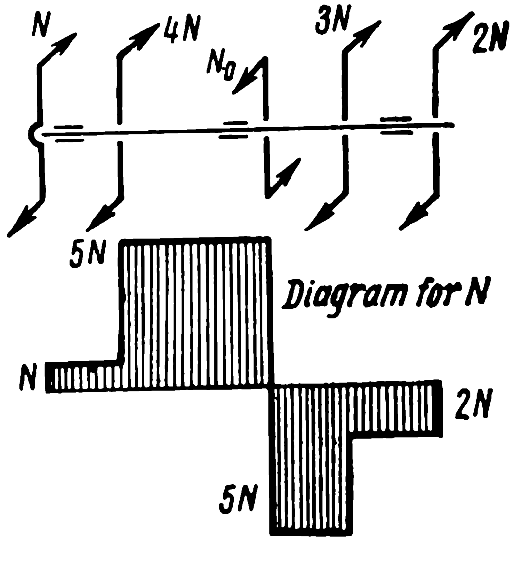

Fig. 43

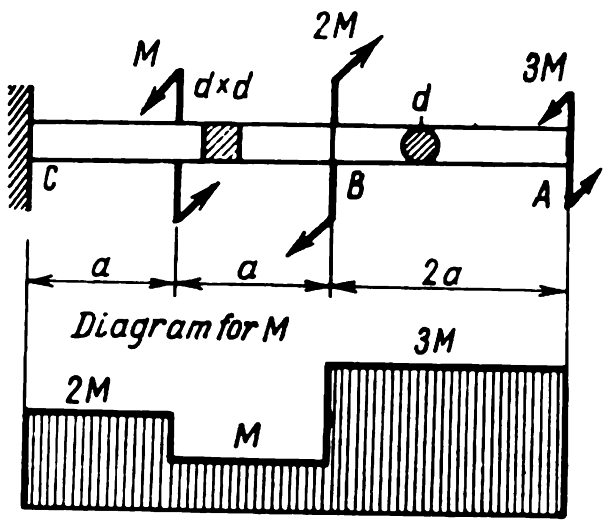

Fig. 44

Find N_0 (kW).

Solution. In accordance with strength condition (88) and taking into account relation (81c) the rated power is

$$N_r\leqslant W_p\,[\tau]\,\omega=\frac{\pi d^3\,[\tau]\,\omega}{16}=\frac{\pi\,(4\times 10^{-2})^3\times 60\times 10^6\times 80}{16}$$

$$\cong 60{,}300\ \text{W}=60.3\ \text{kW}$$

Proceeding from (89) and taking into account (81c) we obtain

$$N_r\leqslant\frac{G\,[\varphi]\,I_p\omega}{l_r}=\frac{G\,[\varphi]\,\pi\,d^4\omega}{32l_r}=\frac{8\times 10^{10}\times 2\times 10^{-2}\pi\,(4\times 10^{-2})^4\times 80}{32}$$

$$\cong 32{,}200\ \text{W}=32.2\ \text{kW}$$

Since from the power diagrams (Fig. 43) $N_r = 5N = \frac{1}{2} N_0$, then $N_0 \leqslant 2N_r = 2 \times 32.2 = 64.4$ kW.

Example 22. Let $d = 4$ cm, $a = 40$ cm; $G = 8 \times 10^5$ kgf/cm² and $\varphi^\circ_{B-C} = 1^\circ$ (Fig. 44).

Find τ_{max} and φ°_{A-C}.

Solution. First we plot the diagram for the torque (Fig. 44). The angle of twist of cross section B with respect to cross section C is calculated from the equation

$$\varphi_{B-C} = 1 \times \frac{\pi}{180} = \frac{Ma}{GI_t} + \frac{2Ma}{GI_t} = \frac{3Ma}{GI_t}$$

Whence

$$M = \frac{\pi G I_t}{3 \times 180 \times a}$$

Since $I_t \cong 0.1404d^4$, $W_t \cong 0.208d^3$, $I_p = \frac{\pi d^4}{32}$, then $W_p = \frac{\pi d^3}{16} \cong$ $\cong 0.2d^3$, i.e. $W_t \cong W_p$ and the dangerous portion is the one with $M_t = 3M$. Therefore

$$\tau_{max} = \frac{3M}{W_p} = \frac{3\pi G \times 0.1404 d^4 \times 16}{3 \times 180 \times a\pi d^3}$$

$$= \frac{G \times 0.1404 \times d \times 16}{180a} = \frac{8 \times 10^5 \times 0.1404 \times 4 \times 16}{180 \times 40} \cong 1000 \text{ kgf/cm}^2$$

The angle of twist section A with respect to section C can be obtained by the formula

$$\varphi^\circ_{A-C} = \varphi^\circ_{B-C} + \varphi^\circ_{A-B} = 1^\circ + \frac{3M \times 2a \times 180}{GI_p\pi}$$

$$= 1^\circ + \frac{2I_t}{I_p} \cong 1^\circ + 2.86^\circ = 3.86^\circ$$

Problems 174 through 177. Find the dimensions of cross sections of rods proceeding from the conditions of required strength and rigidity.

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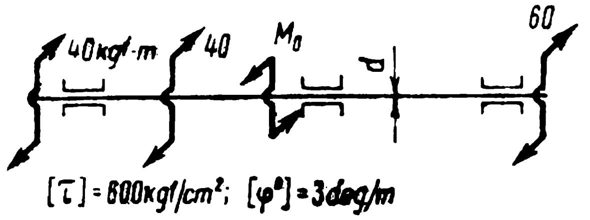

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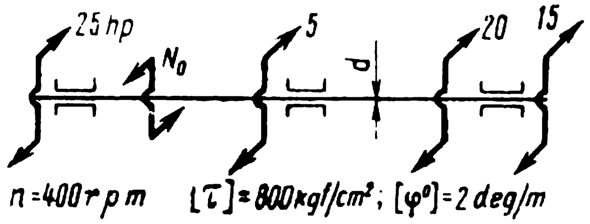

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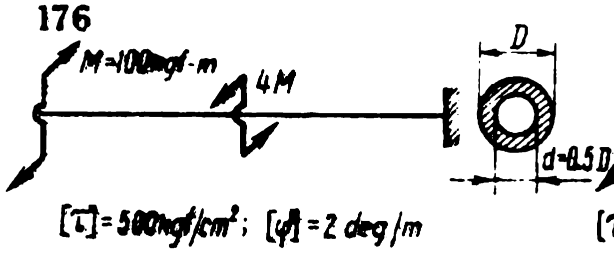

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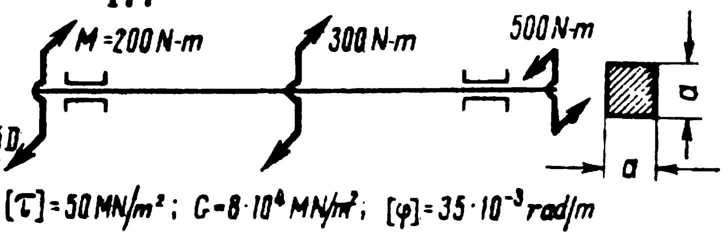
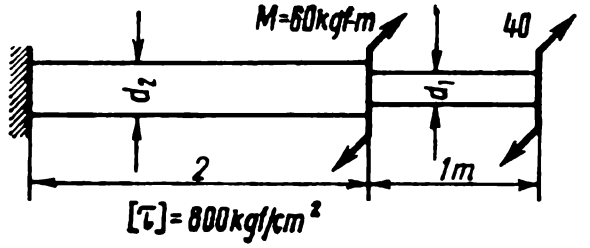


In all the problems of this and subsequent paragraphs, except those solved in the International System of Units (for which G is given in the figure) assume $G = 8 \times 10^5$ kgf/cm^2.

Problems 178 through 183. Find the required dimensions of the cross sections for the rods and the total angle of twist.

In all the problems of this and subsequent paragraphs the numerical values of the lengths of the rods are given in metres.

178



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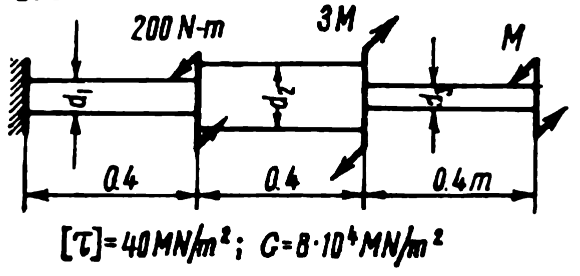

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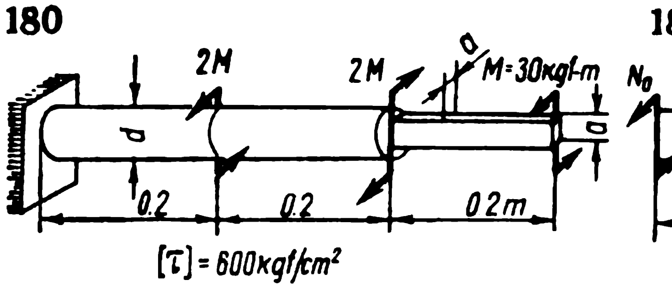

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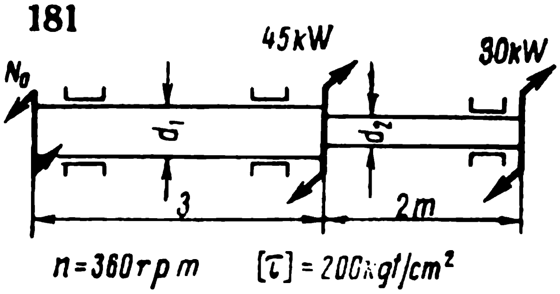

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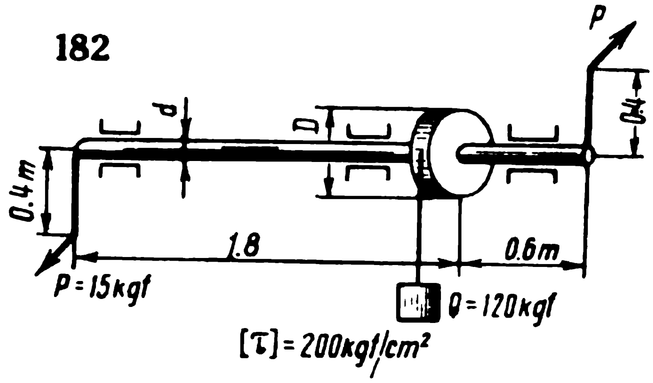

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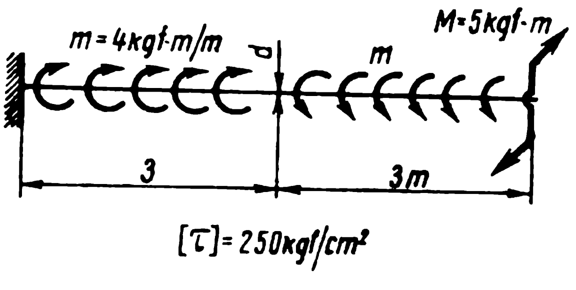
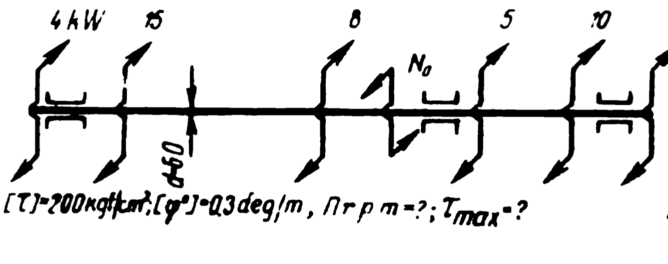


Problems 184 through 201. Find the sought-for quantities indicated on the accompanying figures.

In Problem 193 rod AB rotates at constant speed in a resisting medium which develops a torque reaction uniformly distributed over the length of the rod.

184



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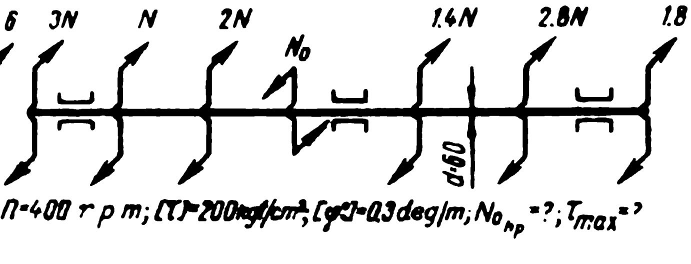

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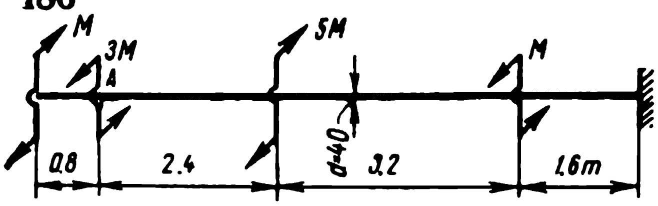
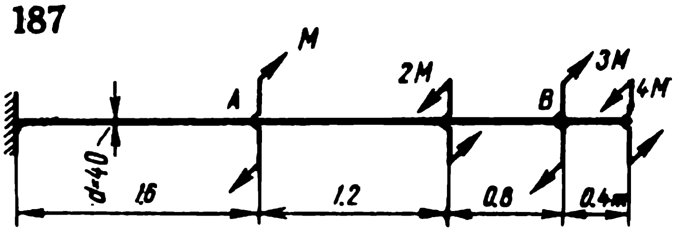


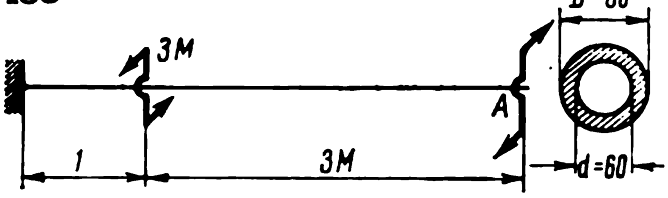
$\varphi_A^\circ = 1^\circ$; $M = ?$; $\tau_{max} = ?$

187



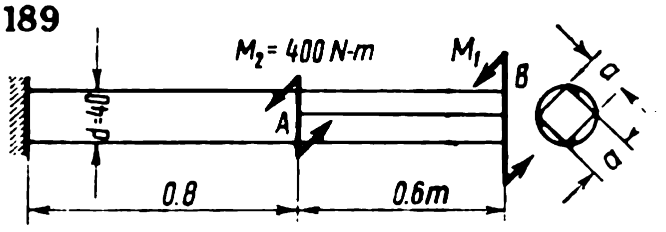
$\varphi_A^\circ = 1^\circ$; $\tau_{max} = ?$; $\varphi_B^\circ = ?$

188



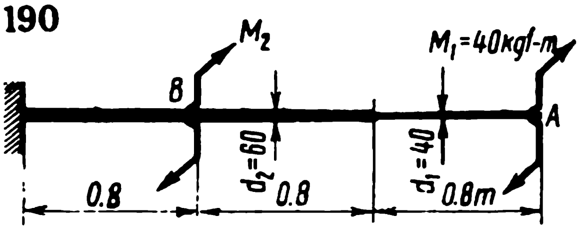
$[\tau] = 200\,kgf/cm^2$; $M = ?$; $\varphi_A^\circ = ?$

189



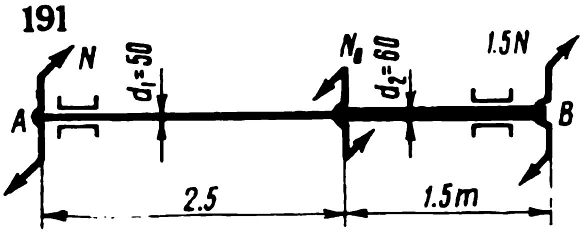
$\varphi_A = 0.5\varphi_B$; $G = 8\times10^4\,MN/m^2$; $M_1 = ?$; $\tau_{max} = ?$

190



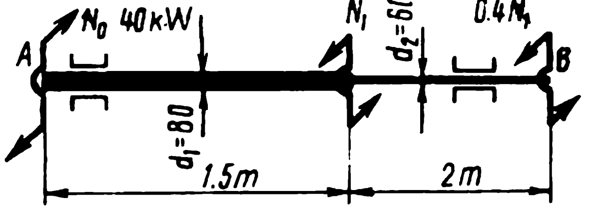
$\varphi_A = 3\varphi_B$; $M_2 = ?$ $\tau_{max} = ?$ $\varphi_A^\circ = ?$

191



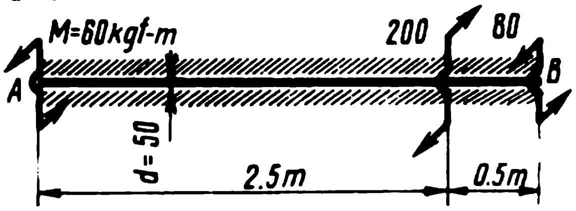
$\omega = 20\,rad/sec$; $[\tau] = 20\,MN/m^2$; $G = 8\cdot10^4\,MN/m^2$;
$N_0 = ?$; $\varphi_{AB} = ?$

192



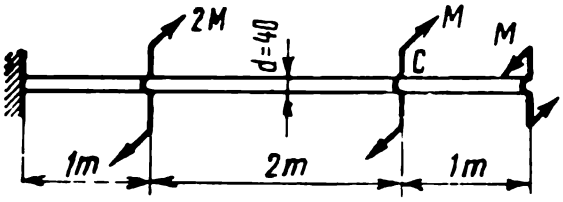
$[\tau] = 200\,kgf/cm^2$; n rpm $= ?$ $\varphi_{AB}^\circ = ?$

193



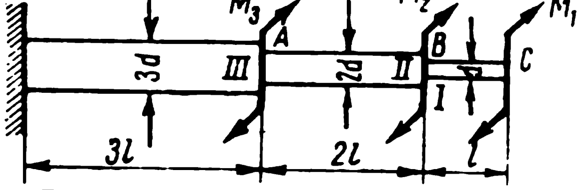
$\tau_{max} = ?$; $\varphi_{AB}^\circ = ?$

194



$\tau_{max} = ?$

195



$\tau_{max_I} = \tau_{max_{II}} = \tau_{max_{III}}$; $\frac{M_2}{M_1} = ?$; $\frac{M_3}{M_1} = ?$
$\frac{\varphi_A}{\varphi_C} = ?$ $\frac{\varphi_B}{\varphi_C} = ?$

In Problem 194 point C on the surface of the rod has been displaced 0.5 mm.

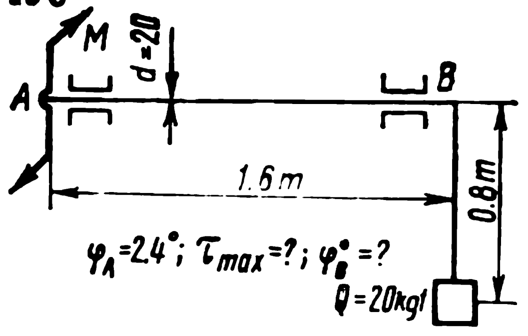
In Problem 197 weight Q has been displaced 2 cm.

In Problem 198 rods AB and CD are absolutely rigid.

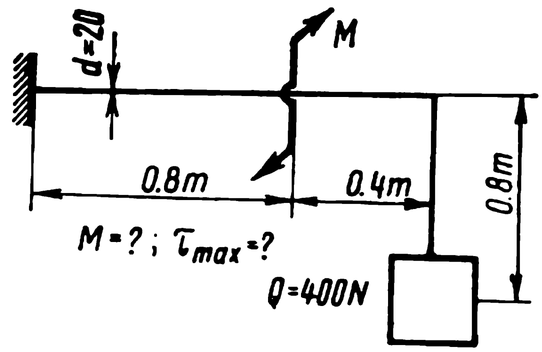
In Problem 200 the rods are practically of equal strength.

In Problem 201 point A lies on an inclined section.

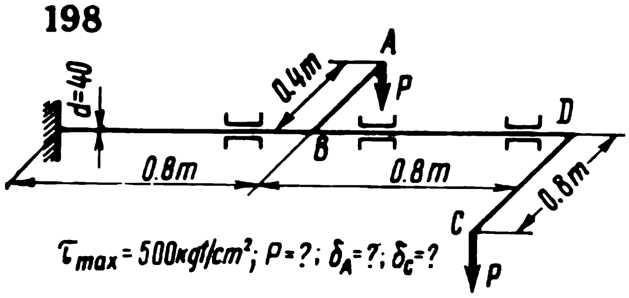
196



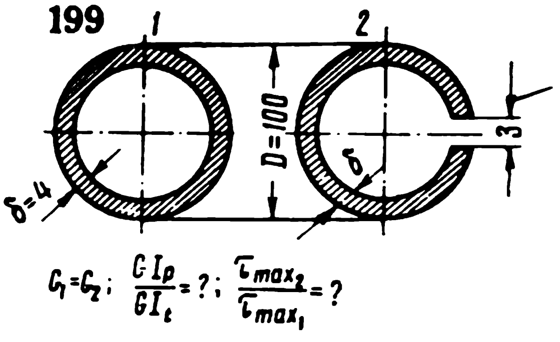
197



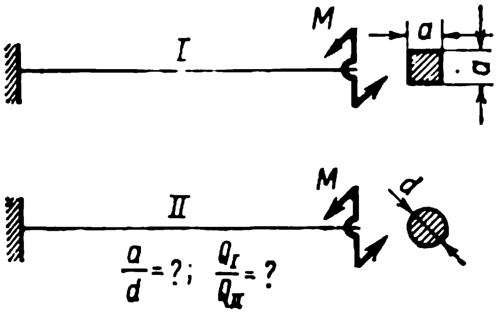
198



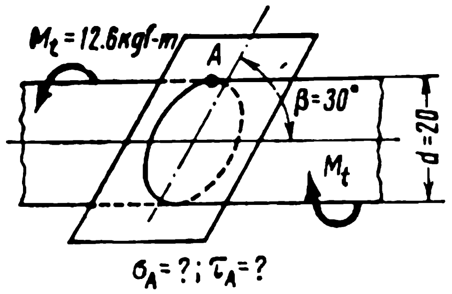
199



200



201



7.4.

Statically Indeterminate Problems

Statically indeterminate problems on torsion, like those on tension and compression, comprise systems in which the reactions of the fixed ends and the internal stresses cannot be found by means of equations of statics alone.

In solving such problems simultaneous use is made of the equations of combined displacements and pertinent equations of statics. The former are based on non-separability of the elements of the system and represent geometric dependences between the displacements of elements making up the system.

If the redundant constraints are absolutely rigid, their deformation equals zero, while if they are elastic, their displacements are found by means of the deformations determined by Hooke's law.

If all the elements of a statically indeterminate system are subject to torsion, the elastic displacements are expressed by angles of twist. If certain elements in a system are subject to torsion, and others to

tension or compression, the displacements for the first are found by the angles of twist, and for the second, through the linear axial strain.

The conditions of statics and those of combined displacements differ for each type of statically indeterminate system. They are essentially the same, however, for certain types of systems and can be expressed by equations that are similar in principle.

As an example, let us consider systems consisting of several straight elements, connected coaxially to one another, with the systems rigidly fixed at the ends and loaded only by external couples producing torsion (Fig. 45). The corresponding equation of statics is the algebraic sum of moments of all the applied and reaction force couples with respect to the geometric axis of the elements. This sum of moments is equated to zero.

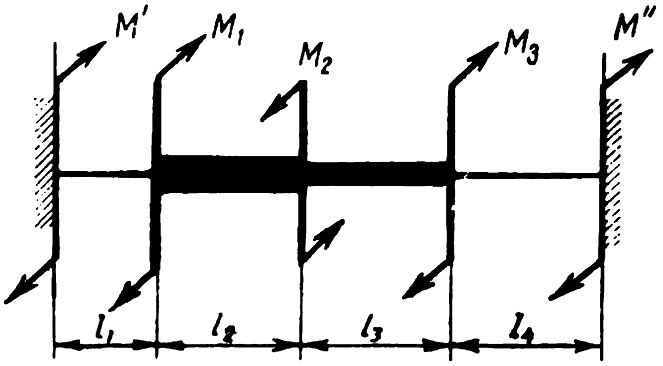


Fig. 45

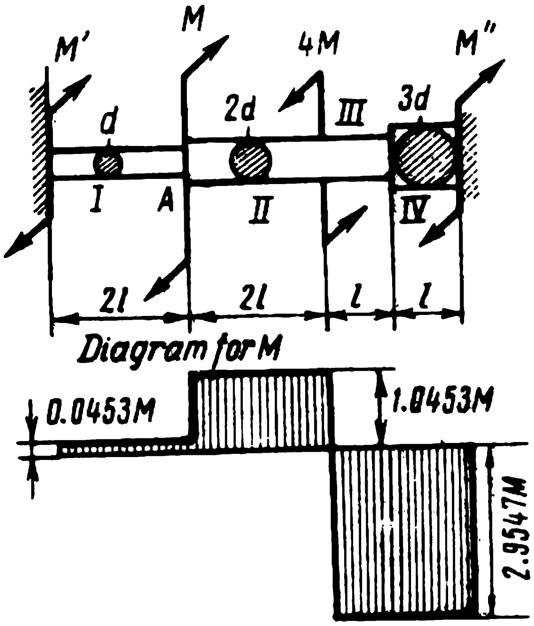


Fig. 46

The equation of combined displacements is the algebraic sum of the angles of twist in all the portions, which also equals zero, since the end sections are fixed.

If one of the rod ends is not fixed rigidly, but elastically, the angle of twist of the elastically fixed end is not zero. It is proportional to the torque reaction. If both ends are fixed elastically, the total angle of twist is equal to the difference between the angles of twist in the fixed sections.

Example 23. Given: M, d, l and G (Fig. 46).

Find τ_{max} and φ_A°.

Solution. We first determine the polar moments of inertia for circular sections in portions *I*, *II*, *III* and *IV*

$$I_{p_I}=I_p=\frac{\pi d^4}{32};\quad I_{p_{II}}=I_{p_{III}}=16I_p;\quad I_{p_{IV}}=81I_p$$

and the angles of twist

$$\varphi_I=\frac{M'2l}{GI_p};\quad \varphi_{II}=\frac{(M'+M)\,2l}{16GI_p};\quad \varphi_{III}=\frac{(M'+M-4M)\,l}{16GI_p}\ \text{and}$$

$$\varphi_{IV}=\frac{(M'+M-4M)\,l}{81GI_p}$$

Since the angle of relative twist of the end sections is zero,

$$\varphi_I + \varphi_{II} + \varphi_{III} + \varphi_{IV} = \frac{l}{GI_p}\Big[2M' + 2(M'+M)\frac{1}{16} + (M'-3M)\frac{1}{16} + (M'-3M)\frac{1}{81}\Big] = 0$$

from which

$$M' = \frac{129}{2851}M \cong 0.0453M$$

From the equation of statics $M'' = 3M - M' = 2.9547M$.

The torque (twisting moment) diagram plotted from the above data is shown in Fig. 46.

Since $\frac{M_{t_{III}}}{M_{t_I}} = \frac{2.9547}{0.0453} \cong 65$ and $\frac{W_{p_{III}}}{W_{p_I}} = \frac{\pi(2d)^3}{\pi d^3} = 8$

the maximum tangential stress is in the cross section of diameter $2d$ (portion *III*).

The tangential stress is found by the formula

$$\tau_{\max} = \frac{M_{t_{III}}}{W_{p_{III}}} = \frac{2.9547M}{\pi d^3} \times 2 \cong 5.909\frac{M}{\pi d^3} \cong 1.88\frac{M}{d^3}$$

The angle of twist, in degrees, for cross section A is found in the following manner:

$$\varphi_A^\circ = \frac{M_{t_I} \times 2l}{GI_p} \times \frac{180}{\pi} = \frac{0.0453M \times 2l \times 180 \times 32}{G\pi d^4 \pi} \cong 52.2\frac{M}{Gd^4}$$

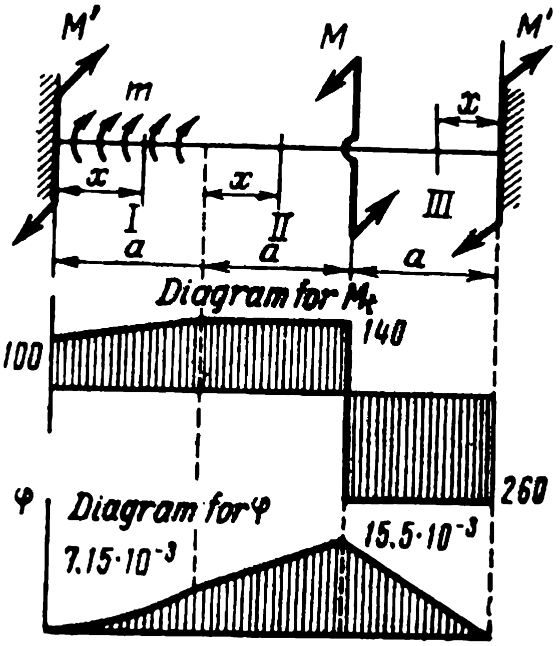


Fig. 47

Example 24. Let $m = 80$ N-m/m; $M = 400$ N-m; $a = 0.5$ m, $[\tau] = 40$ MN/m² and $G = 8 \times 10^4$ MN/m² (Fig. 47).

Find d and draw the diagram for φ.

Solution. From the condition of statics $M' + ma - M + M'' = 0$.

According to the displacements equation

$$\varphi_I + \varphi_{II} + \varphi_{III} = 0$$

Since the rigidity of the cross section of the rod is constant, the preceding equation can be written as follows:

$$M'a + m\int_0^a x\,dx + (M'+ma)a + (M'+ma-M)a = 0$$

Whence

$$3\,M'a = Ma - 2\,ma^2 - m\frac{a^2}{2}$$

and

$$M' = \frac{1}{3}(M - 2.5\,ma) = \frac{1}{3}(400 - 2.5\times 80\times 0.5) \cong 100 \text{ N-m}$$

Substituting M' into the statics equation, we obtain

$$M'' = \frac{1}{3}(2M - 0.5\,ma) = \frac{1}{3}(800 - 0.5\times 80\times 0.5) \cong 260 \text{ N-m}$$

Next we plot the torque (twisting moment) diagram (Fig. 47) from which it is found that the maximum torque $M_t = 260$ N-m.

From the design formula

$$W_p = \frac{\pi d^3}{16} \geqslant \frac{\max M_t}{[\tau]}$$

Whence

$$d \geqslant \sqrt[3]{\frac{16}{\pi}\times\frac{\max M_t}{[\tau]}} \cong 1.72\sqrt[3]{\frac{\max M_t}{[\tau]}} = 1.72\sqrt[3]{\frac{260}{400\times 10^6}}$$
$$\cong 3.2\times 10^{-2} \text{ m} = 3.2 \text{ cm}$$

Then we find the angles of twist

$$\varphi_{I_x} = \frac{1}{GI_p}\int_0^x (M' + mx)\,dx = \frac{M'x + \frac{mx^2}{2}}{GI_p};$$

$$\varphi_{I_{x=0}} = 0;\quad \varphi_{I_{x=a}} = \frac{M'a + \frac{ma^2}{2}}{GI_p} = \frac{100\times 0.5 + \frac{1}{2}\times 80\times 0.25}{8\times 10^{10}\times 0.1\times 3.2^4\times 10^{-8}}$$
$$\cong 0.00715;$$

$$\varphi_{II_x} = \varphi_{I_{x=a}} + \frac{(M' + ma)}{GI_p}x;\quad \varphi_{II_{x=0}} = \varphi_{I_{x=a}};$$

$$\varphi_{II_{x=a}} = \varphi_{I_{x=a}} + \frac{M'a + ma^2}{GI_p} = 0.00715 + \frac{100\times 0.5 + 80\times 0.25}{8\times 10^{10}\times 0.1\times 3.2^4\times 10^{-8}}$$
$$= 0.00715 + 0.00835 = 0.0155;$$

$$\varphi_{III_x} = \frac{M''x}{GI_p};\quad \varphi_{III_{x=0}} = 0;$$

$$\varphi_{III x=a} = \frac{M''a}{GI_p} = \frac{26\times 10^2\times 50}{8\times 10^5\times 0.1\times 3.2^4} \cong 0.0155$$

Now we can plot the diagram for the angle of twist on the basis of the above data (see Fig. 47).

Example 25. A tube of length $4a$ and outside and inside diameters D and d is fixed at its lower end C (Fig. 48). A rod of round cross section

and diameter $d_0 = \frac{D}{2} = \frac{d}{1.6}$ is inserted into the tube to a distance of $2a$. The lower end B of the rod is rigidly fixed in the tube, the upper end E of the tube is rigidly fixed to the rod. A force couple of moment M acts about the geometric axis of the system at the end cross section A of the projecting portion of the rod; the upper cross section E of the tube is subject to the action of a force couple of moment $2M$.

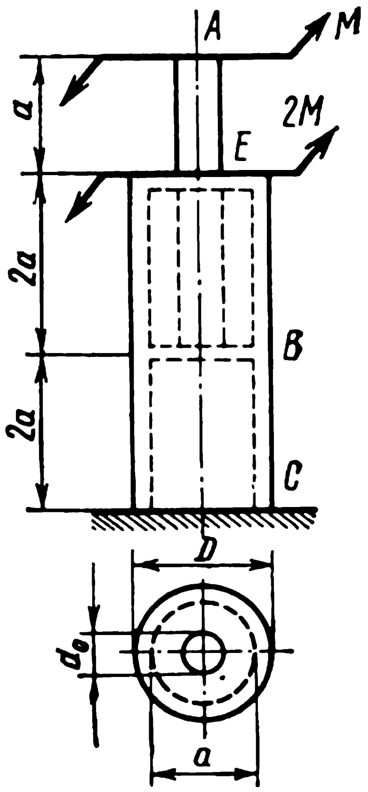


Fig. 48

Find $\tau_{\max_I}$ in the rod and $\tau_{\max_{II}}$ in the tube, as well as φ_{A-C}, if G is known for the material of the rod and tube.

Solution. The torque on the portion A-E of the rod is

$$M_{t_{AE}} = M$$

Since at C where the lower end of the tube is fixed the torque reaction $M_C = 3M$, the torque on the tube portion B-C is $M_{t_{BC}} = 3\ M$.

The statically indeterminate system, consisting of the rod and tube is subject to the torque $3M$ at portion B-E. From the equation of statics it follows that $M_I + M_{II} = 3\ M$ in which M_I is the torque transmitted through the rod and M_{II} is the torque transmitted through the tube.

From the condition of combined displacements, the angles of twist φ_I of the rod and φ_{II} of the tube in the portion B-E are equal to each other, i.e.

$$\frac{M_I 2a}{GI_{p_I}} = \frac{M_{II} 2a}{GI_{p_{II}}}$$

Whence

$$M_{II} = M_I \frac{I_{p_{II}}}{I_{p_I}}$$

Substituting it into the equation of statics we obtain

$$M_I \left(1 + \frac{I_{p_{II}}}{I_{p_I}}\right) = 3M \quad \text{or} \quad M_I = \frac{3M}{1 + \frac{I_{p_{II}}}{I_{p_I}}}; \quad M_{II} = \frac{3M}{1 + \frac{I_{p_I}}{I_{p_{II}}}}$$

Since

$$I_{p_I} = \frac{\pi d_0^4}{32} \text{ and } I_{p_{II}} = \frac{\pi D^4}{32}(1 - \alpha^4) = \frac{16\pi d_0^4}{32}(1 - 0.8^4) \cong 9.45 I_{p_I};$$

$$M_I = \frac{3M}{1 + 9.45} \cong 0.287M \text{ and } M_{II} \cong 2.713M$$

The maximum tangential stresses are: for the rod in the portion A-E: $\tau_{\max_I} = \frac{M}{W_{p_I}} = \frac{16M}{\pi d_0^3} \cong 5.09 \frac{M}{d_0^3}$; for the tube in the portion

$$B\text{-}C:\ \tau_{\max_{II}} = \frac{3M}{\frac{\pi D^3}{16}(1-\alpha^4)} = \frac{3M}{\frac{8\pi d_0^3}{16} \times 0.58} \cong 0.636\tau_{\max_I} = 3.24 \frac{M}{d_0^3}$$

The angle of twist of cross section A with respect to section C can be determined as follows:

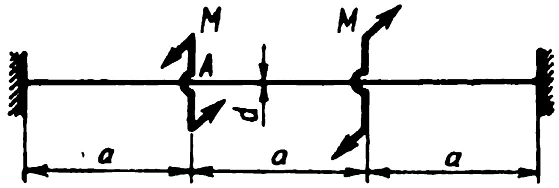
$$\varphi_{A-C} = \varphi_{A-E} + \varphi_{E-B} + \varphi_{B-C} = \frac{Ma}{GI_{p_I}} + 0.287\frac{M2a}{GI_{p_I}} + \frac{3M2a}{G \times 9.45 \times I_{p_I}}$$

$$= 2.21 \frac{Ma}{GI_{p_I}} \cong 22.5 \frac{Ma}{Gd_0^4}$$

Problems 202 through 221. Find the sought-for quantities indicated on the accompanying figures.

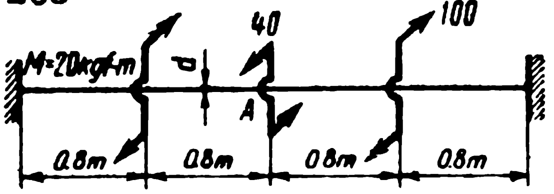
In Problems 216 through 221 assume that $E = 2 \times 10^6$ kgf/cm^2 and $G = 8 \times 10^5$ kgf/cm^2.

202



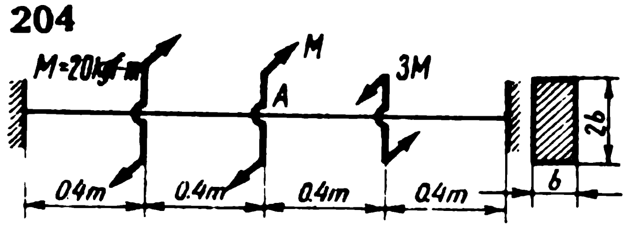
$[\tau]$; $d=?$; $\varphi_A^0=?$

203



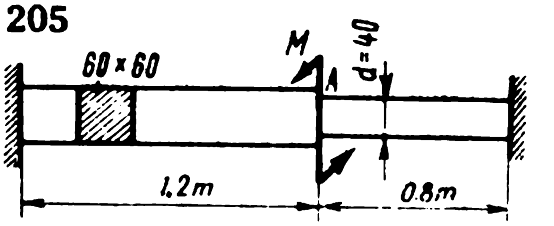
$[\tau]=500$ kgf/cm^2; $d=?$; $\varphi_A^0=?$

204



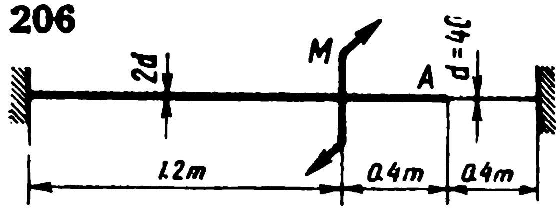
$[\tau]=800$ kgf/cm^2; $b=?$; $\varphi_A^0=?$

205



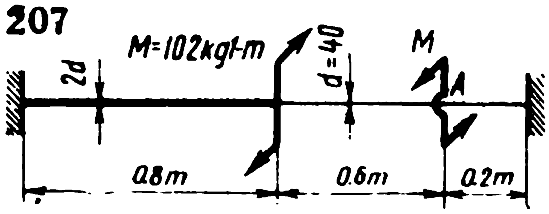
$[\tau]=40$ MN/m^2; $G=8\cdot10^4$ MN/m^2; $M=?$; $\varphi_A=?$; $U=?$

206



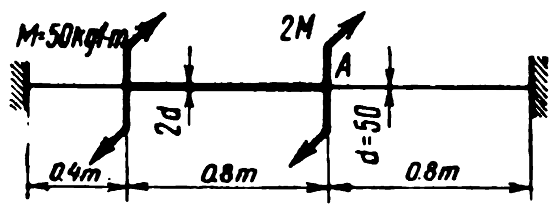
$[\tau]=150$ kgf/cm^2; $M=?$; $\varphi_A^0=?$; $U=?$

207



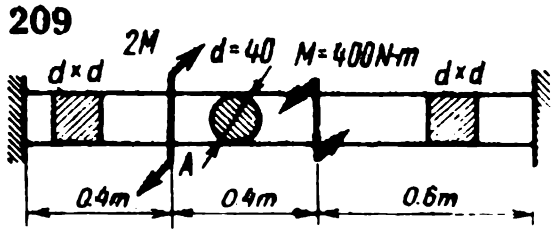
$\tau_{max}=?$; $\varphi_A^0=?$

208



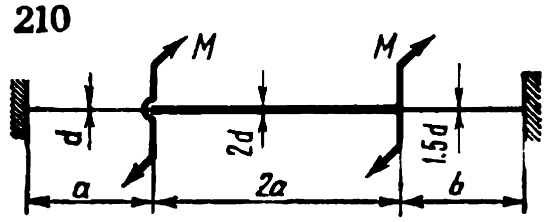
$\tau_{max}=?$; $\varphi_A^0=?$

209



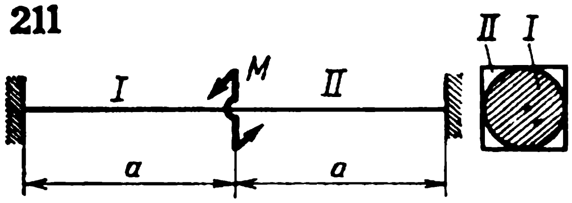
$G=8\cdot10^4$ MN/m^2; $\tau_{max}=?$; $\varphi_A=?$

210



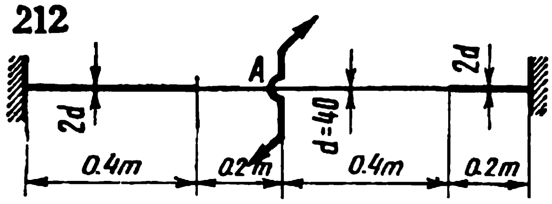
$(\tau_a)_{max} = (\tau_b)_{max}$; $b = ?$

211



$(\tau_I)_{max} = 200\,kgf/cm^2$; $(\tau_{II})_{max} = ?$

212

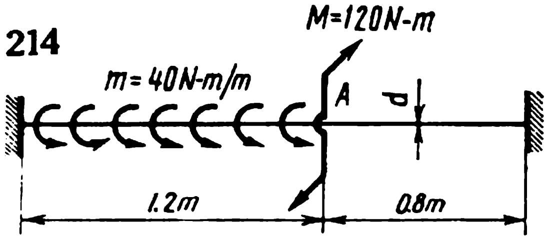


$[\tau] = 20\,MN/m^2$; $G = 8 \cdot 10^4\,MN/m^2$; $M = ?$; $\varphi_A = ?$

213

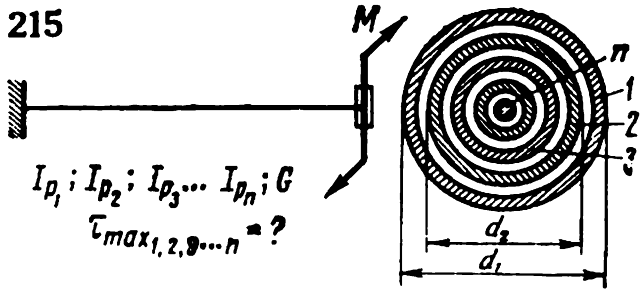
$[\tau] = 250\,kgf/cm^2$; $d = ?$; $\varphi_A^0 = ?$

214



$[\tau] = 10\,MN/m^2$; $G = 8 \cdot 10^4\,MN/m^2$; $d = ?$; $\varphi_A = ?$

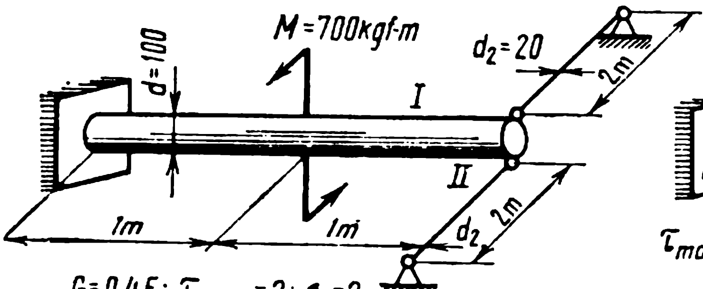
215



I_{p_1}; I_{p_2}; $I_{p_3} \dots I_{p_n}$; G

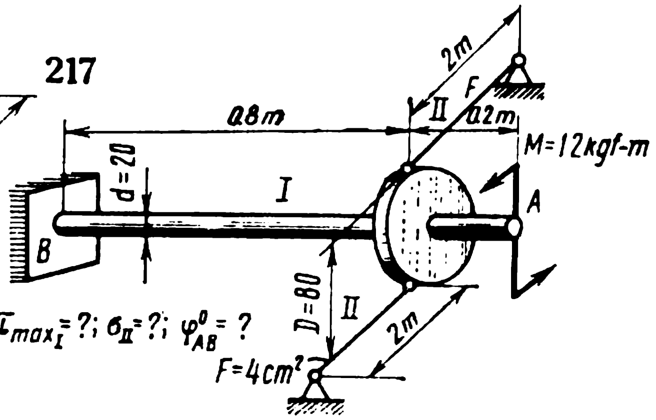
$\tau_{max_{1,2,3 \dots n}} = ?$

216



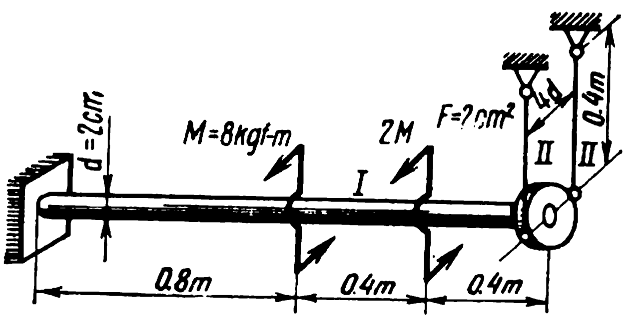
$G = 0.4E$; $\tau_{max_I} = ?$; $\sigma_{II} = ?$

217



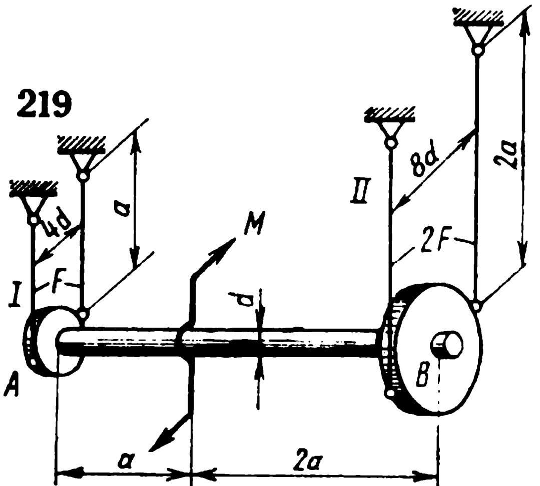
$\tau_{max_I} = ?$; $\sigma_{II} = ?$; $\varphi_{AB}^0 = ?$

218



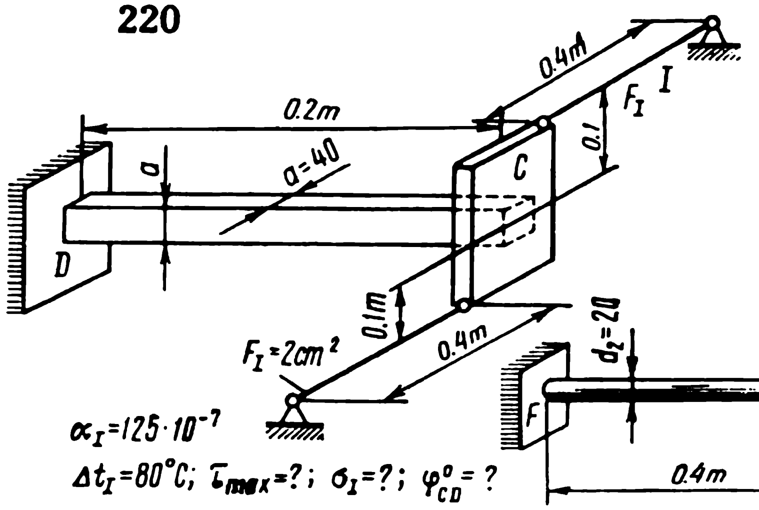
$\tau_{max_I} = ?$; $\sigma_{II} = ?$

219



$F = \frac{\pi d^2}{64}$; $G = 0.4E$; $\tau_{max} = ?$; $\sigma_{I,II} = ?$; $\varphi_{AB}^0 = ?$

220



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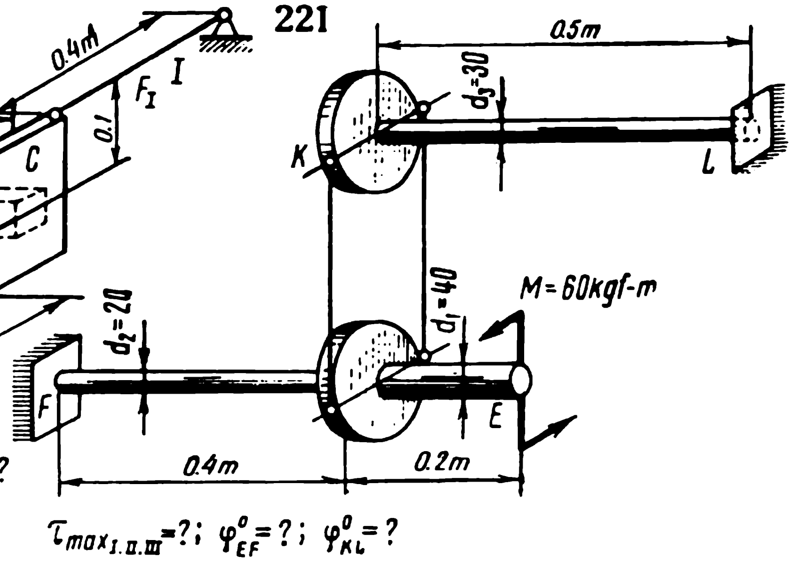


In Problem 215 the cylindrical tubes are arranged concentrically with clearances and are rigidly fixed only at the ends.

CHAPTER 8. TRANSVERSE BENDING

8.1.

Transverse (Shearing) Force and Bending Moment

The shearing force and bending moment are determined by the method of sections.

The shearing force Q_x in any cross section of a beam is equal to the algebraic sum of the projections of all the external forces (both concentrated and distributed) acting on the beam to one side of the cross section under consideration. The forces are to be projected on one of the principal centroidal axes of inertia of the section.

The bending moment M_x in any cross section of the beam is equal to the algebraic sum of moments of all the external forces acting on the beam to one side of the cross section under consideration with respect to one of the principal centroidal axes of inertia of the section.

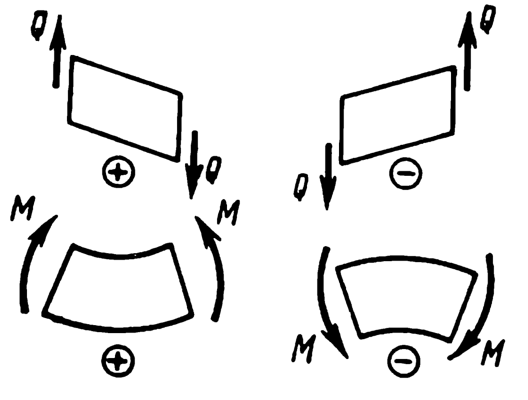


Fig. 49

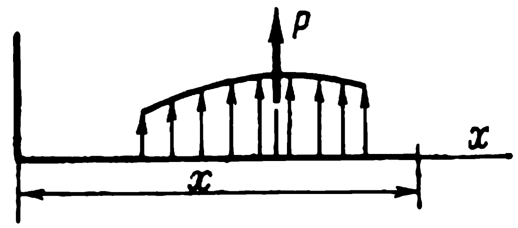


Fig. 50

Q and M are assumed to be positive and negative in accordance with the direction of their action as indicated in Fig. 49.

If the distributed load ends before reaching the cross section under consideration (Fig. 50), it can be replaced with a concentrated force numerically equal to the area of the load diagram and applied at the section passing through the centre of gravity of the area of the distributed load diagram.

For loads varying linearly the areas and positions of the centres of gravity for the cut-away portions are readily determined by the use of well-known geometrical formulas. If the loads vary according to the laws of the quadratic parabola ABC (Fig. 51), the following data from analytical geometry will be useful. The area of parabola $ABC = \frac{2}{3}\, lh$, the centre of gravity O of this area is on the vertical BD;

the area of the parabolic segment $FBE = \frac{2}{3} l_1 h_1$, the centre of gravity O_1 of this area is at a distance $\frac{1}{2} l_1$ from the vertical FH; the area of half the parabola ABD and $DBC = \frac{2}{3} \times \frac{l}{2} h = \frac{1}{3} lh$; the centre of gravity O_2 of this area is at a distance $\frac{3}{8} \times \frac{l}{2} = \frac{3}{16} l$ from the line BD; the area of the right triangle CBG with the parabolic hypotenus BC equals $\frac{1}{3} \times \frac{l}{2} h = \frac{1}{6} lh$, the centre of gravity O_3 of this area is at a distance $\frac{1}{4} \times \frac{l}{2} = \frac{1}{8} l$ from the vertical CG.

It is advisable to approach the section from the end of the beam which is less loaded, and to first plot the diagram for Q and then for M.

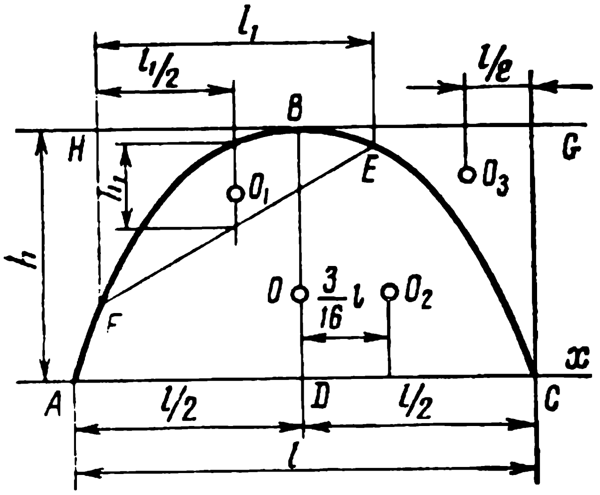


Fig. 51

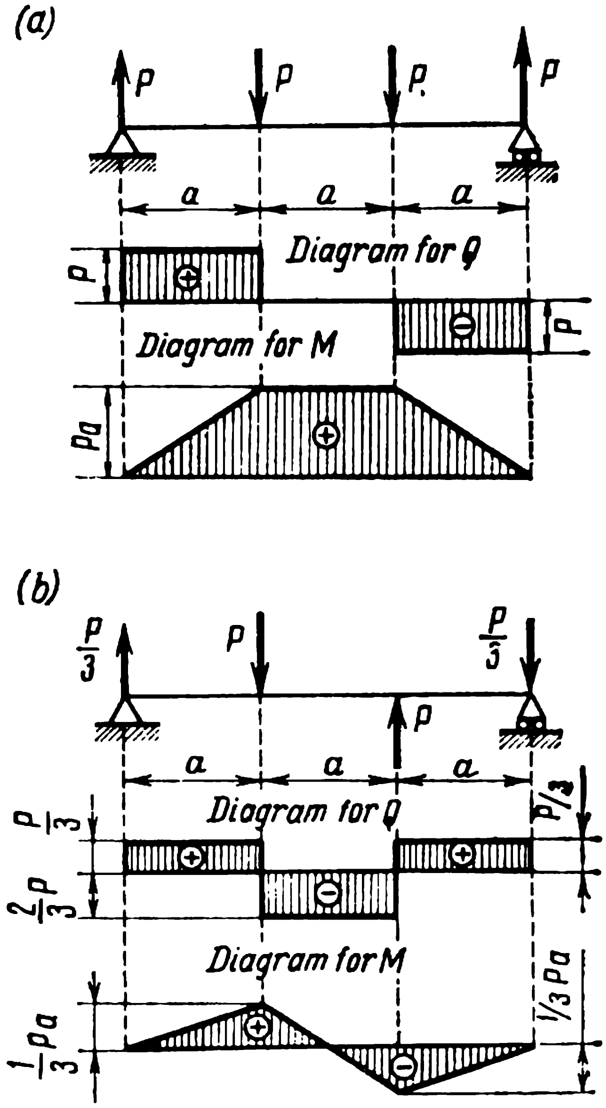


Fig. 52

It follows from the definitions of the quantities Q and M and the conventional sign rule that for beams which are loaded symmetrically and fixed by the same method at their ends, the shearing force, or simply shear diagram is symmetrical about a central point and bending moment diagram is symmetrical about a vertical line (Fig. 52*a*).

On the contrary, for beam loaded symmetrically about a point the shear diagram is symmetrical about a vertical line, whereas the moment diagram is symmetrical about a point (Fig. 52*b*).

According to the definition of Q at a section with an applied concentrated force the shear diagram should have a vertical step equal to this external force (Fig. 52*a* and *b*).

Likewise, from the definition of M it follows that at a section in which a force couple is applied the moment diagram should have a vertical step equal to the moment of this external force couple (Fig. 53).

In plotting and checking shear and moment (Q and M) diagrams for beams not subject to distributed couples producing bending, use should be made of the differential relationships (90) and (91) between M, Q, q and the consequences which follow from them. Thus

$$Q=\frac{dM}{dx}; \tag{90}$$

$$q=\frac{dQ}{dx}=\frac{d^2M}{dx^2} \tag{91}$$

The principal consequences of equations (90) and (91) are:

1. The transverse (shearing) force is geometrically interpreted as the tangent of the angle between the tangent to the bending moment diagram at the section under consideration and the x-axis (beam axis), whereas the intensity of a distributed load is interpreted as the tangent of the angle between the same axis and the tangent to the shear diagram.

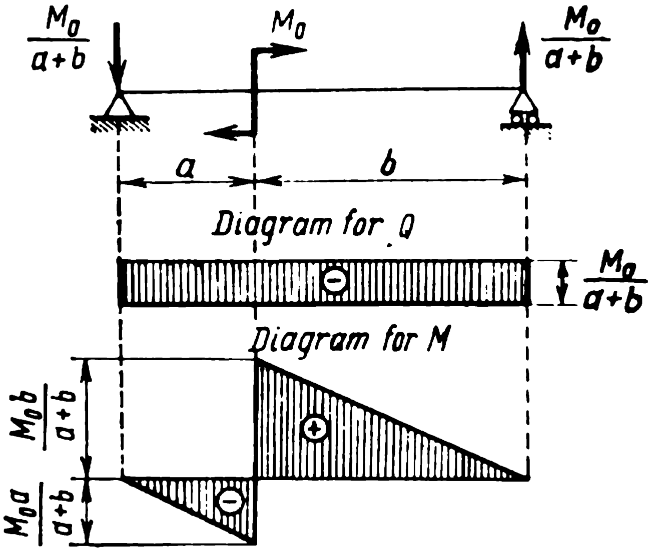


Fig. 53

2. If the functions for the variation of distributed loads are algebraic, then in each portion of the beam the degree of the transverse (shearing) force function is one unit higher than that of the distributed load function in the same portion of the beam, and the degree of the bending moment function is one unit higher than that of the transverse force function.

3. At a section of the beam in which the transverse (shearing) force equals zero, the bending moment is extremal, and at a section in which the transverse force passes through zero in a vertical step, the bending moment diagram has an abrupt change in curvature.

4. At a section of the beam in which the shear diagram has a vertical step, not passing through zero, the bending moment diagram has a sharp bend.

5. If the shear diagram is symmetrical about a point over the entire length of the beam or over some portion of it, the bending moment diagram is symmetrical about a line at the respective portions of the beam, and vice versa.

6. In each portion of the beam, the change in the magnitude of the bending moment between any two sections is equal to the area of the shear diagram between the same two sections.

7. If the x-axis is directed to the left from the right-hand end of the beam then

$$Q = -\frac{dM}{dx}$$

8. The convexity of a curvilinear diagram of the bending moment faces the direction of the intensity of the distributed load.

It may prove useful to remember that in the cross section coinciding with the axis of symmetry of a beam transverse force (symmetrical about a point) equals zero, and in the section coinciding with the axis of the beam passing through the point about which it is symmetrically loaded the bending moment (symmetrical about a line) equals zero. If an external concentrated force is applied along the axis of symmetry, the transverse (shearing) forces in sections to the left and to the right of the axis of symmetry are numerically equal to half the applied force.

PLOTTING SHEAR AND MOMENT Q AND M DIAGRAMS FROM THEIR EQUATIONS

Example 26. Let $M_1 = 2$ tnf-m; $M_2 = 12$ tnf-m; $q_1 = 2$ tnf/m; $q_2 = 4$ tnf/m; $P = 12$ tnf; $a = 3$ m; $b = 2$ m; $c = 1$ m and $d = 4$ m (Fig. 54).

Plot the diagrams for Q and M.

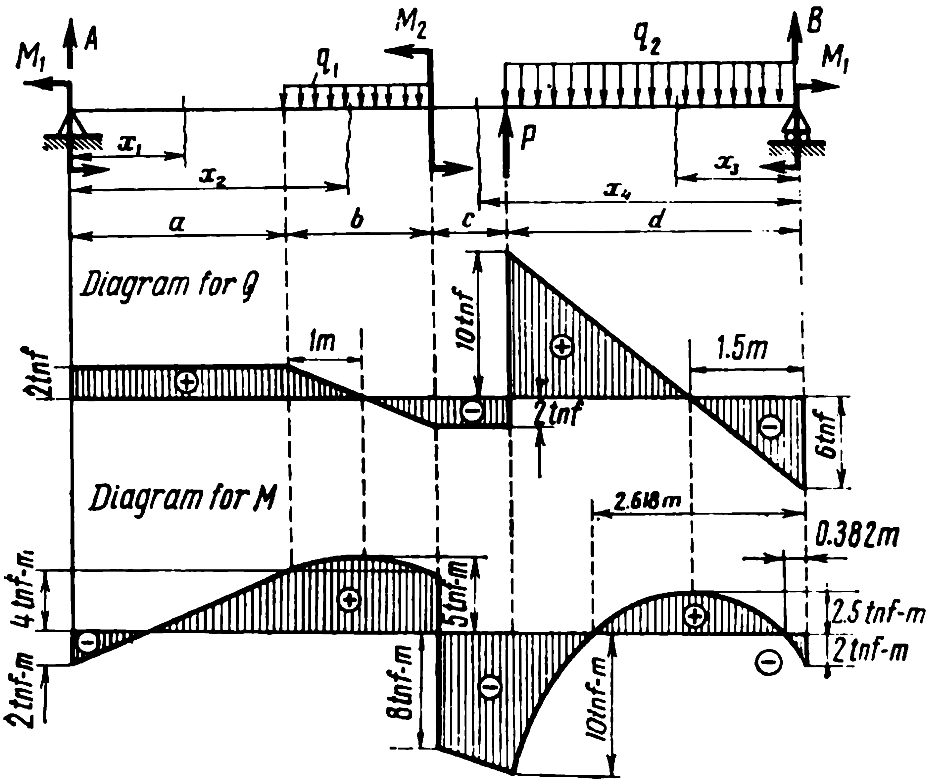


Fig. 54

Solution. Using the equations of statics we find the reactions A and B of the supports as the sums of moments about the right- and left-hand supports. Thus

$$A(a+b+c+d)-q_1b\left(\frac{b}{2}+c+d\right)-M_2+Pd-q_2d\frac{d}{2}=0;$$

$$A\times 10-2\times 2\times 6-12+12\times 4-4\times 4\times 2=0;\ A=2\ \text{tnf};$$

$$B(d+c+b+a)-q_2d\left(\frac{d}{2}+c+b+a\right)+P(c+b+a)$$
$$+M_2-q_1b\left(\frac{b}{2}+a\right)=0;$$

$$B\times 10-4\times 4\times 8+12\times 6+12-2\times 2\times 4=0;\ B=6\ \text{tnf}$$

To simplify the expressions determining Q and M, we consider the sections in the portions a and b from the left, and in the portions d and c, from the right. Then

$$0\leqslant x_1\leqslant a;$$

$$Q_{x_1}=A=2\ \text{tnf};\quad M_{x_1}=-M_1+Ax_1=-2+2x_1;$$

$$M_{x_1=0}=-2\ \text{tnf-m};\quad M_{x_1=a=3}=-2+2\times 3=4\ \text{tnf-m};$$

$$a\leqslant x_2\leqslant a+b;$$

$$Q_{x_2}=A-q_1(x_2-a)=2-2(x_2-3);\quad Q_{x_2=a=3}=2\ \text{tnf};$$

$$Q_{x_2=a+b=5}=2-2\times 2=-2\ \text{tnf};$$

$$M_{x_2}=-M_1+Ax_2-q_1\frac{(x_2-a)^2}{2}=-2+2x_2-(x_2-3)^2;$$

$$M_{x_2=a=3}=-2+6=4\ \text{tnf-m};$$

$$M_{x_2=a+b=5}=-2+2\times 5-2^2=4\ \text{tnf-m}$$

Since

$$Q_{x_2}=-2x_2+8=0\ \text{at}\ x_2=4\ \text{m, then}\ \max M_{x_2=4}=-2$$
$$+2\times 4-1=5\ \text{tnf-m}$$

$$0\leqslant x_3\leqslant d;$$

$$Q_{x_3}=-B+q_2x_3=-6+4x_3;\ Q_{x_3=0}=-6\ \text{tnf};\quad Q_{x_3=d=4}$$
$$=-6+4\times 4=10\ \text{tnf};$$

$$M_x=-M_1+Bx_3-q_2\frac{x_3^2}{2}=-2+6x_3-2x_3^2;$$

$$M_{x_3=0}=-2\ \text{tnf-m};\quad M_{x_3=d=4}=-2+6\times 4-2\times 4^2=-10\ \text{tnf-m}$$

Since

$$Q_{x_3}=-6+4x_3=0\quad\text{at}\quad x_3=\frac{3}{2}\ \text{m, then}$$

$$\max M_{x_3=\frac{3}{2}} = -2+6\times\frac{3}{2}-2\times\frac{9}{4}=2.5 \text{ tnf-m};\quad d\leqslant x_4\leqslant d+c;$$

$$Q_{x_4} = -B+q_2d-P = -6+4\times 4-12 = -2 \text{ tnf};$$

$$M_{x_4} = -M_1+Bx_4-q_2d\left(x_4-\frac{d}{2}\right)+P(x_4-d)$$

$$= -2+6x_4-16(x_4-2)+12(x_4-4);$$

$$M_{x_4=d=4} = -2+6\times 4-16\times 2 = -10 \text{ tnf-m};$$

$$M_{x_4=d+c=5} = -2+6\times 5-16\times 3+12\times 1 = -8 \text{ tnf-m}$$

Next we find in which sections $M_{x_3} = -2+6x_3-2x_3^2 = 0$;

$$x_3^2-3x_3+1=0;\quad x_3=\frac{3}{2}\pm\sqrt{\frac{9}{4}-1}=\frac{3}{2}\pm\frac{\sqrt{5}}{2}\cong 1.5\pm 1.118;$$

$$x_{3_1}=2.618 \text{ m} \quad\text{and}\quad x_{3_2}=0.382 \text{ m}$$

Diagrams for Q and M plotted from the above data are shown in Fig. 54.

Example 27. Given: q, l and $P = 0.2ql$ (Fig. 55).

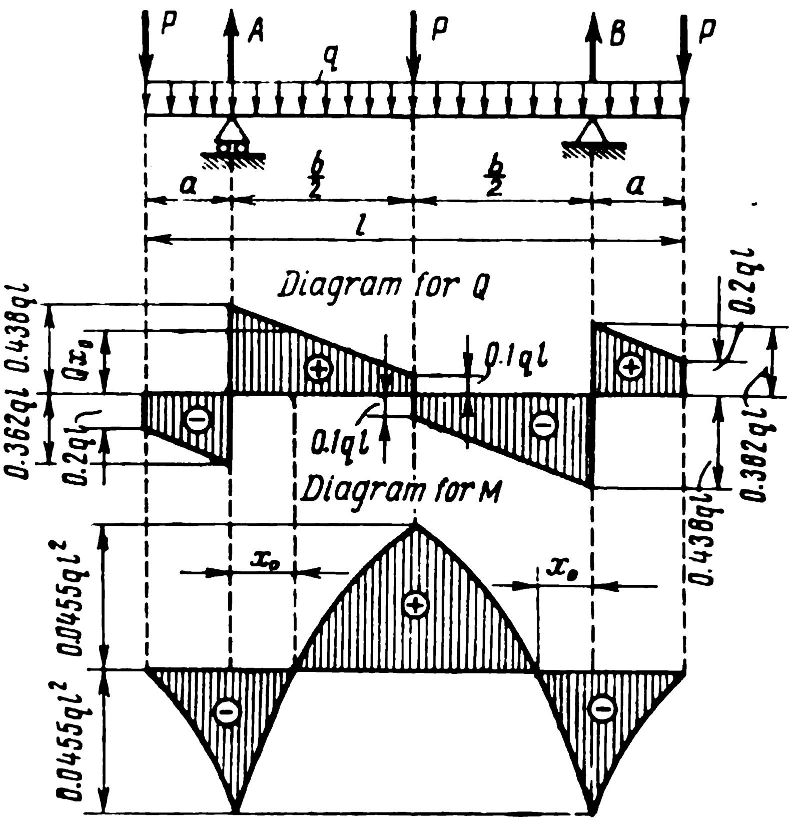


Fig. 55

Find the optimum length a of the cantilever portions and plot the diagrams for Q and M.

Solution. The optimum length of cantilever portions at the ends of a beam is one at which the maximum bending moment has its mini-

mum possible value. The optimum cantilever length is found by equating the absolute values of the bending moment in the section above the support (M_{sp}) and of the maximum bending moment within the span between the supports (M_{max}).

Since the beam is symmetrical with respect to the central section, M_{max} is in the middle of the beam within the span and the bending moments in the sections above the supports (M_{sp}) are equal to each other.

The condition for determining the optimum length of the cantilever portion is expressed as $|M_{sp}| = |M|_{max}$

First we find the moments. The support reactions are

$$A=B=\frac{3}{2}P+\frac{ql}{2};$$

$$M_{sp}=-Pa-\frac{qa^2}{2}; \quad M_{max}=-P\frac{l}{2}-\frac{ql^2}{8}+A\frac{l-2a}{2}$$

$$=-\frac{Pl}{2}-\frac{ql^2}{8}+\left(\frac{3}{2}P+\frac{ql}{2}\right)\frac{l-2a}{2}=\frac{Pl}{4}+\frac{ql^2}{8}-\frac{3Pa}{2}-\frac{qla}{2}$$

From the condition $|M_{sp}| = |M|_{max}$ it follows that

$$Pa+\frac{qa^2}{2}=\frac{Pl}{4}+\frac{ql^2}{8}-\frac{3Pa}{2}-\frac{qla}{2}$$

or

$$\frac{a^2}{l^2}+\left(\frac{5P}{ql}+1\right)\frac{a}{l}-\frac{1}{4}\left(\frac{2P}{ql}+1\right)=0$$

Whence

$$\frac{a}{l}=\frac{1}{2}\left[-\left(\frac{5P}{ql}+1\right)+\sqrt{\left(\frac{5P}{ql}+1\right)^2+\frac{2P}{ql}+1}\right]$$

The minus sign before the radical is dropped because the ratio $\frac{a}{l}$ cannot be a negative value.

Now we can plot the diagrams for Q and M.

Since $\frac{P}{ql} = 0.2$, then

$$a=\frac{\sqrt{5.4}-2}{2}l \cong 0.162\,l; \quad b=l-2a \cong l\,(1-0.324)=0.676\,l$$

$$\text{and } A=B=\frac{3}{2}0.2\,ql+\frac{ql}{2}=0.8\,ql$$

Since a uniformly distributed load acts over the full length of the beam, the shear (Q) diagram will be linear and the moment (M) diagram parabolic for all portions of the beam. Since the beam is symmetrical about a vertical line, the diagram for Q will be symmetrical about a point and for M will be symmetrical about a line.

At the left-hand free end of the beam $Q = -P = -0.2\,ql$.

In the extreme right-hand section of the left cantilever portion

$$Q = -P - qa \cong -(0.2 + 0.162)\, ql = -0.362\, ql$$

There is a vertical step over the support on the diagram for Q due to reaction A.

Therefore, at the left end of the span between the supports

$$Q = 0.362\, ql + 0.8\, ql = 0.438\, ql$$

Approaching the mid section of the beam from the left

$$Q = -0.2\, ql + 0.8\, ql - 0.5\, ql = 0.1\, ql$$

The shear (Q) diagram of Fig. 55 has been plotted from the above data.

At the left-hand free end of the beam the bending moment $M = 0$.

In the section over the support

$$M_{sp} \cong -0.2\, ql \times 0.162\, l - \frac{q}{2}(0.162\, l)^2 = -0.0455\, ql^2$$

In the middle of the beam

$$M_{\max} = |M_{sp}| = 0.0455\, ql^2$$

To find the distance x_0 at which $M = 0$ in the beam span, we proceed as follows.

From the similar triangles we can write

$$\frac{Q_{x_0} - 0.1\, ql}{0.438\, ql - 0.1\, ql} = \frac{\frac{b}{2} - x_0}{\frac{b}{2}}$$

or

$$\frac{Q_{x_0} - 0.1\, ql}{0.338\, ql} = \frac{0.338\, l - x_0}{0.338\, l}$$

Whence

$$Q_{x_0} = (0.438\, l - x_0)\, q$$

The area of the trapezoid of height x_0 on the shear diagram Q is equal to the change of bending moment M in passing from the section over the support to the section a distance x_0 from the support, i.e.

$$\frac{0.438\, ql - Qx_0}{2} \times x_0 = 0.0455\, ql^2$$

or

$$(0.438\, l + 0.438\, l - x_0)\, x_0 = 0.091\, l^2$$

Whence

$$x_0^2 - 0.876\, lx_0 + 0.091\, l^2 = 0$$

and

$$x_0 = l\,(0.438 \pm \sqrt{0.438^2 - 0.091}) = l\,(0.438 \pm \sqrt{0.101})$$

Since x_0 cannot be greater than $\frac{b}{2} = 0.338l$, only one root can satisfy the conditions of the problem, viz.

$$x_0 = l\,(0.438 - \sqrt{0.101}) \cong 0.122l$$

The moment (M) diagram of Fig. 55 has been plotted from these data.

Problem 28. Given: P and a (Fig. 56*a*).

Plot the diagrams for Q and M.

Solution. Since the bending moment in the section B passing through the unsupported hinge joint is zero, the beam can be dealt with as two beams (Fig. 56*b*), of which the left-hand one AB is a simple freely supported beam and the right-hand one BC, a cantilever beam. The right-hand end B of the simple beam rests upon the free left-hand end B of the cantilever beam. These two beams can be dealt with separately (Fig. 56*c*). In the simple beam the support reactions are $A = B = P$.

The effect of the simple beam on the cantilever beam is the action of the downward concentrated force $B = P$ on the left-hand end of the latter.

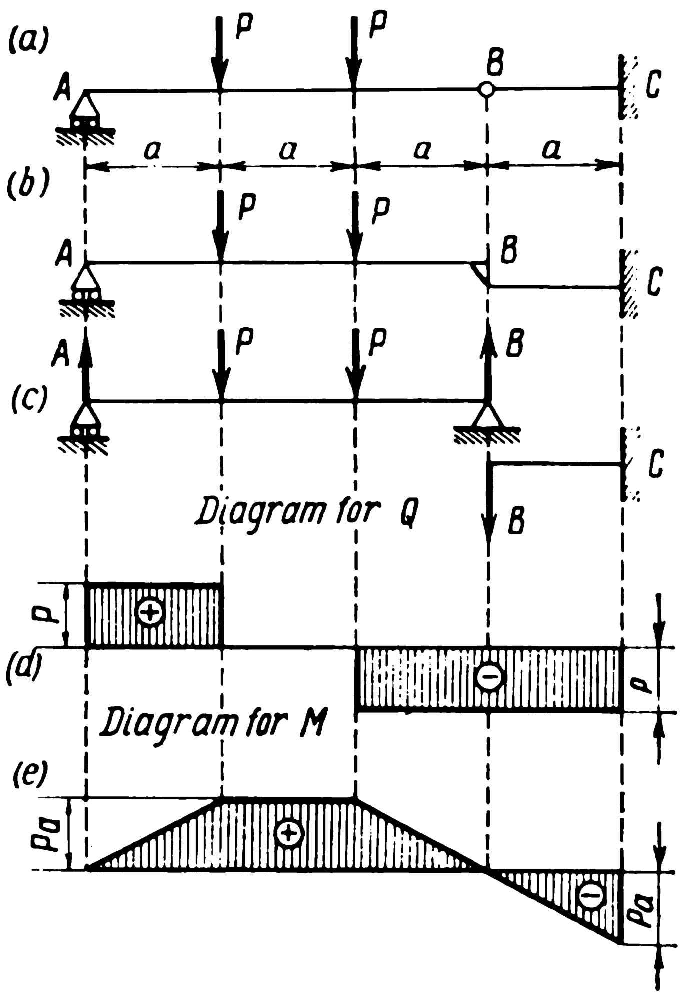


Fig. 56

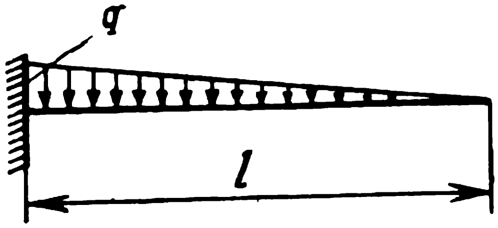
The problem is solved separately for each of the beams.

In the left-hand portion of the simple beam $Q = A = P$. In the middle portion $Q = 0$ and in the right-hand portion $Q = -P$. In the cantilever beam $Q = -P$. The shear (Q) diagram is shown in Fig. 56*d*. In the portions of the beam where $Q =$ const, the bending moment varies linearly, and in the portion in which $Q = 0$, $M =$ const. The variations of M over the portions can easily be found from the areas of the shear diagram Q. Fig. 56*e* illustrates the bending moment diagram.

Problems 222 through 270. Plot the diagrams for the transverse (shearing) force Q and the bending moment M.

In Problems 251, 253, 255, 256 and 260 plot the diagrams for Q and M with the optimum lengths a of the cantilever portions.

222



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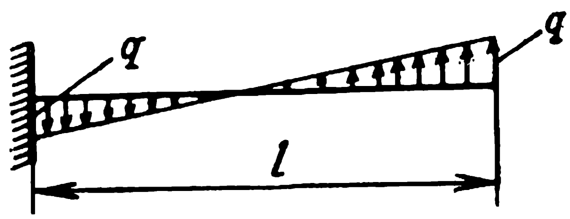

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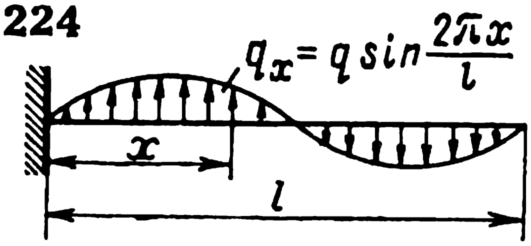

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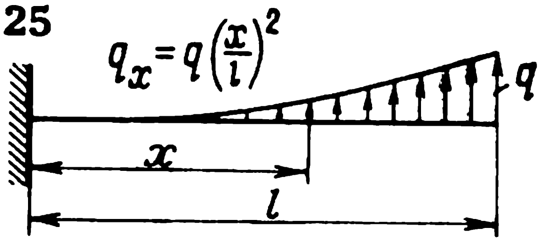

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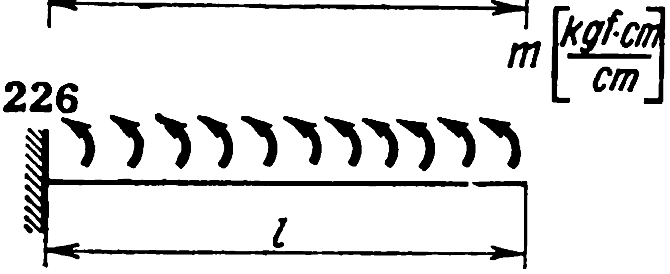

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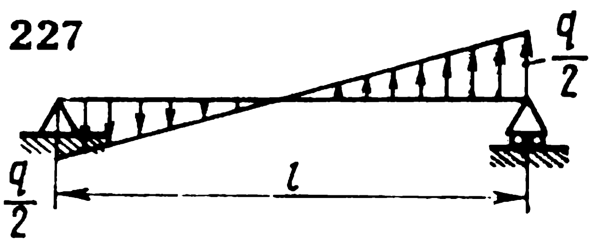

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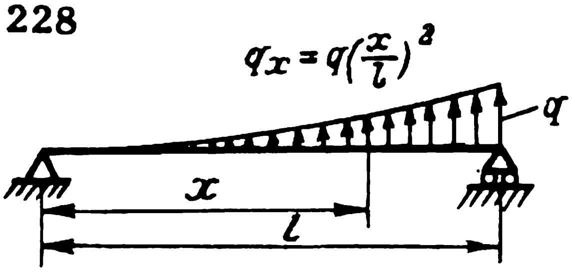

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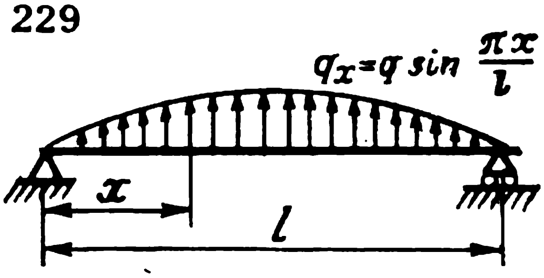

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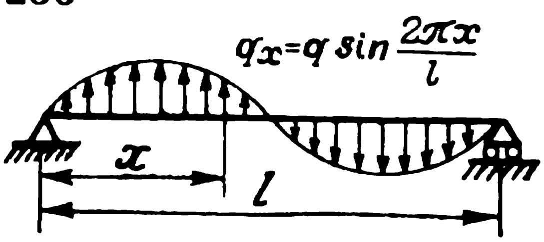

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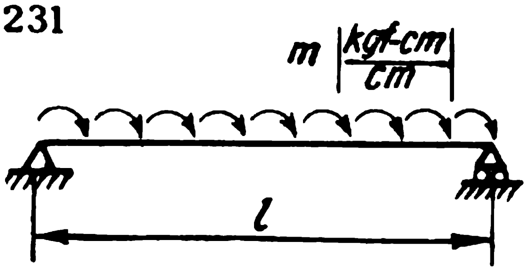

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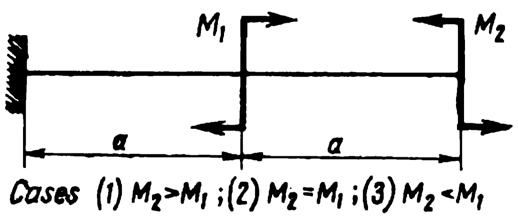

Cases (1) $M_2 > M_1$; (2) $M_2 = M_1$; (3) $M_2 < M_1$

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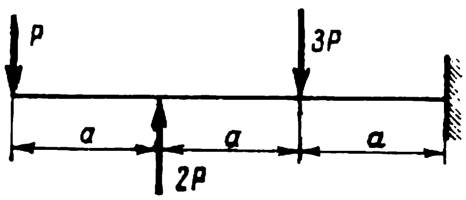

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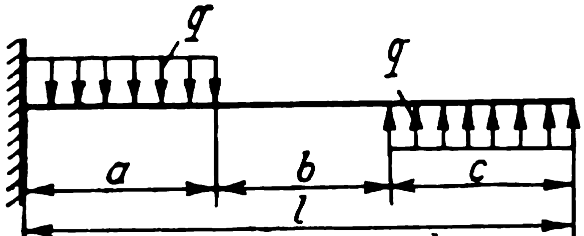

(1) $a=b=c=\frac{l}{3}$, (2) $a=c=\frac{l}{2}$, $b=0$

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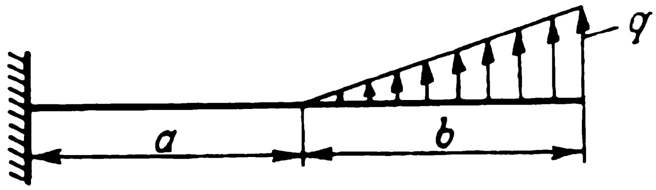

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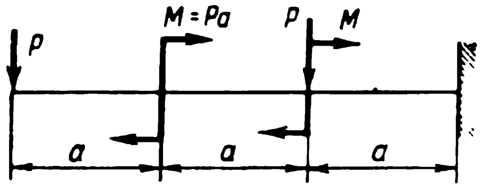

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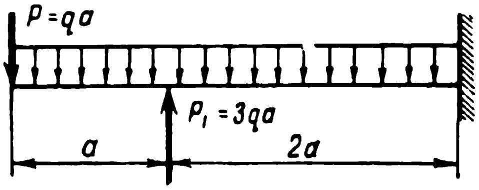

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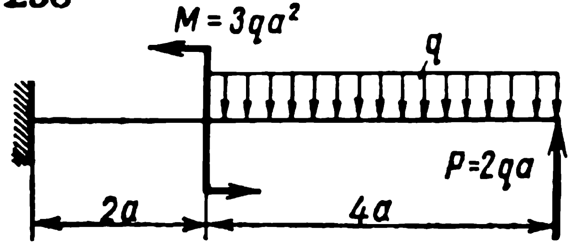

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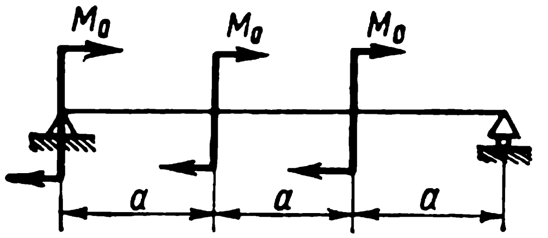

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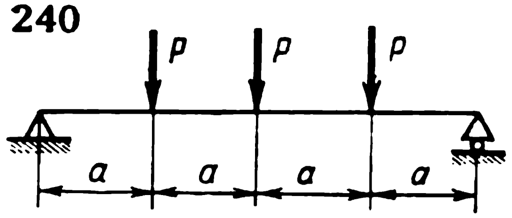

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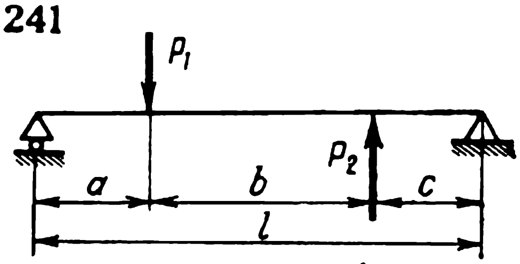

(1) $P_1=P_2=P$, $a=b=c=\frac{l}{3}$
(2) $P_1=2P_2$, $a=c$, $b=2a$

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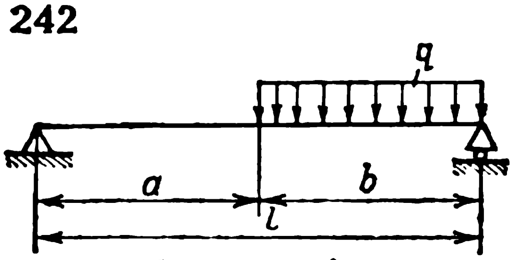
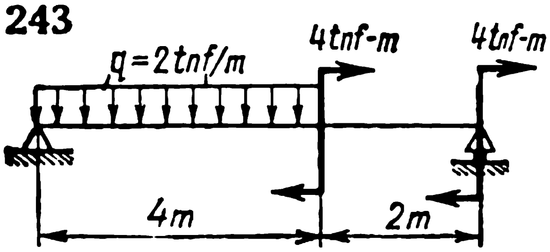


(1) $a=b=\frac{l}{2}$; (2) $a=\frac{l}{4}$; $b=\frac{3}{4}l$;
(3) $a=\frac{3}{4}l$, $b=\frac{l}{4}$

243



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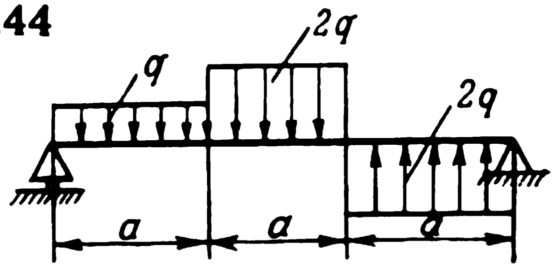

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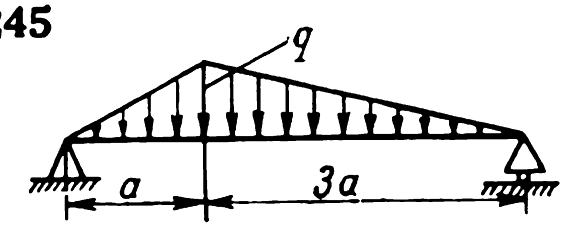

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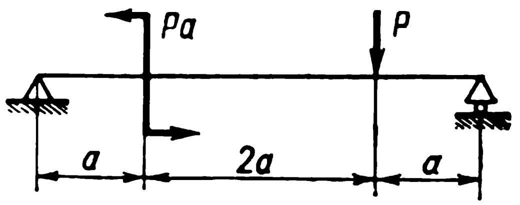

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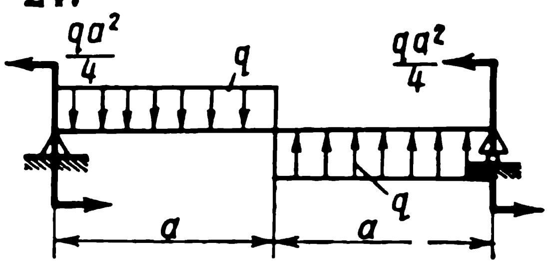

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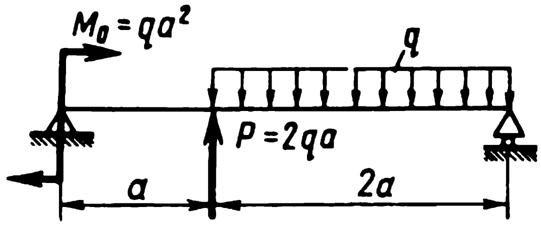

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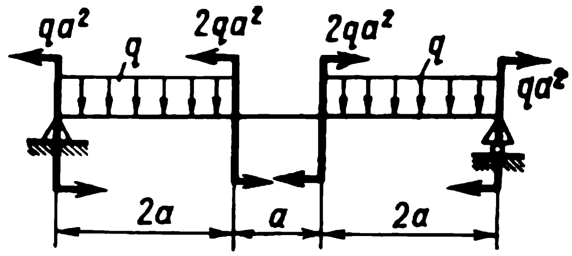

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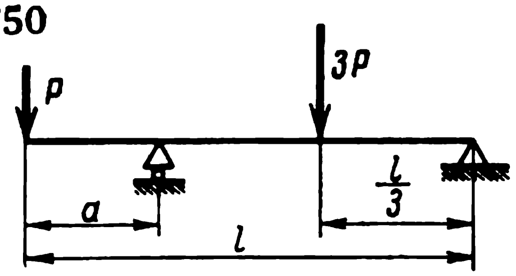

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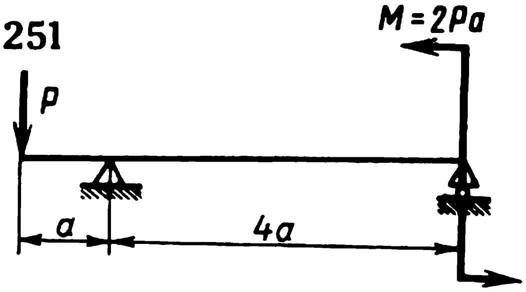

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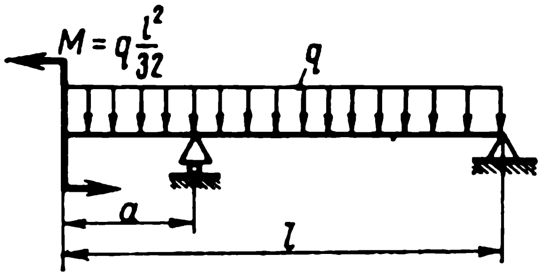

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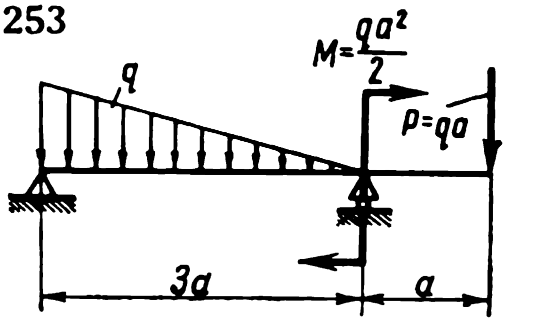

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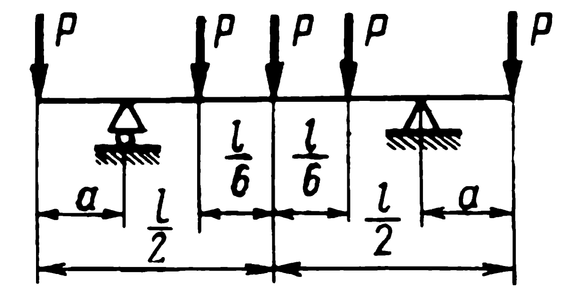

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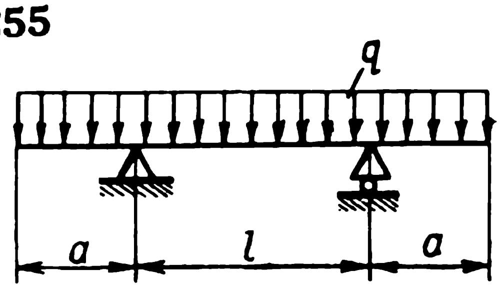

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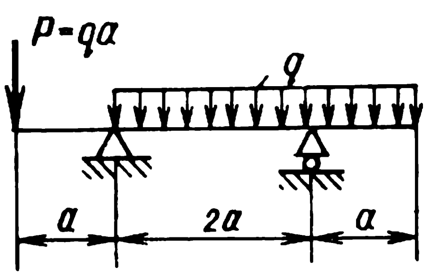

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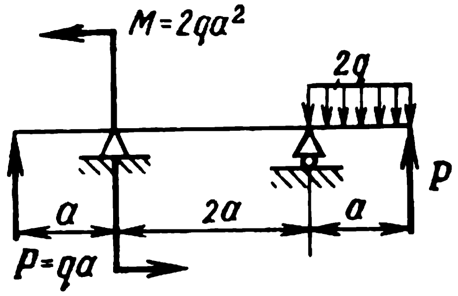

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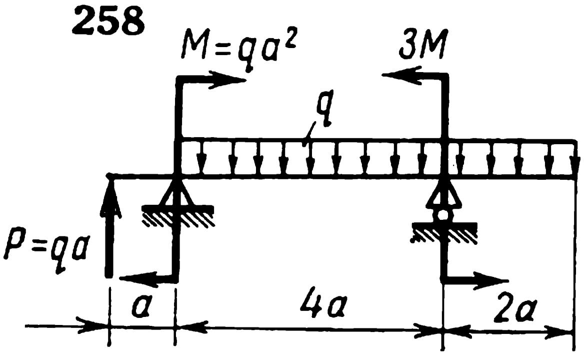

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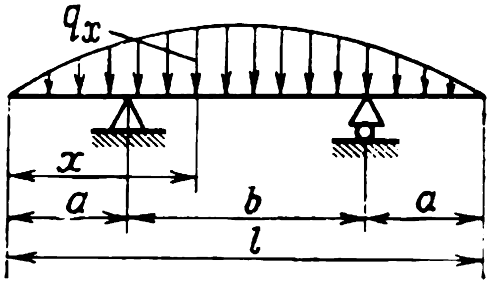

$$q_x=\frac{4qx}{l}\left(1-\frac{x}{l}\right)$$

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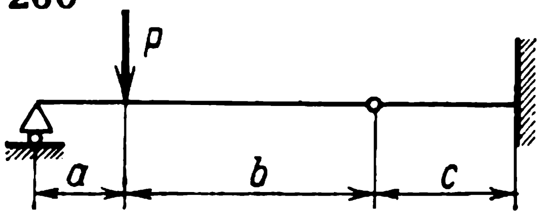

$b>c$

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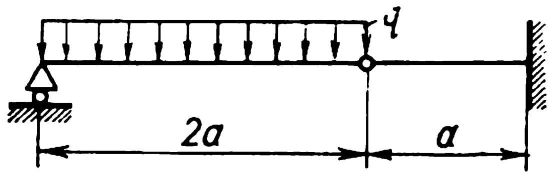

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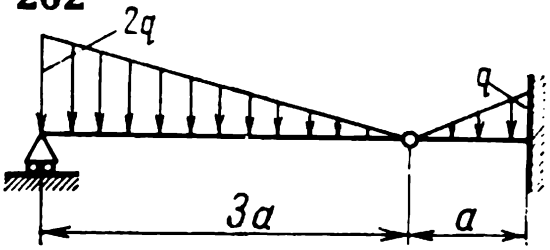

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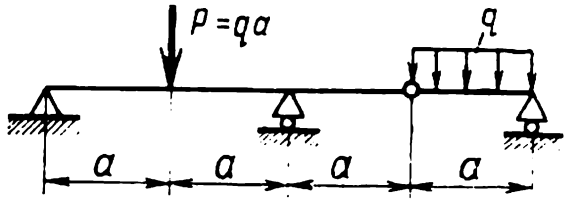

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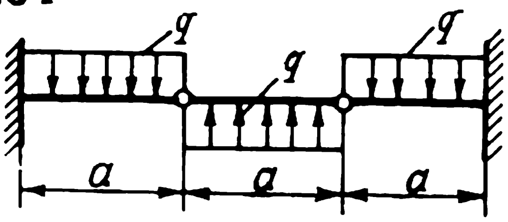

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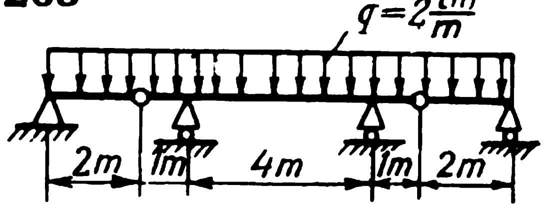

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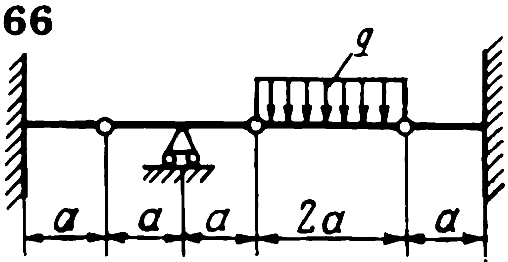

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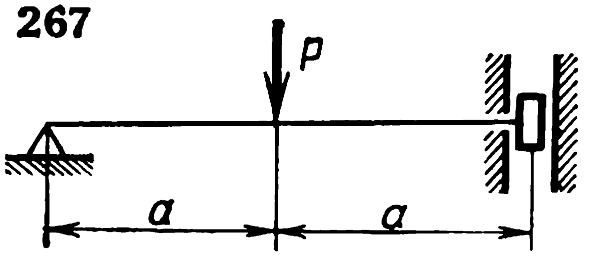

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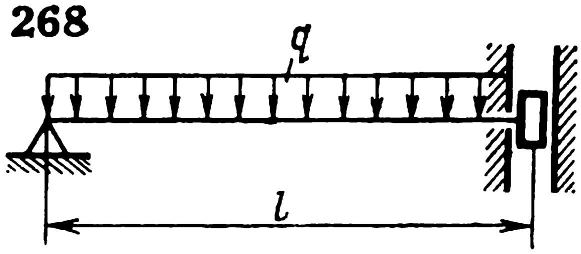

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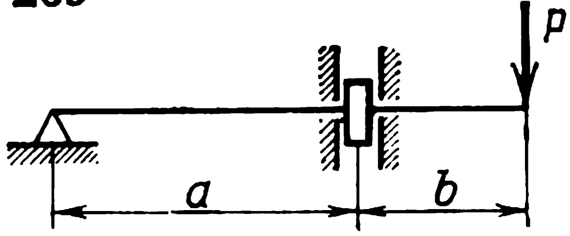

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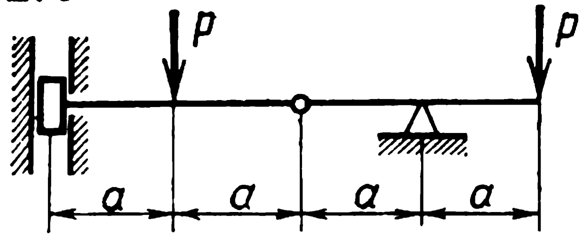


In Problem 252 plot the diagram for M without determining the reactions of the supports and shearing forces Q.

GRAPHICAL METHOD OF CONSTRUCTING SHEAR AND MOMENT (Q AND M) DIAGRAMS

Diagrams of the bending moments and transverse (shearing) forces can also be constructed by a purely graphical method. This method is especially convenient when a complex system of external forces is applied to a beam.

If the construction is carefully drawn and the scale is properly selected the accuracy of the results obtained by the use of the graphical method is adequate for all practical purposes. This method of plotting the diagrams for M and Q is based on the properties of a parallel force polygon and a funicular (string) polygon.

In graphically determining the bending moment M_x and shearing force Q_x in any arbitrary section of the beam, it is necessary to take into account the length scale $\frac{1}{\xi}$ to which the beam has been drawn (ξ cm of beam length corresponds to 1 cm on the drawing) and the force scale $\frac{1}{\eta}$ to which the force polygon has been drawn (η kgf corresponds to 1 cm of the vector in the force polygon).

Since the pole distance H on the force polygon is drawn to the scale of the forces, whereas the vertical length y between the funicular polygon and its closing string in the beam section under consideration is drawn to the scale of the lengths, the true values of M_x and Q_x are

$$M_x = y\,(\xi)\,H\,(\eta)\ \text{kgf-cm},\qquad Q_x = \overline{Q}_x\,(\eta)\ \text{kgf}$$

in which Q_x = vector in the force polygon, corresponding to the algebraic sum of vectors of the external forces to one side of the beam section under consideration.

The closing string (strings) of the funicular polygon is (are) drawn in accordance with the following rules which correspond to the method of fixing the beam.

(1) The closing string at the free end of the beam is tangent to the funicular polygon.

(2) At a hinged support of the beam end and at an unsupported hinge joint the closing string intersects the funicular polygon.

(3) At the hinged support in the span of the beam the closing string is a broken line (the closing strings intersect at one point in approaching the section from the right and from the left).

If a funicular polygon is drawn for a given load and a closing string (or strings) is (are) drawn according to the method of supporting the beam we will obtain the bending moment diagram.

The scale for this diagram will be a quantity indicating that 1 cm of vertical length of the diagram corresponds to H ($\xi\eta$) kgf-cm of bending moment.

Before constructing the shear diagram we must determine the vectors that correspond to the reactions of the supports. On the force polygon these vectors are intercepted by lines parallel to the closing string (strings) of the funicular polygon drawn from the pole. The shear diagram is constructed by transferring the vectors of the force polygon to the respective points of lines of zero values of Q, which are parallel to the geometric axis of the beam. The scale for this diagram is $\frac{1}{\eta}$ which indicates that 1 cm of vertical length of the diagram corresponds to η kgf of the transverse (shearing) force.

If a beam is subject to a distributed load the latter is divided into portions by lines perpendicular to the geometric axis of the beam. The area of each of the portions is represented by a vector applied at its centre of gravity. Both the force polygon and the funicular polygon are constructed with the aid of these vectors using them as vectors of concentrated forces. The polygonal moment diagram (M) thus obtained is made more accurate by drawing a curve inscribed into the polygon; the stepped shear (Q) diagram is made more accurate by drawing a curve or a straight line (depending on the load distribution) passing through points of the horizontal lines of the stepped diagram, these points being opposite the beginning and end of each portion of the area of the distributed load.

If a beam is subject to concentrated couples of forces, the moments M of these forces are plotted on the bending moment diagram at the respective sections in the form of vertical intercepts of length $y = \frac{M}{H\,(\xi\eta)}$.

For the purpose of unification with the analytical method, in const-

ructing diagrams for M and Q graphically, vertical distances above the closing string of the funicular polygon for M and above the

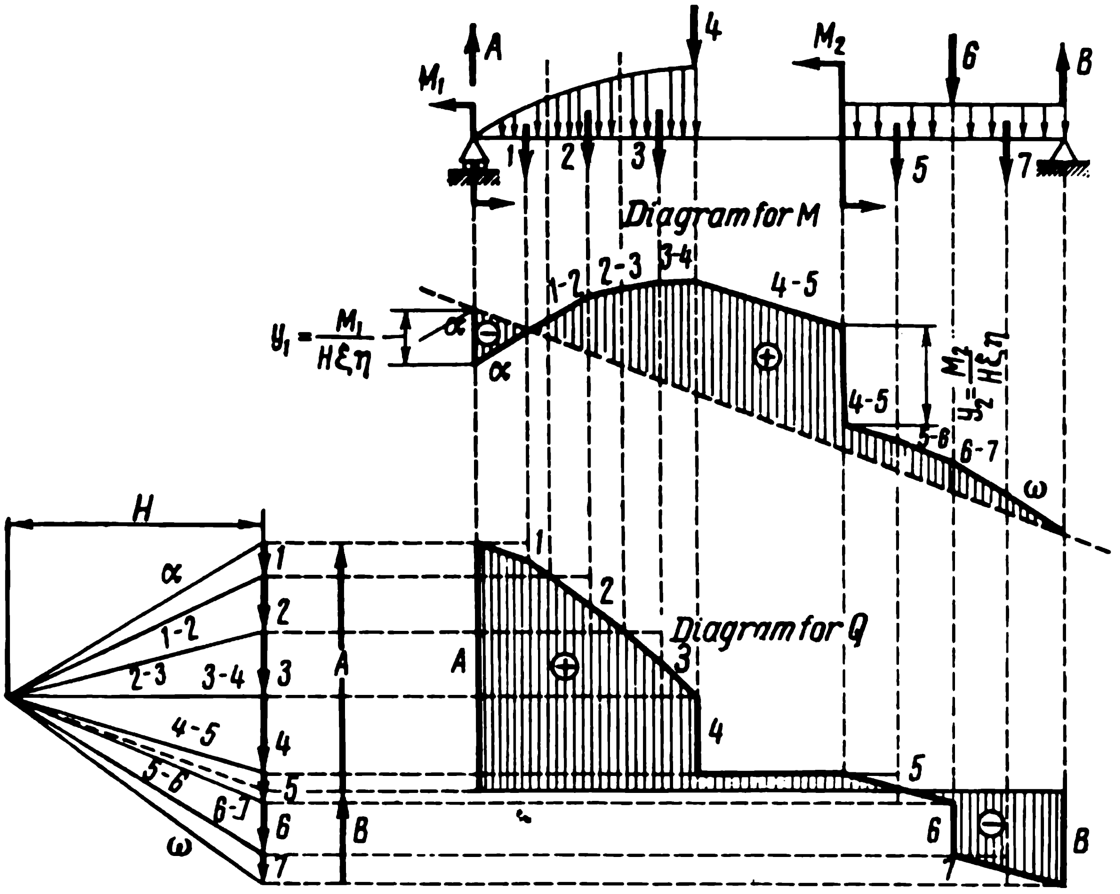


Fig. 57

parallel to the geometric axis of the beam, which is assumed to be the line of zero values of Q, shall be taken as positive values.

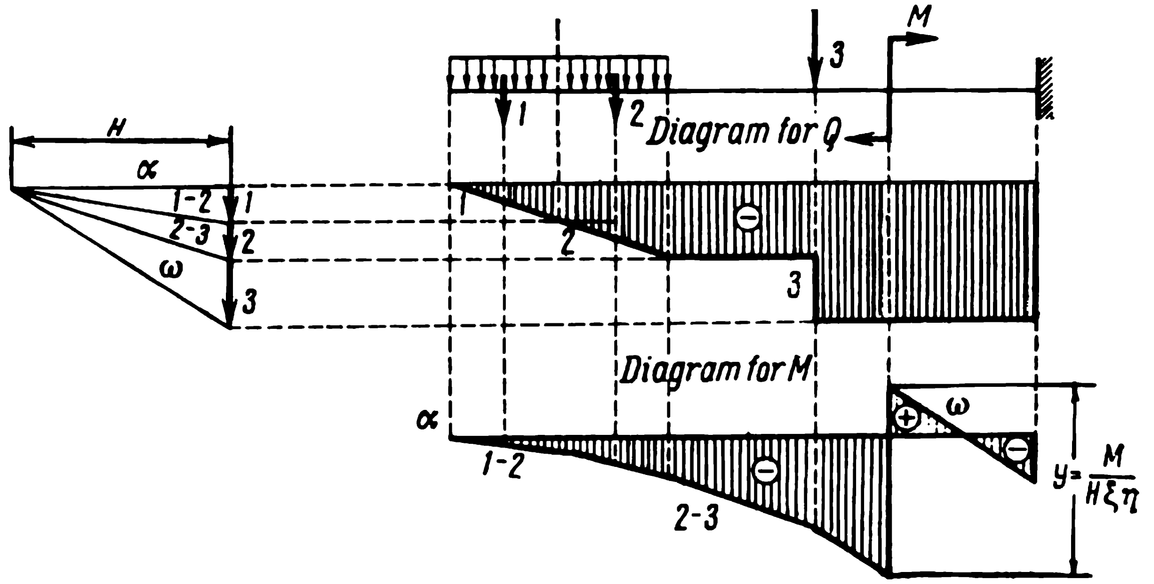


Fig. 58

If we approach the beam section under consideration from the left and plot load vectors on the force polygon in this same order, then

to obtain the accepted signs for M and Q the pole is to be located to the left of the force polygon.

Examples 29, 30, 31, 32 and 33 explain the graphical method of constructing diagrams for M and Q for beams which are fixed and loaded in various ways.

Example 29 (Fig. 57).

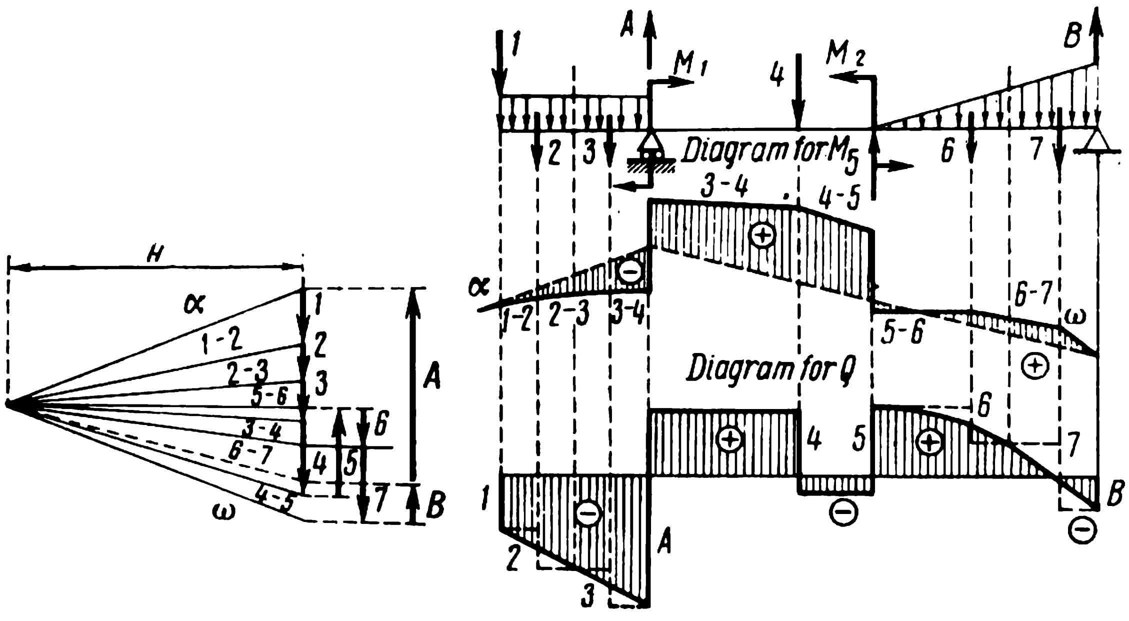


Fig. 59

Example 30 (Fig. 58).
Example 31 (Fig. 59).
Example 32 (Fig. 60).

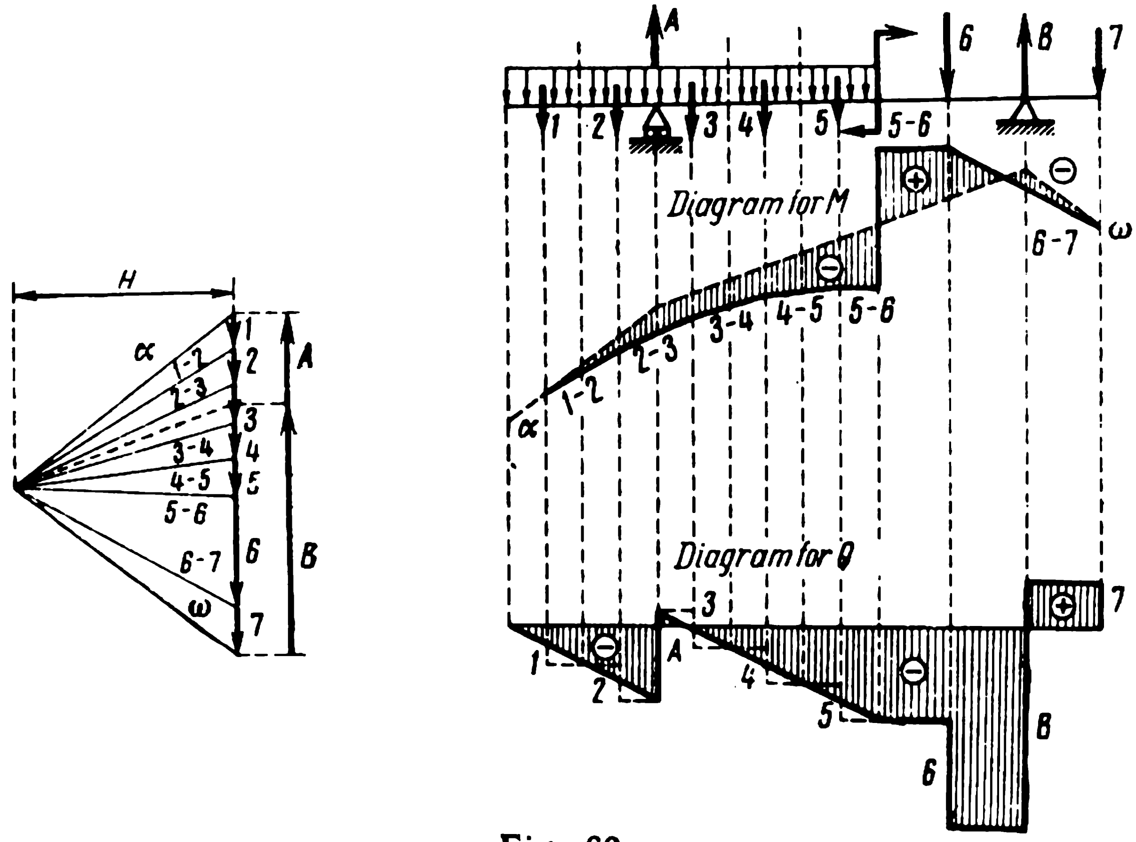


Fig. 60

Example 33 (Fig. 61).

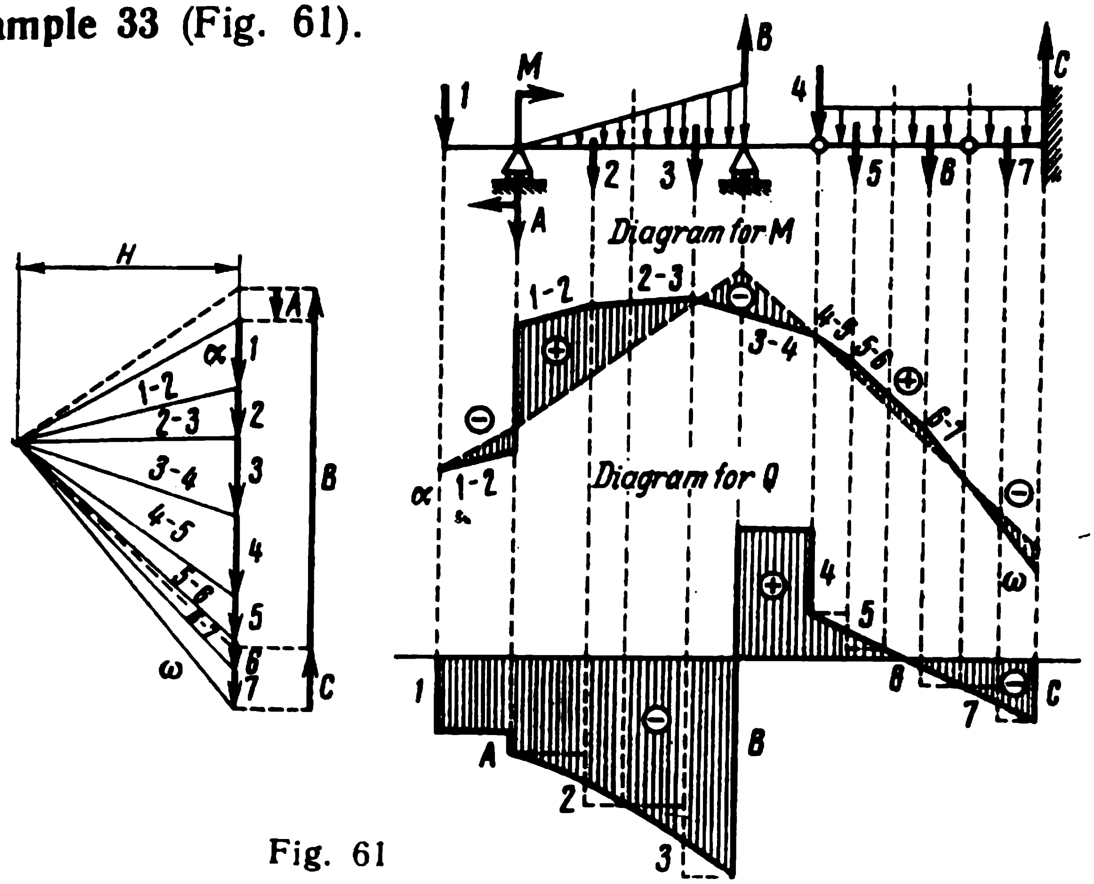
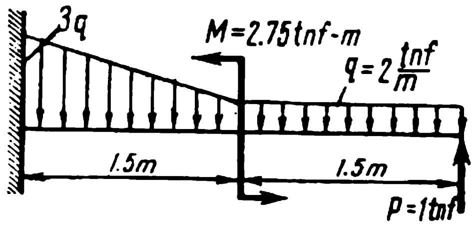


Fig. 61

Problems 271 through 275. Using the graphical method construct the diagrams for transverse (shearing) force Q and bending moment M.

271



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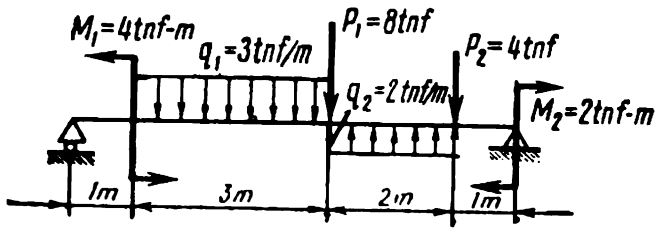

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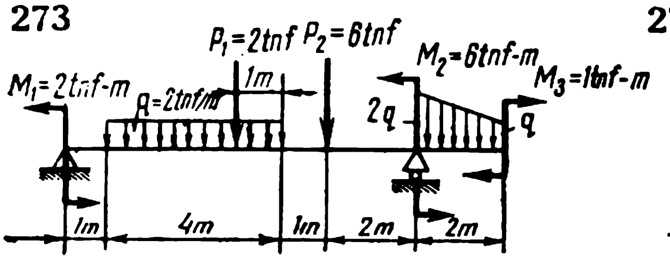

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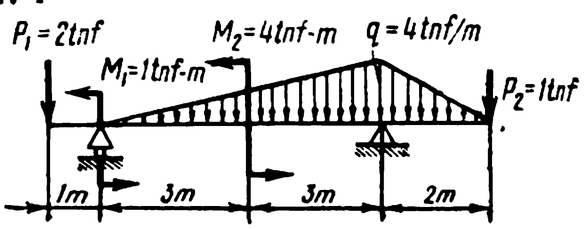

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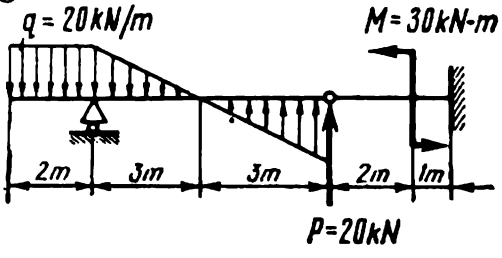

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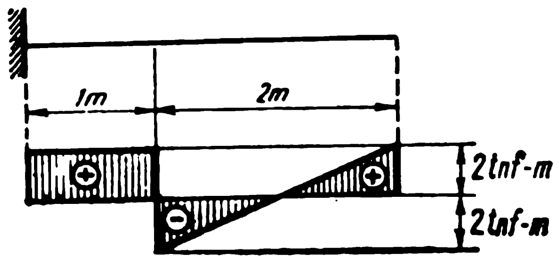

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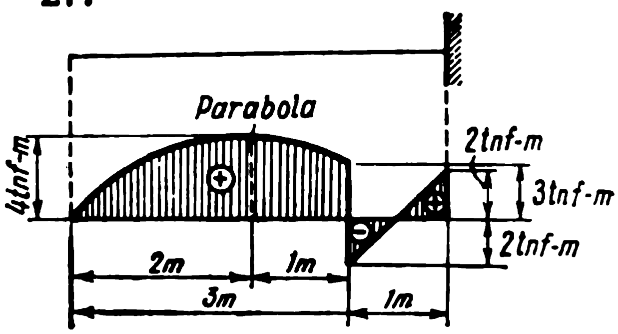

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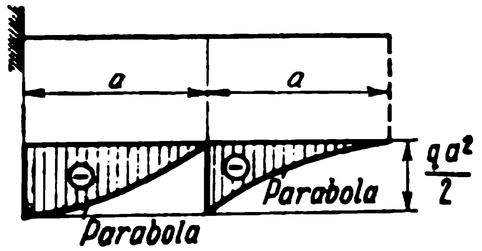

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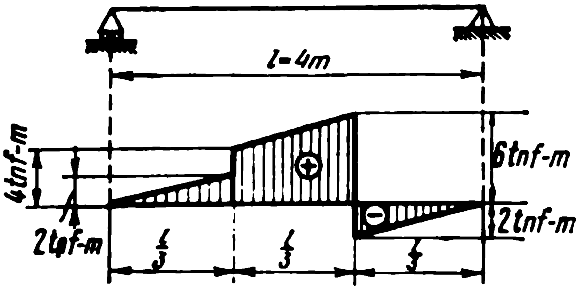

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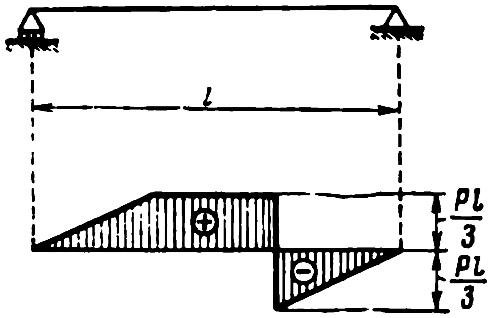

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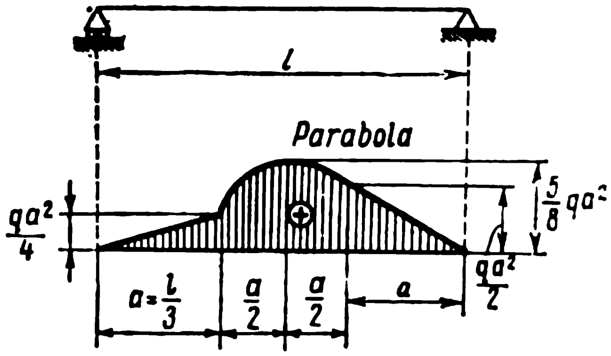

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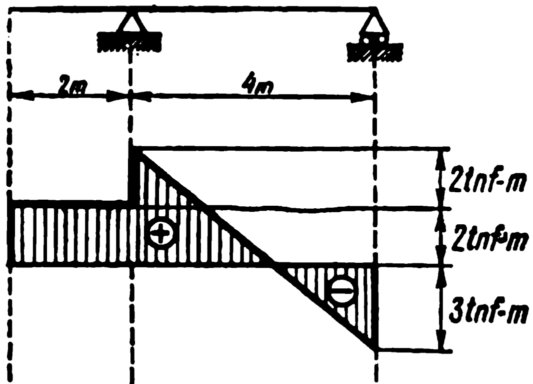

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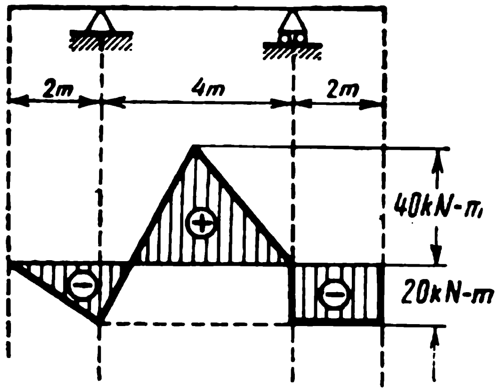

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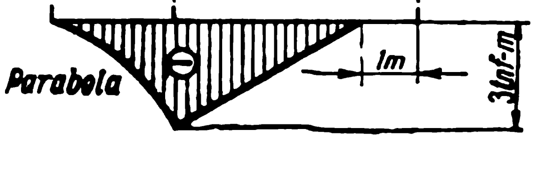

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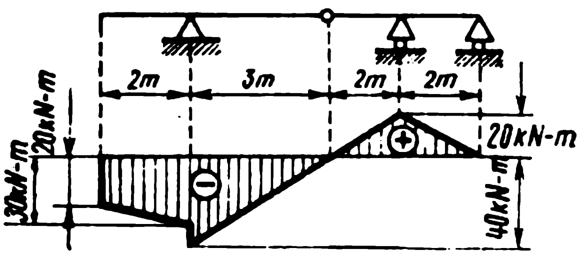

Problems 276 through 285. Construct the shear (Q) diagrams and determine the loads acting on the beams from the given bending moment (M) diagrams (see page 117).

8.2.

Normal Stresses and Selection of the Cross Section of Beams

The normal stresses at an arbitrary point of a cross section of a beam in bending are found from the formula

$$\sigma = \frac{My}{I} \tag{92}$$

in which M = bending moment in the cross section under consideration

I = moment of inertia of the area of this section about the neutral axis

y = distance from the point under consideration in the cross section to the neutral axis.

The maximum tensile and compressive stresses in the given cross section of a beam occur at points most remote from the neutral axis. They are found by the use of the formulas

$$\sigma_{max} = \frac{My_1}{I} = \frac{M}{W_1}; \tag{93}$$

$$\sigma_{min} = -\frac{My_2}{I} = -\frac{M}{W_2} \tag{94}$$

in which y_1 and y_2 = distances from the neutral axis to the most remote stretched and compressed fibres

$W_1 = \frac{I_1}{y_1}$ and $W_2 = \frac{I_2}{y_2}$ = equatorial or axial section moduli of the beam cross section (or section moduli in bending) for the stretched and compressed fibres, respectively.

If in the cross section

$$y_1 = y_2 = \frac{h}{2}$$

in which h = height of the section (for example, for sections symmetrical with respect to the neutral axis or sections whose centre of gravity is at the midpoint of the height, for instance the cross section of a rail),

then

$$W_1 = W_2 = W = \frac{2I}{h} \quad \text{and} \quad \sigma_{max} = -\sigma_{min} = \frac{M}{W}$$

The required dimensions of the cross section of a beam subject to bending are selected on the basis of the normal stresses developed at points most remote from the neutral axis.

For beams of materials having equal strength in tension and compression, i.e. when $[\sigma_t] = [\sigma_c]$, the design formula for the required cross section of a beam subject to bending can be written in the following form:

$$W = \frac{M_{max}}{[\sigma_b]} \tag{95}$$

where W = minimum section modulus of the cross section of the beam about the neutral axis
M_{max} = maximum absolute value of the bending moment
$[\sigma_b]$ = permissible stress of the material of the beam in bending.

Any deviation from equation (95) must not exceed ± 5 per cent. In selecting the cross sections of rolled beams greater deviations towards an increased safety factor are permissible. For beams made of materials having different strengths in tension and compression, two conditions must be met:

$$W_1 = \frac{M_{max}}{[\sigma_t]}; \tag{96}$$

$$W_2 = \frac{M_{max}}{[\sigma_c]} \tag{97}$$

The condition of equal strength of the beam material in the extreme fibres at the dangerous section requires that the cross section of a beam made of a material having equal strengths in tension and compression should be symmetrical with respect to the neutral axis, and the cross section of a beam made of a material having unequal strengths in tension and compression should be asymmetrical with respect to the neutral axis. In the latter case attempts should be made to comply with the following proportion:

$$\frac{[\sigma_c]}{[\sigma_t]} = \frac{W_1}{W_2} = \frac{y_2}{y_1} \tag{98}$$

Along with the strength condition, the beam should also satisfy the condition of the most economical cross section.

Since the strength of the cross section of a beam in bending is determined by the value of its section modulus W and the beam weight is proportional to the area of its cross section F, the efficiency of a beam cross section can be evaluated by the ratio $\xi = \frac{W}{F^{3/2}}$ which is called the *specific section modulus per unit area*.

The greater this ratio with equal areas, the more economical is the cross section.

Example 34. Let $q = 11$ kgf/cm; $P = 1$ tnf; $l = 4$ m; $c = 1$ m and $[\sigma_b] = 1600$ kgf/cm^2 (Fig. 62).

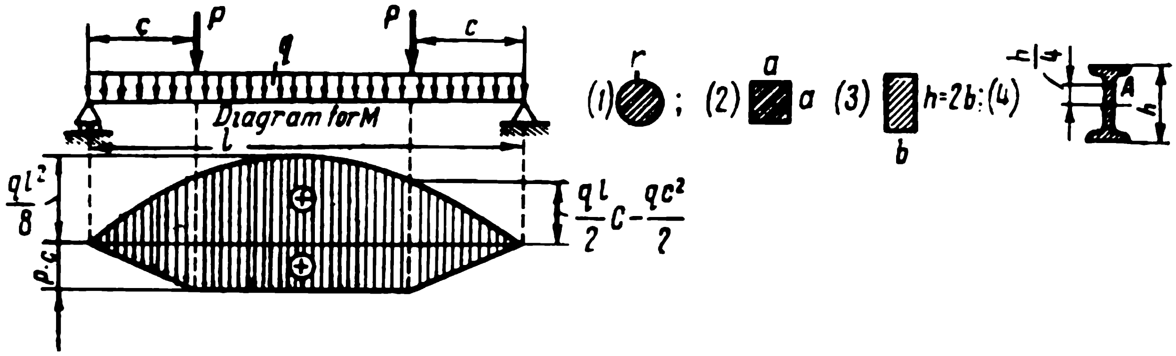

Fig. 62

Find the required dimensions of rolled bars of round, square, rectangular and I-sections; the ratios of the weights of beams of these sections; and the normal stress at point A of the section directly under one of the forces for the I-beam.

Solution. Since the beam is symmetrical with respect to the middle cross section, the maximum bending moment will be at this section.

The moment (M) diagram due to the distributed load is parabolic with $M_{\max q} = \frac{ql^2}{8}$ and that due to the concentrated forces is trapezoidal with $M_{\max p} = Pc$.

Therefore

$$M_{\max} = \frac{ql^2}{8} + Pc = \frac{1.1 \times 4^2}{8} + 1 \times 1 = 3.2 \text{ tnf-m}$$

From the design formula (95), the required section modulus of the cross section should be

$$W = \frac{M_{\max}}{[\sigma_b]} = \frac{3.2 \times 10^5}{16 \times 10^2} = 200 \text{ cm}^3$$

1. For a round section

$$W_1 = \frac{\pi d^3}{32} = 200 \text{ cm}^3; \quad d = \sqrt[3]{\frac{6400}{\pi}} = 12.68 \text{ cm};$$

$$F = \frac{\pi d^2}{4} = 126.8 \text{ cm}^2$$

2. For a square section

$$W_2 = \frac{a^3}{6} = 200 \text{ cm}^3; \quad a = \sqrt[3]{200 \times 6} = 10.63 \text{ cm};$$

$$F = a^2 \cong 113 \text{ cm}^2$$

3. For a rectilinear section

$$W_3=\frac{bh^2}{6}=\frac{h^3}{12}=200\ \text{cm}^3;\quad h=\sqrt[3]{2400}=13.39\ \text{cm};$$

$$F=bh=\frac{h^2}{2}=89.6\ \text{cm}^2$$

4. Data on I-beams are taken from tables of rolled steel shapes listed in handbooks. Thus for I-beam No. 20, W = 184 cm³ and for No. 20ª, W = 203 cm³.

First we check I-beam No. 20:

$$\frac{\sigma_{\max}-[\sigma_b]}{[\sigma_b]}100=\frac{\frac{M_{\max}}{W_{\text{No. }20}}-\frac{M_{\max}}{W}}{\frac{M_{\max}}{W}}100=\frac{W-W_{\text{No. }20}}{W_{\text{No. }20}}100$$

$$=\frac{200-184}{184}100=8.7\%\ \text{overstress}$$

Since the overstress exceeds 5 per cent, I-beam No. 20 cannot be used.

Next we check beam No. 20ª

$$\frac{\sigma_{\max}-[\sigma_b]}{[\sigma_b]}100=\frac{W-W_{\text{No.}20^{\text{a}}}}{W_{\text{No. }20^{\text{a}}}}=\frac{200-203}{203}100$$

$$=-1.5\%\ \text{understress}$$

Hence, I-beam No. 20ª proves suitable. For this beam the cross-sectional area F_4 = 28.9 cm², the moment of inertia with respect to the neutral axis I = 2030 cm⁴ and the height h = 20 cm.

Since the weight of the beam is proportional to the cross-sectional area, the ratios of the beam weights are equal to the ratios of their cross-sectional areas.

Taking the round cross-sectional area as unity we obtain $F_1 : F_2 : F_3 : F_4 \cong 1 : 0.89 : 0.71 : 0.23$.

Thus, even with an excessive cross-sectional area (1.5 per cent understress), the I-beam is approximately 77 per cent lighter than the round beam.

Next we determine the bending moment in the cross section of the I-beam directly under one of the forces:

$$M=\frac{ql}{2}c-\frac{qc^2}{2}+Pc=\frac{1.1\times4\times1}{2}-\frac{1.1\times1}{2}+1\times1=\frac{5.3}{2}\ \text{tnf-m}$$

At point A of this cross section for which $y=\frac{h}{4}=5$ cm, the normal stress is a compressive one (the beam bends downward) and can be found by the use of formula (92)

$$\sigma_A=\frac{My}{I}=\frac{-5.3\times10^5\times5}{2\times2030}\cong-653\ \text{kgf/cm}^2$$

Example 35. A U-beam (Fig. 63) is made of a material with $[\sigma_c] = 3\,[\sigma_t]$. The known dimensions are: $b = 20$ cm and $t = 1$ cm.

Find the rational height h of the cross section.

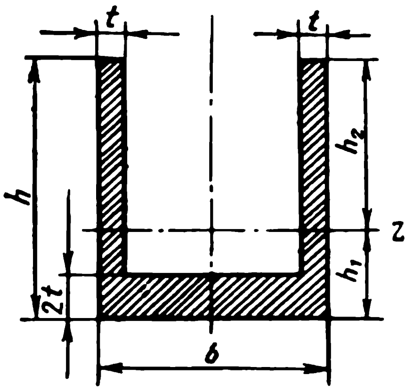

Fig. 63

Solution. Assume z to be the neutral axis with its distances from the extreme fibres being h_1 and h_2.

From the condition of the most rational use of the material (98) (condition of equal strength of the extreme fibres) it follows that

$$\frac{h_1}{h_2}=\frac{[\sigma_t]}{[\sigma_c]}=\frac{1}{3}$$

Since

$$h_1+h_2=h; \quad \text{then } h_1=\frac{h}{4} \text{ and } h_2=\frac{3}{4}h$$

The neutral axis z is a centroidal axis and therefore the statical moment of the cross-sectional area with respect to this axis equals zero, i.e.

$$S_z=-(b-2t)\,2t\left(\frac{h}{4}-t\right)+2ht\left(\frac{h}{2}-\frac{h}{4}\right)$$

$$=-36\left(\frac{h}{4}-1\right)+\frac{h^2}{2}=0$$

Whence

$$h^2-18h+72=0 \quad \text{and} \quad h=9\pm\sqrt{9}=9\pm 3$$

Hence the height of the cross section should be either $h = 12$ cm or $h = 6$ cm.

Problems 286 through 306. Find the axial section moduli W_z (z = horizontal centroidal axis) of the cross sections of the beams. The dimensions for Problem 306 are to be taken from Problem 301.

Problems 307 through 313. Find the normal stresses at points of cross sections of beams (the points and cross sections are indicated in the problems).

Notation: $\sigma_{\substack{max\\min}}$ = maximum and minimum normal stress in the dangerous section

max σ_A = normal stress at point A of the dangerous section

σA_{mn} = normal stress at point A of section mn

γ = weight per unit volume of the beam material

Problems 314 through 329. Determine the required dimensions of cross sections of beams.

In Problem 319 find the required number of parallel rectangular bars of the given dimensions.

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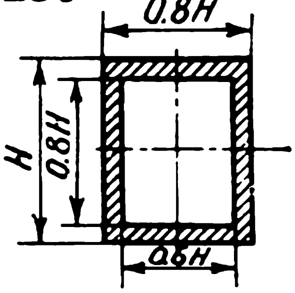

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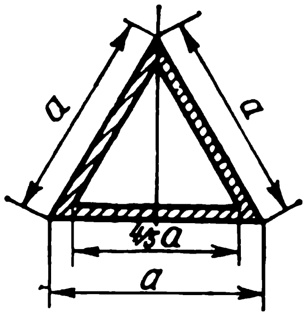

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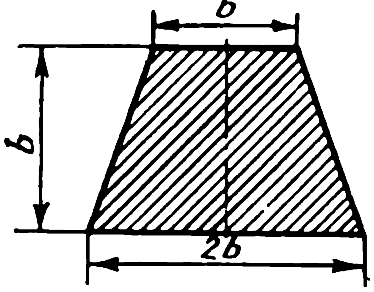

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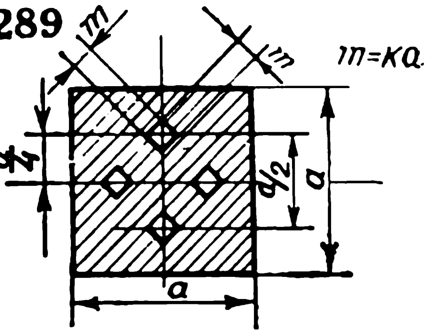

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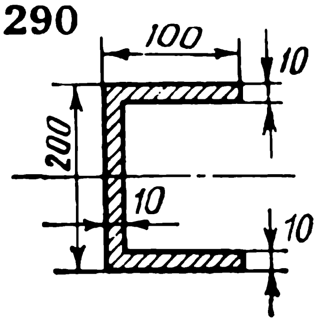

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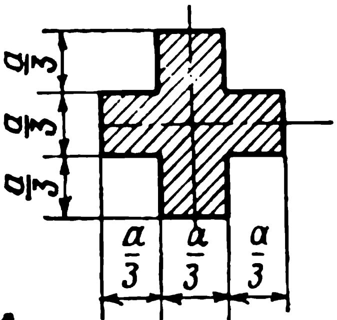

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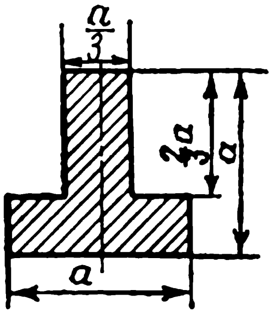

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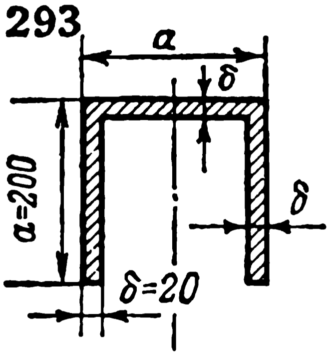

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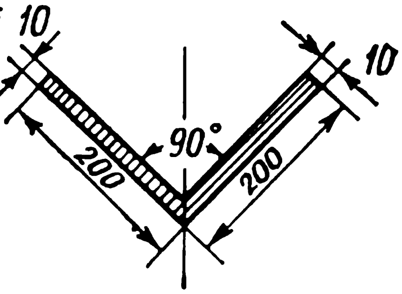

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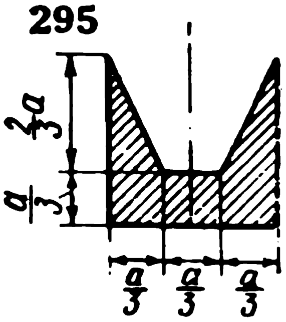

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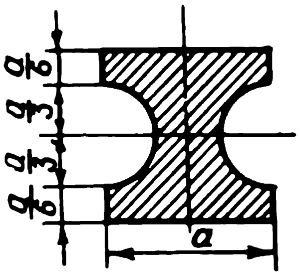

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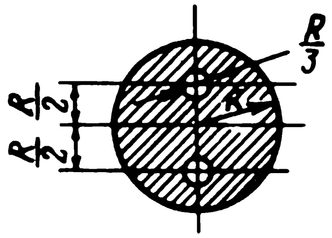

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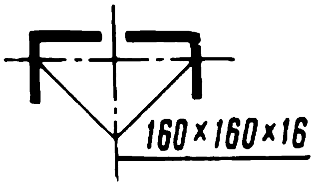

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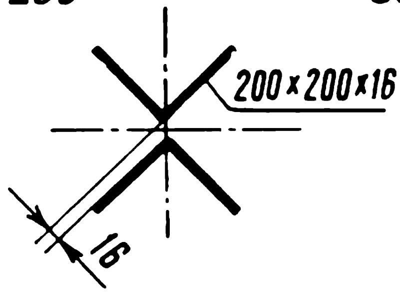

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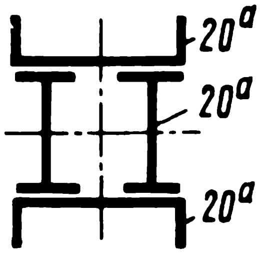

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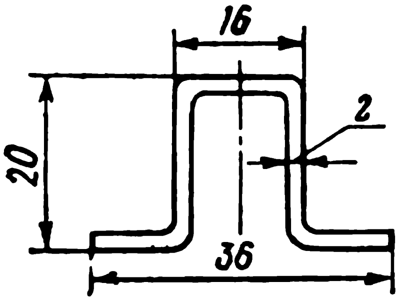

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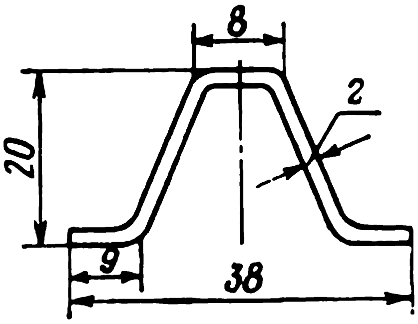

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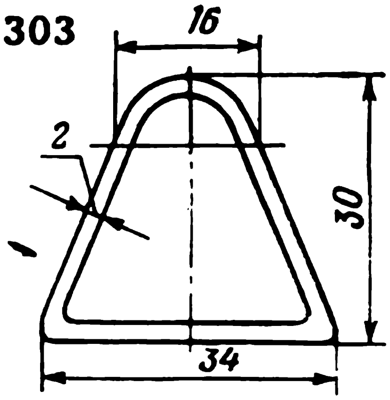

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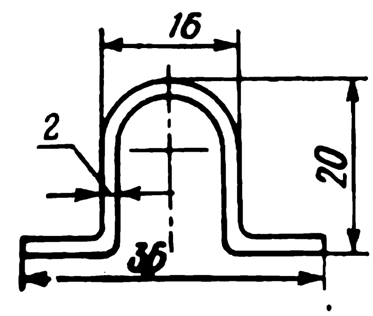

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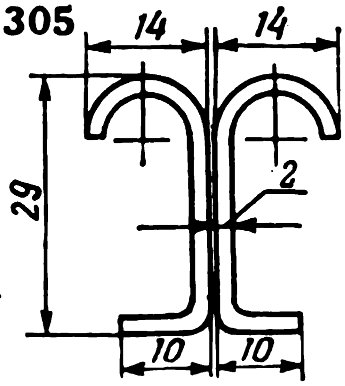

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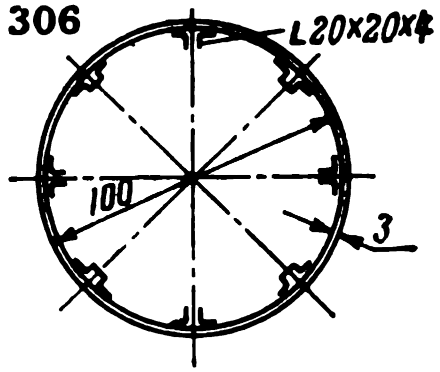

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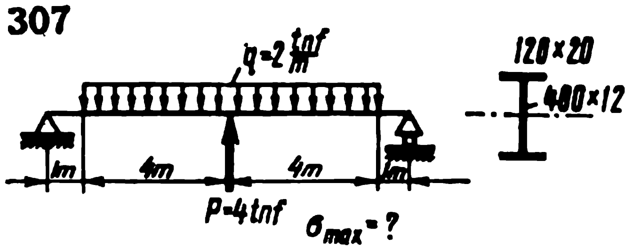

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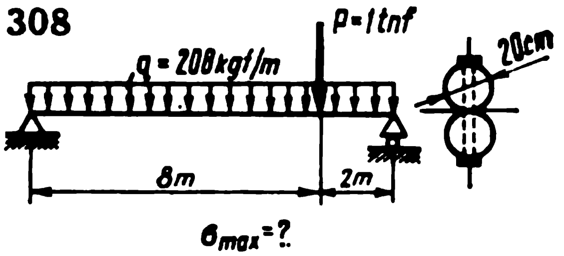

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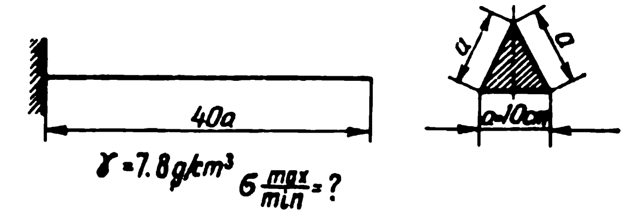

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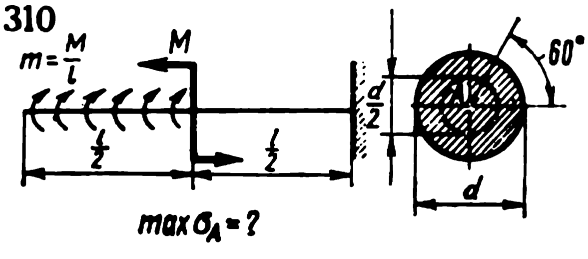

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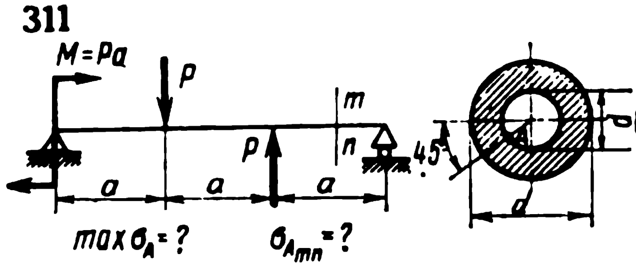

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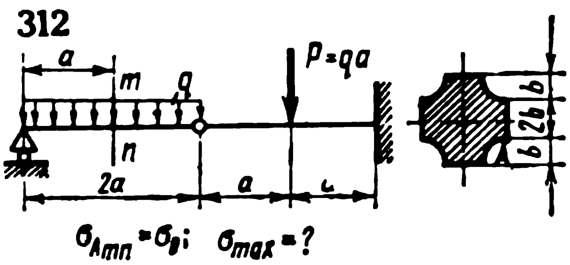

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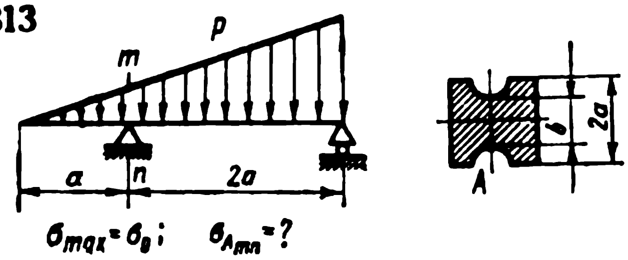

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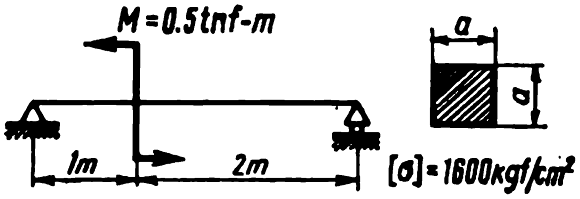

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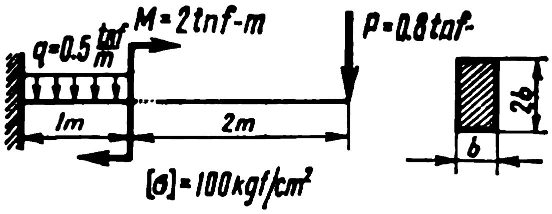

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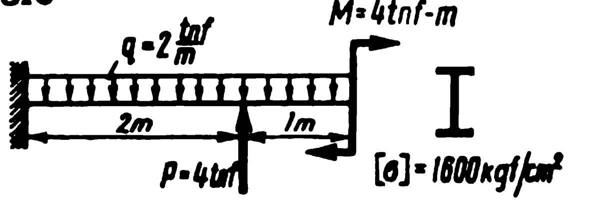

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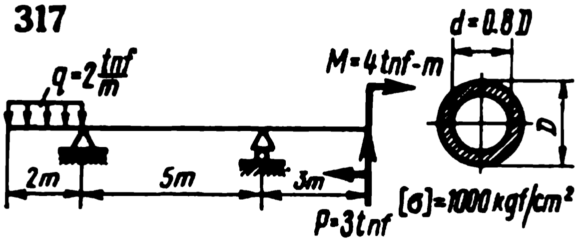

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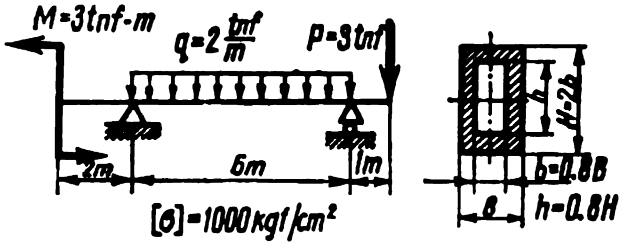

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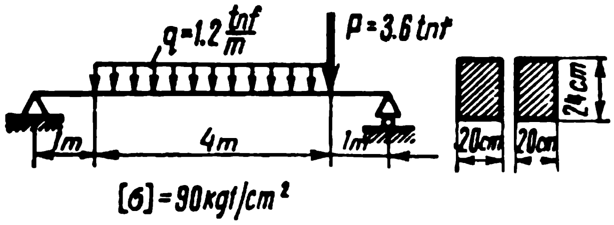

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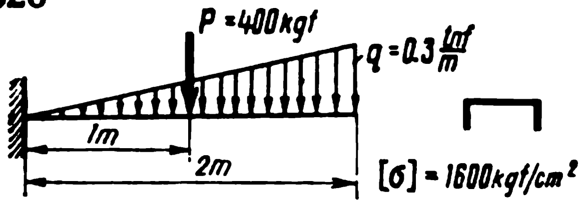

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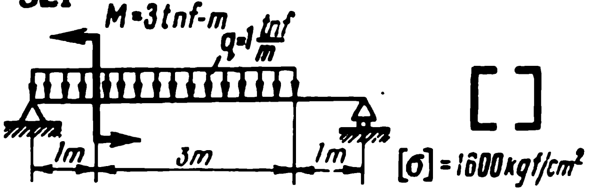

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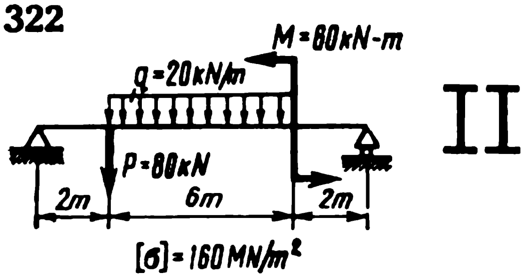

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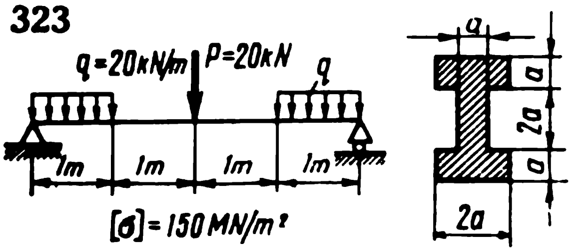

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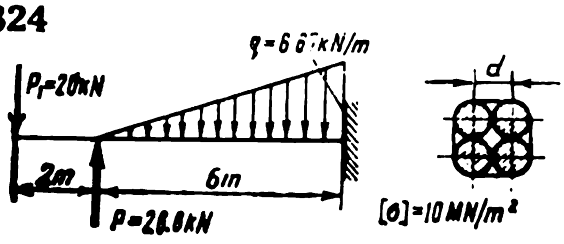

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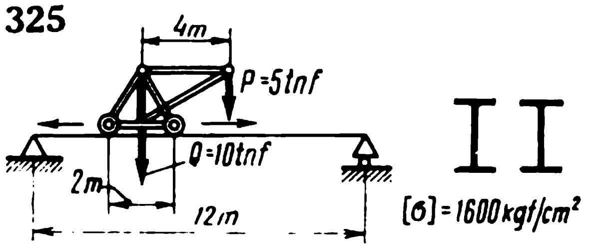

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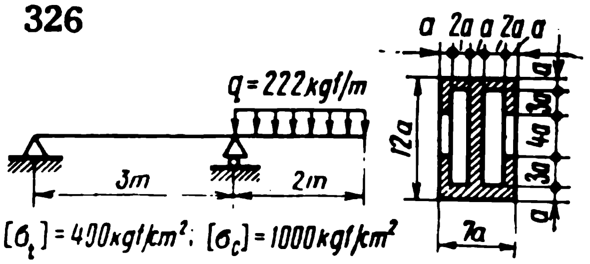

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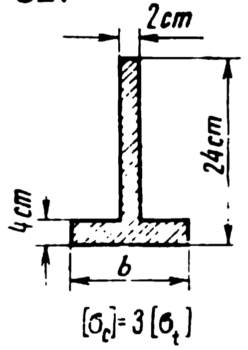

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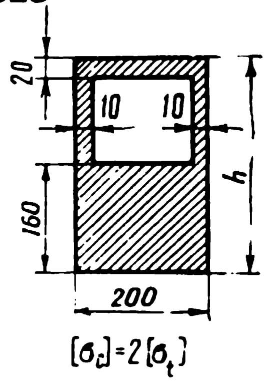

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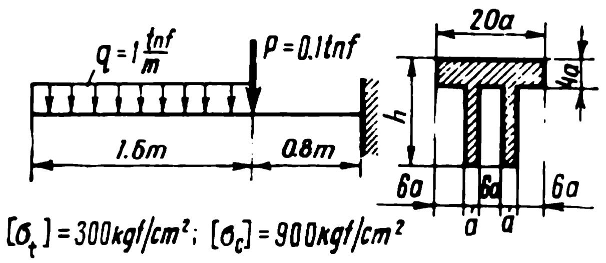

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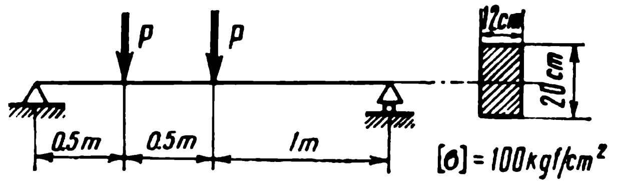

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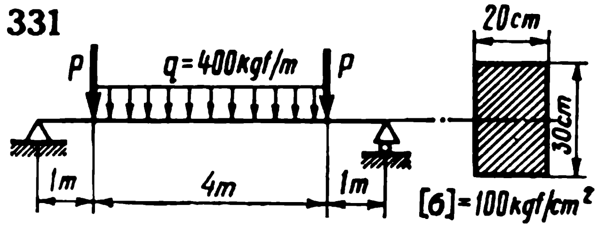

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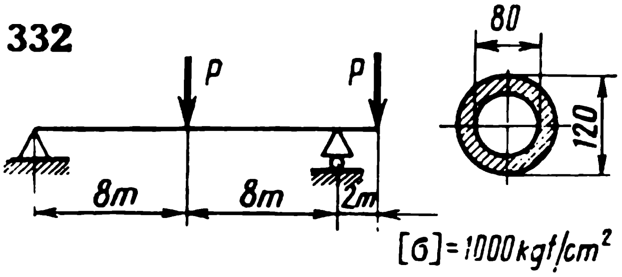

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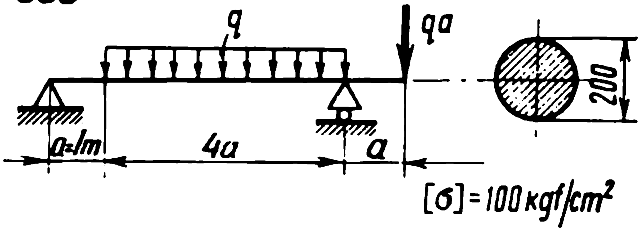

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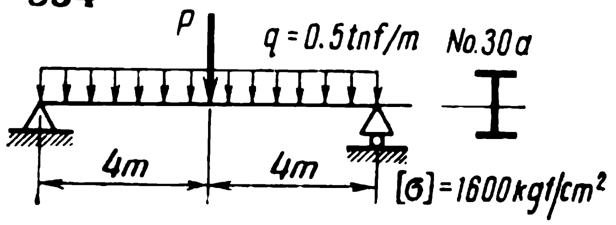

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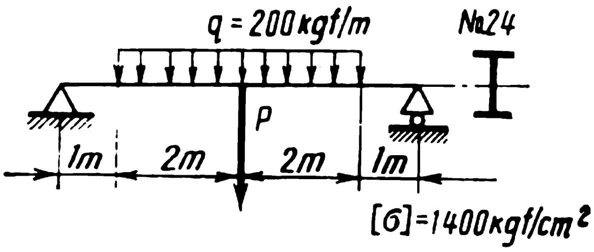

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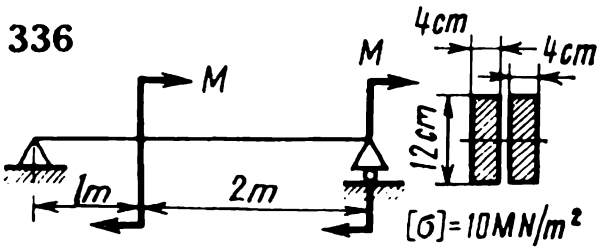

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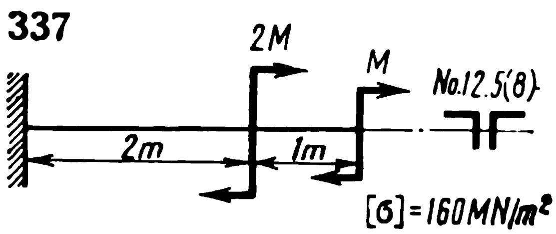

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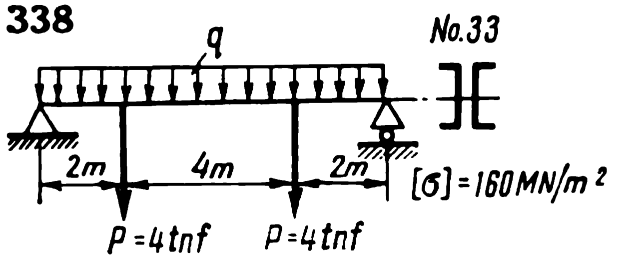

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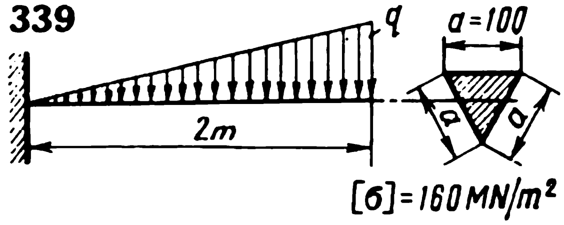

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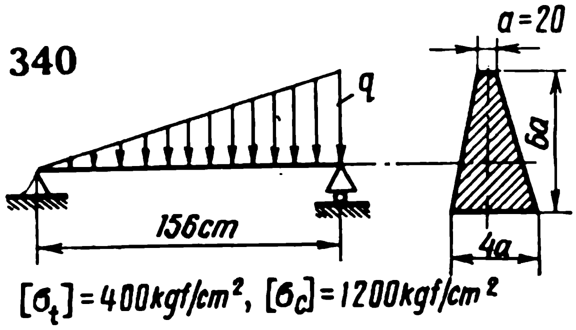

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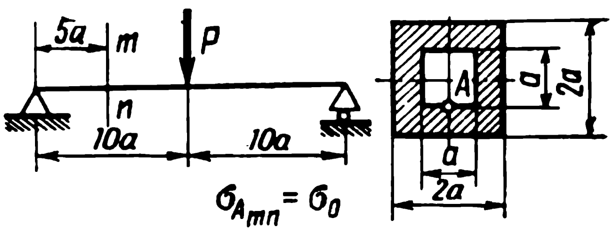

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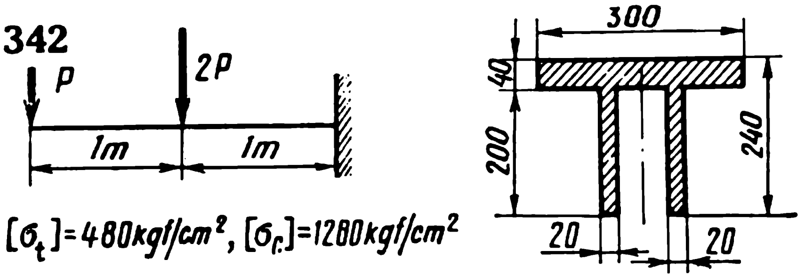


In Problem 325 select the appropriate size No. of I-beams required for the case of the most unfavourable position of the movable loads.

In Problems 327 and 328 find dimensions b and h, respectively, from the condition of equal strength of the stretched and compressed fibres.

Problems 330 through 342. Find the permissible loads on beams of the given dimensions.

8.3.

Shearing Stresses, Shear Centre and Checking the Strength of Beams on the Basis of Shearing Stresses

At an arbitrary point of a rectangular cross section of a beam (Fig. 64), the shearing stress is found, by using Zhuravsky's formula

$$\tau = \frac{QS}{bI} \qquad (99)$$

in which Q = transverse (shearing) force in the section under consideration

$S = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$ = static moment about neutral axis z of the part of the cross-sectional area lying to one side of level y at which the stress is being determined

b = width of the section

$I = \frac{bh^3}{12}$ = moment of inertia of the cross-sectional area about axis z.

The maximum shearing stresses occur at points of the central axis. Their magnitudes are

$$\tau_{max} = \frac{3}{2}\frac{Q}{F} \tag{100}$$

in which $F = bh$ is the cross-sectional area of the beam.

Using formula (99) the components of the shearing stresses normal to the neutral axis can be approximately calculated for beams of non-rectangular section as well, taking b as the width of the section at the level of the point being considered. The resultant shearing stresses at points along the outline of a section are tangent to this outline, and at other points in the section they are inclined towards the plane of action of the forces.

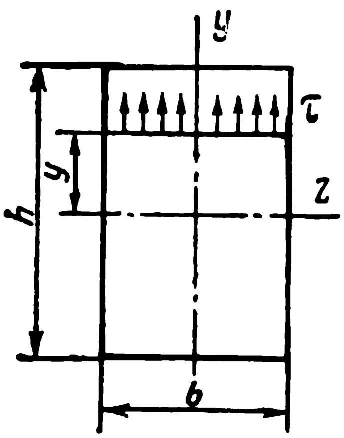


Fig. 64

It is extremely difficult to find the shearing stresses in beams of arbitrary cross-sectional shapes. Their approximate determination is based on certain arbitrary assumptions about the direction of the shearing stresses inside the section.

For beams of thin-walled open shape of the type illustrated in Fig. 65, the shearing stresses are assumed to be tangent to the central line of the wall of the section and along the normal (t) to it, and are assumed to be distributed uniformly. In this case the formula used to find the shearing stress is of the following form:

$$\tau = \frac{QS}{tI} \tag{101}$$

in which S is the static moment about neutral axis z of the portion of the section which is to one side of the normal t to the central line at level y of the point being considered.

If the cross section of the beam is asymmetrical with respect to the principal centroidal axis y, perpendicular to neutral axis z, then shearing stresses develop that produce a torque in this section. To avoid torsion of the beam the transverse (shearing) force should be applied at a point called the *shear*, or *flexural*, *centre* instead of at the centre of gravity.

For beams of arbitrary cross sections it may be quite difficult to determine the position of the shear centre. For a thin-walled section symmetrical with respect to neutral axis z (Fig. 65), the shear centre

is on the z-axis, and its distance from the centre of gravity of the section is

$$e=\frac{1}{I}\int_F \frac{S}{t}\rho\, dF \qquad (102)$$

in which ρ = arm of the element of shearing force $\tau\, dF$ about the centre of gravity of the section

F = area of the whole cross section.

If the wall thickness t of the shape is constant, equation (102) can be used in the simple form

$$e=\frac{1}{I^s}\int_s S^{s'}\rho\, ds \qquad (103)$$

Here I^s = equatorial moment of inertia of the central line arc of the entire section about axis z

$S^{s'}$ = static moment about axis z of arc s' of the central line of the portion of the section lying to one side of level y of an arbitrary point.

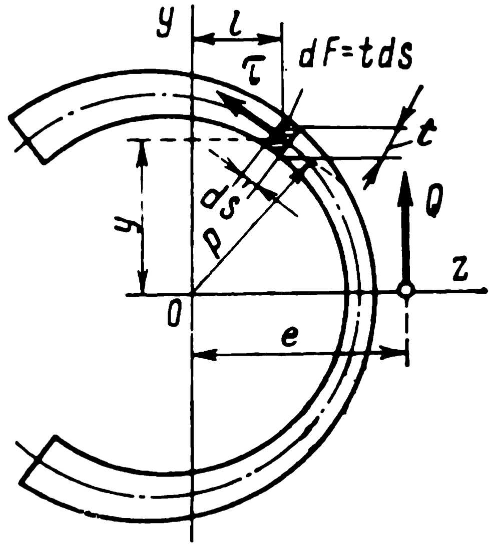


Fig. 65

Integration is performed along arc s and not over the cross-sectional area.

If the shape of the section is built up of several parts of different but constant thickness t, e can be found by summing up equations (102) formulated for each part separately.

The strength of a beam is checked on the basis of the shearing stresses at points of the cross section to which the maximum transverse (shearing) force Q_{max} (in absolute value) is applied.

Such checking calculations should be done for short beams, beams with thin and high wall sections, beams made of materials with a low shear strength and beams which carry heavy loads near the supports.

The following formula is used to check such beams:

$$\tau_{max}=\frac{Q_{max}S_0}{b_0 I}\leqslant[\tau] \qquad (104)$$

in which S_0 = static moment about the neutral axis of the portion of the cross section lying to one side of the line of action of τ_{max}

b_0 = width of the section at line of action of τ_{max}.

For most sections τ_{max} is developed at points of the neutral axis.

The permissible shearing stress $[\tau]$ is commonly taken equal to 0.5-0.7 of $[\sigma]$. Thus for steel Ст. ОС and Ст. 2 (according to the USSR Std), $[\tau]$ = 900 kgf/cm^2; for Ст. 3, $[\tau]$ = 1000 kgf/cm^2; for pine and spruce $[\tau]$ = 20 kgf/cm^2.

Example 36. Let $M_1 = 40$ kN-m; $M_2 = 20$ kN-m; $M_3 = 10$ kN-m; $a = 1$ m; $b = 4$ cm and $h = 12$ cm (Fig. 66).

Find σ_A and τ_A in section mn.

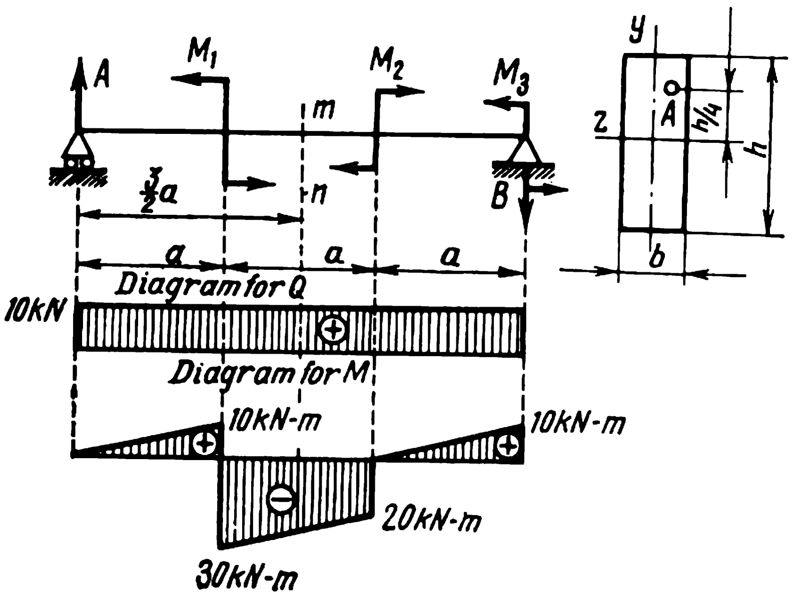


Fig. 66

Solution. The support reaction is

$$A = B = \frac{M_1 - M_2 + M_3}{3a} = \frac{40 - 20 + 10}{3} = 10 \text{ kN}$$

The diagrams for Q and M are shown in Fig. 66.

In section mn, $M = -\frac{30+20}{2} = -25$ kN-m; $Q = 10$ kN.

Since point A is in the stretched zone of the cross section, from formula (92) we obtain

$$\sigma_A = \frac{[M]\,y}{I}$$

For a rectangular section

$$I = \frac{bh^3}{12} = \frac{4 \times 12^3}{12} = 576 \text{ cm}^4$$

For point A being considered

$$y = \frac{h}{4} = 3 \text{ cm}$$

Therefore

$$\sigma_A = \frac{25 \times 10^3 \times 0.03}{576 \times 10^{-8}} \cong 130 \times 10^6 \text{ N/m}^2 = 130 \text{ MN/m}^2$$

From formula (99)

$$\tau_A=\frac{6Q}{bh^3}\left(\frac{h^2}{4}-y^2\right)=\frac{6Q}{h^3b}\left(\frac{h^2}{4}-\frac{h^2}{16}\right)$$

$$=\frac{9}{8}\frac{Q}{bh}=\frac{9}{8}\times\frac{10\times10^3}{0.04\times0.12}=2.34\times10^6\ \text{N/m}^2=2.34\ \text{MN/m}^2$$

Example 37. Let $Q = 8$ tnf; $h = 12$ cm; $h_0 = 8$ cm; $b = 6$ cm and $b_0 = 4$ cm (Fig. 67).

Construct the diagram of conventional shearing stresses perpendicular to the neutral axis (using Zhuravsky's formula).

Solution. First we find the shearing stresses at points *1* of the extreme fibres of the section, at points *2* of the extreme fibres of the rectangular hole through the beam, at points *3* most remote from the neutral axis on the walls of the hole and at points *4* on neutral axis z (Fig. 67).

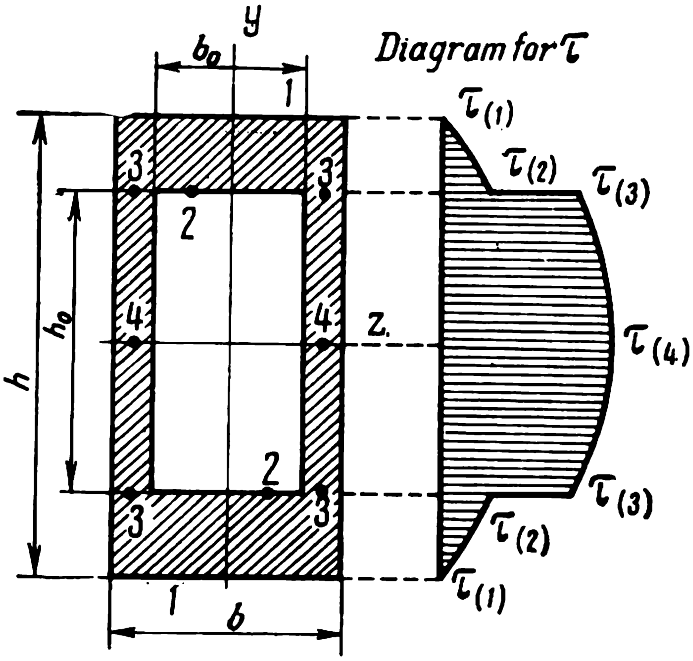


Fig. 67

Use is made of formula (99). The moment of inertia about neutral axis z of the cross-sectional area of the given shape is

$$I=\frac{bh^3}{12}-\frac{b_0h_0^3}{12}=\frac{1}{12}(6\times12^3-4\times8^3)=\frac{2080}{3}\ \text{cm}^4$$

For points *1*, $S_1 = 0$, hence $\tau_{(1)}$ also equals zero.
For points *2*

$$S_{(2)}=b\frac{h-h_0}{2}\left(\frac{h}{2}-\frac{h-h_0}{4}\right)=6\times2\,(6-1)=60\ \text{cm}^3$$

and

$$\tau_{(2)}=\frac{QS_{(2)}}{bI}=\frac{8\times10^3\times60\times3}{6\times2080}\cong115.4\ \text{kgf/cm}^2$$

Since the width of the section at points *3* is equal to $b - b_0$, then

$$\tau_{(3)}=\frac{QS_{(2)}}{(b-b_0)\,I}=\tau_{(2)}\frac{b}{b-b_0}\cong115.4\times\frac{b}{2}=346.2\ \text{kgf/cm}^2$$

For points *4*

$$S_{(4)}=S_{(2)}+(b-b_0)\frac{h_0}{2}\times\frac{h_0}{4}=60+2\times4\times2=76\ \text{cm}^3$$

and

$$\tau_{(4)}=\frac{QS_{(4)}}{(b-b_0)\,I}=\frac{8\times 10^3\times 76\times 3}{2\times 2080}\cong 438.5\ \text{kgf/cm}^2$$

These results are used to plot the diagram for τ (Fig. 67).

Example 38. Find the position of shear centre A in a thin-walled section in the form of a part of a circular ring with the central angle 2α, radius of the central line r and of constant thickness t (Fig. 68).

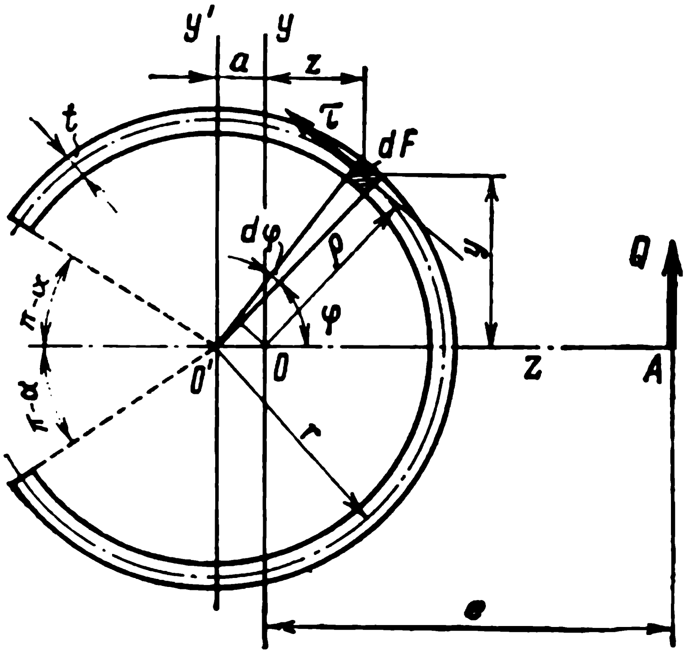


Fig. 68

Solution. As is evident from the drawing, an element of the arc of the central line of the ring $ds=rd\varphi$, an element of the area of the ring wall $dF = tds=rtd\varphi$; and the coordinates of the centre of gravity of an element of area dF in the centroidal axes yoz are

$$y=r\sin\varphi \text{ and}$$

$$z=r\cos\varphi - a$$

in which a = distance between the y-axis and y'-axis passing through the centre of curvature of section O'.

Determine the position of the centre of gravity O of the section on axis z. Since y is a centroidal axis of the section, the static moment of the cross-sectional area is

$$S_y=\int_F z\,dF=2rt\int_0^{\alpha}(\cos\varphi-a)\,d\varphi=2rt\,(r\sin\alpha-a\alpha)=0$$

Therefore

$$a=\frac{r\sin\alpha}{\alpha}$$

Arm ρ of shearing force $\tau t\,ds$ with respect to the centre of gravity of the section is

$$\rho=r-a\cos\varphi=r\left(1-\frac{\sin\alpha}{\alpha}\cos\varphi\right)$$

The moment of inertia I^s of the arc of the central line of the section about neutral axis z is

$$I^s=\int_s y^2\,ds=2r^3\int_0^{\alpha}\sin^2\varphi\,d\varphi=\frac{r^3}{2}(2\alpha-\sin 2\alpha)$$

The static moment $S^{s'}$ about axis z of the portion of the arc of the section's central line, which is situated to one side of an arbitrarily taken element, denoted by the angle φ, is

$$S^{s'}=\int_{s'} y\,ds=r^2\int_{\varphi}^{\alpha}\sin\varphi\,d\varphi=r^2(\cos\varphi-\cos\alpha)$$

From formula (103), the distance e from the centre of gravity of the section to the shear centre A is

$$e=\frac{1}{I^s}\int_s S^{s'}\rho\,ds=\frac{2}{r^3(2\alpha-\sin 2\alpha)}$$

$$\times 2\times r^4\int_0^{\alpha}\left(1-\frac{\sin\alpha}{\alpha}\cos\varphi\right)(\cos\varphi-\cos\alpha)\,d\varphi$$

or

$$e=\frac{2r}{2\alpha-\sin 2\alpha}\left(\sin\alpha-2\alpha\cos\alpha+\frac{1}{\alpha}\sin^2\alpha\cos\alpha\right)$$

The distance from the centre of curvature O' of the section to shear centre A is

$$a+e=\frac{r}{\alpha}\sin\alpha+\frac{2r}{2\alpha-\sin 2\alpha}\left(\sin\alpha\right.$$

$$\left.-2\alpha\cos\alpha+\frac{1}{\alpha}\sin^2\alpha\cos\alpha\right)$$

$$=r\frac{4(\sin\alpha-\alpha\cos\alpha)}{2\alpha-\sin 2\alpha}$$

Example 39. Let $P=4$ tnf; $M=2$ tnf-m; $a=0.5$ m; $l=4$ m; $[\sigma]=1600$ kgf/cm² and $[\tau]=1000$ kgf/cm² (Fig. 69).

Find the required size No. of the I-beam.

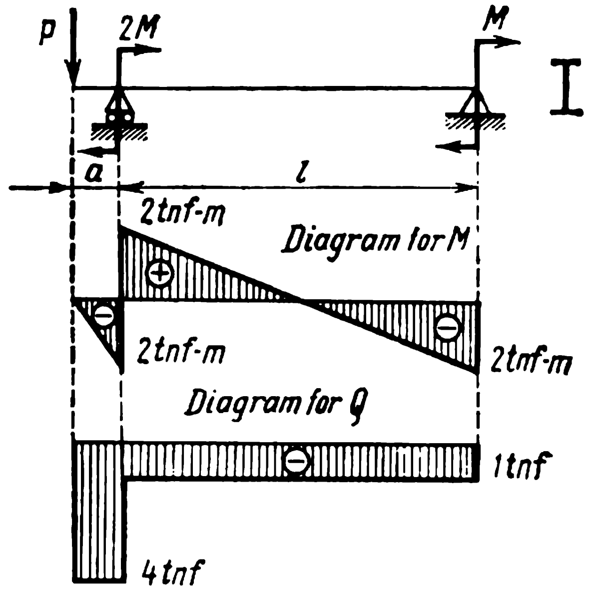


Fig. 69

Solution. The diagram for bending moment M can be readily plotted without calculations, as illustrated in Fig. 69.

On the cantilever $Q=\text{const}=-4$ tnf. In the span between the supports $Q=\text{const}<0$. From the moment (M) diagram it follows that

$$\Delta M=2\text{ tnf-m}=-Q\times 2\text{ m}$$

From which $Q=-1$ tnf.

The diagram for Q is shown in Fig. 69.

Thus,

$$M_{max}=2\text{ tnf-m};\quad |Q|_{max}=4\text{ tnf}$$

From the design formula

$$W=\frac{M_{\max}}{[\sigma]}=\frac{2\times 10^5}{16\times 10^2}=125\ \text{cm}^3$$

From the data for std rolled steel shapes:

$W = 109$ cm^3 for I-beam No. 16;
$W = 143$ cm^3 for I-beam No. 18

First we check whether I-beam No. 16 is suitable:

$$\frac{\sigma_{\max}-[\sigma]}{[\sigma]}\times 100=\left(\frac{125}{109}-1\right)\times 100\cong 15\%\ \text{(overstress)}$$

Then we check I-beam No. 18:

$$\frac{\sigma_{\max}-[\sigma]}{[\sigma]}\times 100=\left(\frac{125}{143}-1\right)\times 100\cong -12.5\ \%\ \text{(understress)}$$

The latter beam is selected. For this beam

$$\sigma_{\max}=\frac{M_{\max}}{W}=\frac{2\times 10^5}{143}\cong 1400\ \text{kgf/cm}^2$$

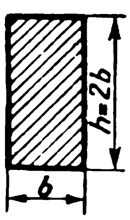
From the data for standard rolled steel shapes: $S = 81.4$ cm^3; $I = 1290$ cm^4 and $d = 0.51$ cm.
Hence

$$\tau_{\max}=\frac{Q_{\max}S}{dI}=\frac{4\times 10^3\times 81.4}{0.51\times 1290}\cong 495\ \text{kgf/cm}^2<[\tau]$$

Thus the selected I-beam (No. 18) has sufficient strength for withstanding both normal and shearing stresses.

Problems 343 through 351. Draw diagrams of the shearing stresses τ perpendicular to the neutral (horizontal) axis for the cross-sectional areas of the beams shown, evaluating the stresses as decimal fractions of the maximum stress ($\tau_{\max} = \tau_0$).

343



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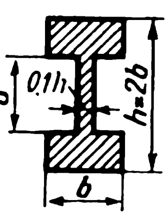

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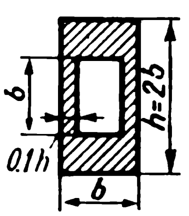

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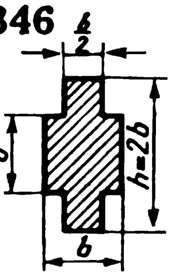

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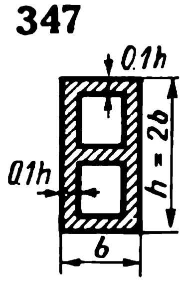

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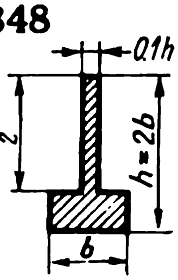

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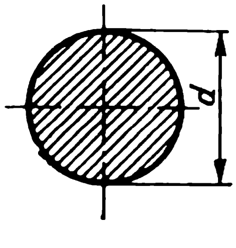

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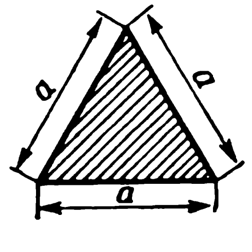

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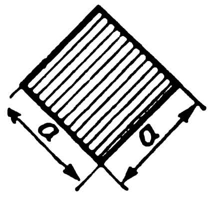

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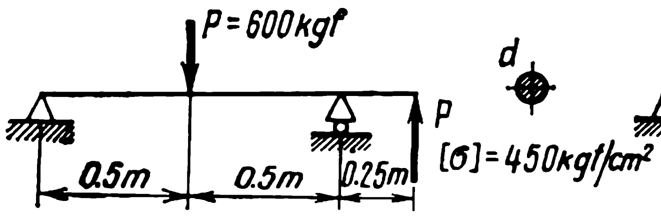

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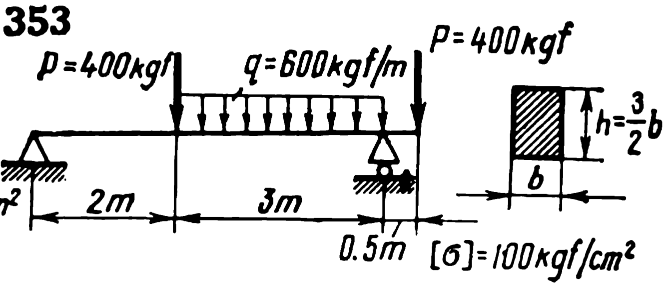

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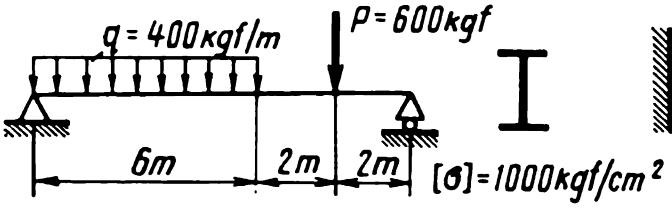

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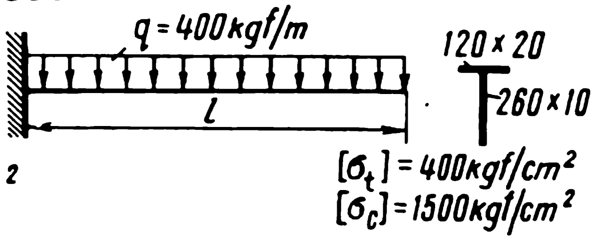

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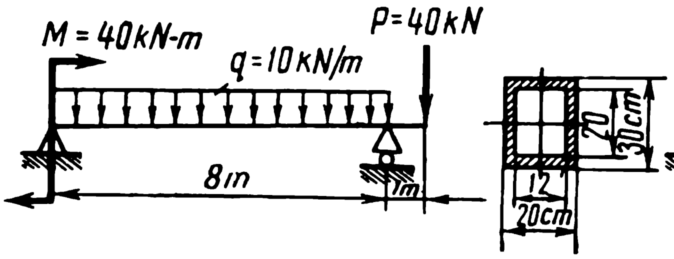

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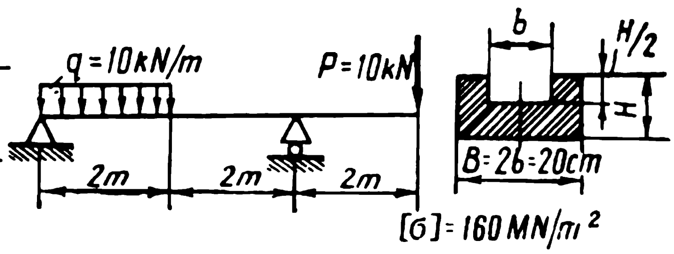

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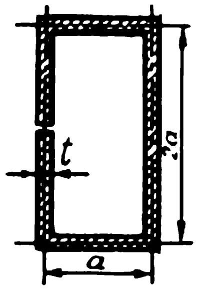

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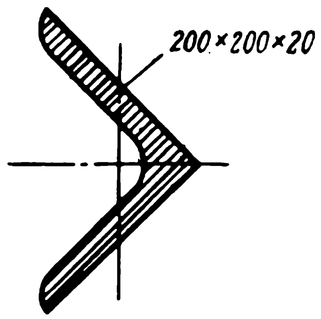

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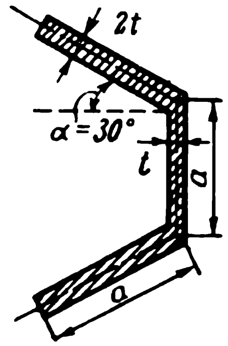

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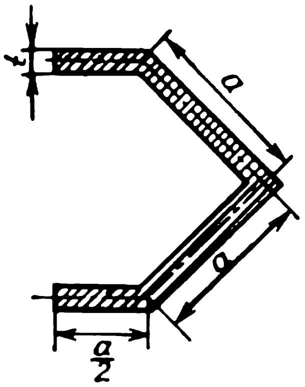

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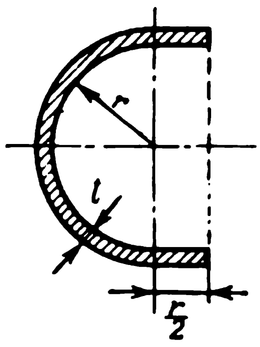

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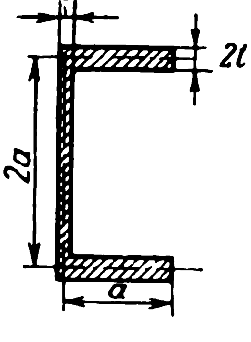


Problems 352 through 357. Find the maximum shearing stresses τ_{max}, having first determined (where required) the necessary dimensions of the beams or the safe loads on the basis of the given permissible normal stresses (see page 135).

Problems 358 through 363. Find distance e of the shear centre from the centre of gravity of the sections (consider the sections to be thin-walled) (see page 135).

8.4.

Principal Stresses and Overall Checking of the Strength of Beams

The normal σ and shearing τ stresses at an arbitrary point of the cross section of a beam a distance y from neutral axis z are found by formulas (92) and (99).

An element near this point having infinitely close cross sections a distance dx from each other and infinitely close longitudinal sections

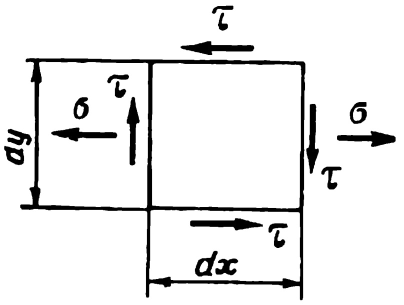


Fig. 70

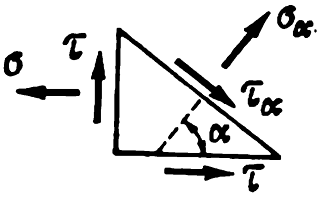


Fig. 71

parallel to the neutral layer and situated a distance dy from each other is in a plane or biaxial state of stress of the kind illustrated in Fig. 70.

The normal σ_α and shearing τ_α stresses in the inclined plane of the section passing through this point (Fig. 71) equal

$$\left.\begin{aligned}\sigma_\alpha &= \sigma\cos^2\alpha - \tau\sin 2\alpha;\\ \tau_\alpha &= \frac{\sigma}{2}\sin 2\alpha + \tau\cos 2\alpha\end{aligned}\right\} \qquad (105)$$

Two perpendicular inclined planes are the principal planes of stresses at the given point of the beam if

$$\tan 2\alpha = -\frac{2\tau}{\sigma} \qquad (106)$$

The principal stresses σ_1 and σ_3 are found by the equation

$$\sigma_{\substack{1\\3}} = \frac{1}{2}\left(\sigma \pm \sqrt{\sigma^2 + 4\tau^2}\right) \qquad (107)$$

The extremal shearing stresses are determined by the formula

$$\tau_{\substack{max\\min}} = \pm\frac{1}{2}\sqrt{\sigma^2+4\tau^2} \tag{108}$$

The magnitudes and directions of the principal stresses for the four possible versions of states of stress for elements of the beam are illustrated in Fig. 72.

If the maximum bending moment and the maximum transverse force or values of M and Q close to the maximum ones act simultaneously

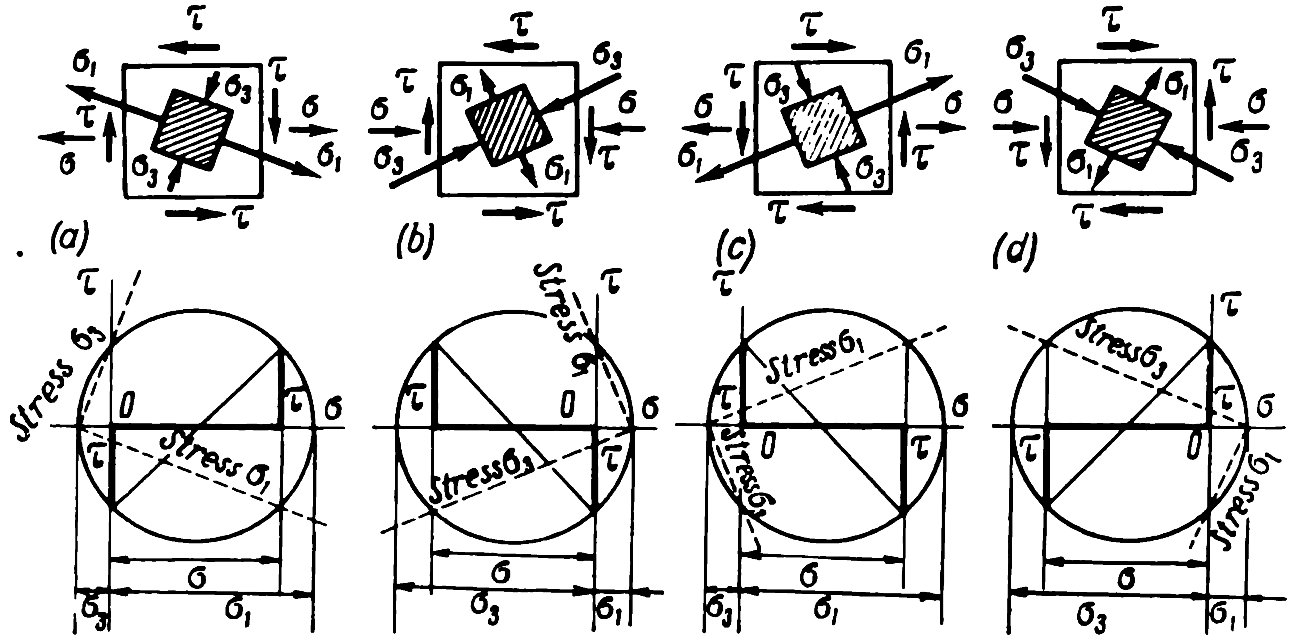


Fig. 72

at the same cross section of the beam, the beam strength should be checked at this section on the basis of the principal stresses.

Beam strength is checked on the basis of the principal stresses only for beams with cross sections having a thin wall which becomes abruptly thicker near the extreme fibres. Strength is checked at the points of transition from a thin to a thick cross section.

Beams of ductile materials are checked by the third strength theory, using the formula

$$\sigma_{eq_{III}} = \sigma_1 - \sigma_3 = \sqrt{\sigma^2+4\tau^2} \leqslant [\sigma] \tag{109}$$

Beams of brittle materials are checked by the first strength theory from the condition

$$\sigma_1 = \frac{1}{2}(\sigma + \sqrt{\sigma^2+4\tau^2}) \leqslant [\sigma_t] \tag{110}$$

An overall strength calculation of a statically determinate beam is given in Example 40.

Example 40. Let $P = 4$ tnf; $q = 3$ tnf/m; $a = 0.8$ m; $l = 4$ m; $[\sigma] = 1600$ kgf/cm^2 and $[\tau] = 1000$ kgf/cm^2 (Fig. 73).

Determine the size No. of the I-beam.

Solution. 1. Finding the support reactions:

$$A=\frac{P(a+l)+\frac{q}{2}l^2}{l}=\frac{4\times 4.8+3\times 8}{4}=10.8 \text{ tnf};$$

$$B=\frac{\frac{q}{2}l^2-Pa}{l}=\frac{3\times 8-4\times 0.8}{4}=5.2 \text{ tnf}$$

2. Plotting the diagrams for Q and M.
On the cantilever portion $0\leqslant x_1\leqslant a$:

$$Q_{x_1}=-P=-4 \text{ tnf};\quad M_{x_1}=-Px_1=-4x_1;\quad M_{x_1=0}=0;$$
$$M_{x_1=a}=-4\times 0.8=-3.2 \text{ tnf-m}$$

In the span between the supports $0\leqslant x_2\leqslant l$:

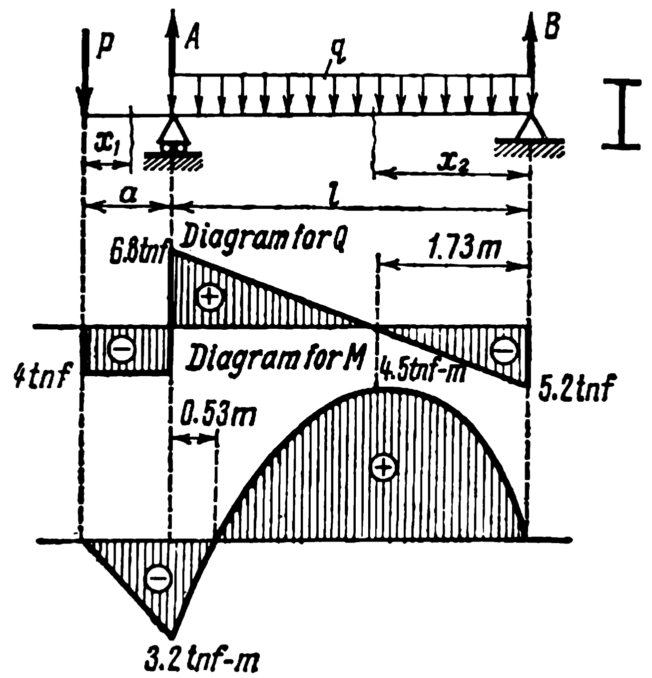


Fig. 73

$$Q_{x_2}=-B+qx_2=-5.2+3x_2;$$
$$Q_{x_2=0}=-5.2 \text{ tnf};$$
$$Q_{x_2=l}=-5.2+3\times 4=6.8 \text{ tnf};$$
$$M_{x_2}=Bx_2-\frac{q}{2}x_2^2=5.2x_2-\frac{3}{2}x_2^2;\quad M_{x_2=0}=0;$$
$$M_{x_2=l}=5.2\times 4-\frac{3}{2}\times 16=-3.2 \text{ tnf-m}$$

Since $Q_{x_2}=-5.2+3x_2=0$ at $x_2=\frac{5.2}{3}\cong 1.73$ m, then

$$M_{\substack{\max\\ x_2=1.73}}=5.2\times 1.73-3\frac{1.73^2}{2}\cong 4.5 \text{ tnf-m};$$
$$M_{x_2}=5.2x_2-\frac{3}{2}x_2^2=0 \text{ when } x_2=\frac{5.2\times 2}{3}=3.47 \text{ m}$$

These results are used to plot the diagrams for Q and M (Fig. 73).
3. Selecting the size of the I-beam.
Since $M_{\max}=4.5$ tnf-m, $W=\frac{M_{\max}}{[\sigma]}=\frac{4.5\times 10^5}{16\times 10^2}\cong 281 \text{ cm}^3$
From the data for rolled steel shapes W = 254 cm^3 for I-beam size No. 22a

$$\frac{\sigma_{\max}-[\sigma]}{[\sigma]}100=\frac{W-W_{\text{No. }22^{a}}}{W_{\text{No. }22^{a}}}\times 100$$
$$=\frac{281-254}{254}\times 100=10.6\%>5\% \text{ (overstress)}$$

For I-beam size No. 24, $W = 289\ \text{cm}^3$. Then

$$\frac{\sigma_{max}-[\sigma]}{[\sigma]}\times 100=\frac{W-W_{\text{No. 24}}}{W_{\text{No. 24}}}\times 100=\frac{281-289}{289}100\cong -2.77\%$$

(understress)

Hence we select I-beam No. 24 for which $W = 289\ \text{cm}^3$, $I = 3460\ \text{cm}^4$, $S_0 = 163\ \text{cm}^3$, $h = 24$ cm, $b = 11.5$ cm, $t = 0.95$ cm, $d = b_0 = 0.56$ cm, $h_0 = h - 2t = 22.1$ cm (Fig. 74). The maximum normal stress in the extreme fibre of the dangerous section of this I-beam will be:

$$\sigma_{max}=\frac{M_{max}}{W}=\frac{4.5\times 10^5}{289}\cong 1560\ \text{kgf/cm}^2$$

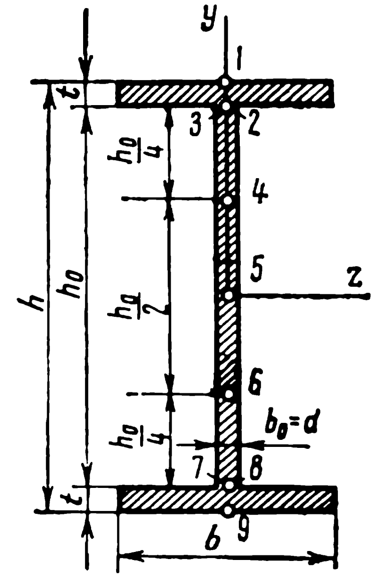


Fig. 74

4. Checking the section of the beam on the basis of the shearing stresses.

Since $Q_{max} = 6.8$ tnf, then

$$\tau_{max}=\frac{Q_{max}S_0}{b_0 I}=\frac{6.8\times 10^3\times 163}{0.56\times 3460}\cong 570\ \text{kgf/cm}^2<[\tau]$$

5. Plotting the diagrams for the normal σ, shearing τ, principal $\sigma_{1,3}$ and extremal shearing $\tau_{\substack{max\\min}}$ stresses in the most unfavourable section of the beam and determining the directions of these stresses.

With respect to the principal stresses the unfavourable section is directly over the left-hand support (if approached from the right) in which $M = -3.2$ tnf-m and $Q = 6.8$ tnf.

We shall determine the stresses at the nine points of the section shown in Fig. 74.

The normal stresses at an arbitrary point a distance y from the neutral axis are

$$\sigma=-\frac{My}{I}=\frac{3.2\times 10^5}{3460}\cong 92.5y \tag{a}$$

The static moment of the area of the flange with respect to axis z is

$$S_{fl}=bt\frac{h-t}{2}=11.5\times 0.95\frac{24-0.95}{2}\cong 126\ \text{cm}^3$$

The static moment of the part of the web area to one side of ordinate y is

$$S_w=\frac{b_0}{2}\left(\frac{h_0^2}{4}-y^2\right)=\frac{0.56}{2}\left(\frac{22.1^2}{4}-y^2\right)=0.28\,(122-y^2)$$

The static moment of the part of the section to one side of ordinate y is

$$S=S_{fl}+S_w=126+0.28\,(122-y^2)\cong 160-0.28y^2$$

The shearing stresses for the points of the flange are calculated by

formula (99). Thus

$$\tau=\frac{Q\left(\frac{h^2}{4}-y^2\right)}{2I} \tag{b}$$

and for the points of the web

$$\tau=\frac{QS}{b_0 I} \tag{c}$$

Formulas (a), (b), (c), (108), (107) and (106) have been used to calculate the quantities σ, τ, $\tau_{\substack{max\\min}}$, $\sigma_{1,3}$, $\tan 2\alpha$, α_1 and α_2 for y corresponding to the nine points of the section.

The data thus obtained are listed in the following table:

Point No.	y, cm	σ (kgf/cm²)	τ (kgf/cm²)	$\tau_{\substack{max\\min}}$ (kgf/cm²)	σ_1 (kgf/cm²)	σ_3 (kgf/cm²)	tan 2α	α_1	α_2
1	12.00	1110	0	±555	1110	0	0.00	0°0′	90°0′
2	11.05	1020	20	±510	1020	0	−0.0392	−1°07′	88°53′
3	11.05	1020	440	±670	1180	−160	−0.863	−20°24′	69°36′
4	5.52	510	530	±590	840	−340	−2.08	−32°10′	57°50′
5	0.00	0	560	±560	560	−560	∞	−45°0′	45°0′
6	−5.52	−510	530	±590	340	−840	2.08	32°10′	122°10′
7	−11.05	−1020	440	±670	160	−1180	0.863	20°24′	110°24′
8	−11.05	−1020	20	±510	0	−1020	0.0392	1°07′	91°07′
9	−12.00	−1110	0	±555	0	−1110	0.00	0°0′	90°0′

The diagrams for the stresses are given in Fig. 75.

The directions of the principal stresses at the points being considered in the section are illustrated in Fig. 76.

A graphic determination of the magnitudes and directions of the principal stresses σ_1 and σ_3 at points *4*, *5* and *6* is shown in Fig. 77.

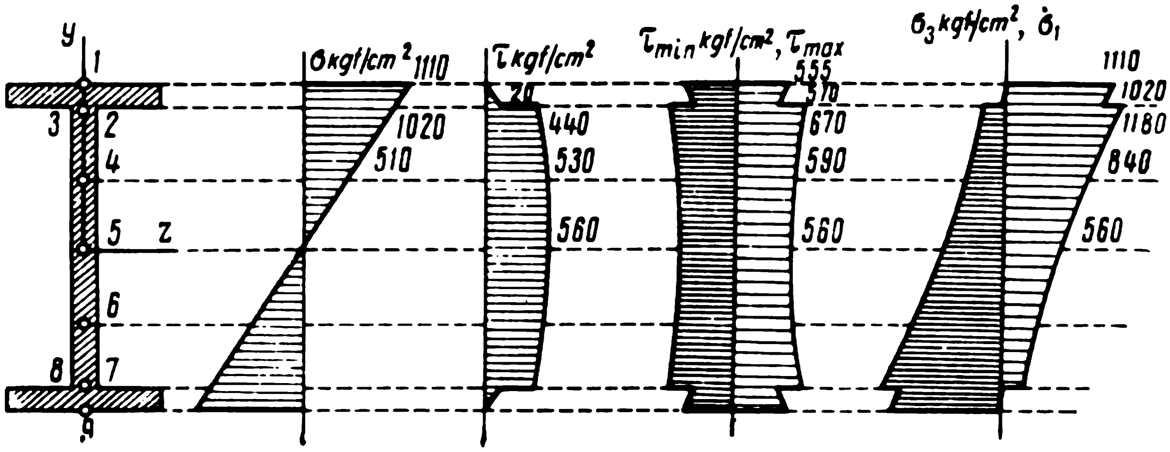


Fig. 75

6. Checking the beam strength by the principal stresses.

Point *3* is the most dangerous one in the unfavourable section. At this point $\sigma_1 = 1180$ kgf/cm² and $\sigma_3 = -160$ kgf/cm². We shall

check the beam strength at this point by the third strength theory, using the inequality

$$\sigma_1 - \sigma_3 \leqslant [\sigma]$$

Since $1180 + 160 = 1340 < 1600$, the selected section is sufficiently strong on the basis of the principal stresses as well.

Problems 364 through 367. Determine (both analytically and graphically) the magnitudes and directions of the maximum and minimum principal stresses σ_1 and σ_3 in the given sections of beams.

Notation: max σ_1 and min σ_3 are the maximum and minimum principal stresses in the most unfavourable section of the beam along its length at places where abrupt changes occur in the width of the section; σ_{1mn} and σ_{3mn} are the principal stresses at the same points of section mn.

In Problem 366 the plane of loading passes through the line of shear centres of the cross sections.

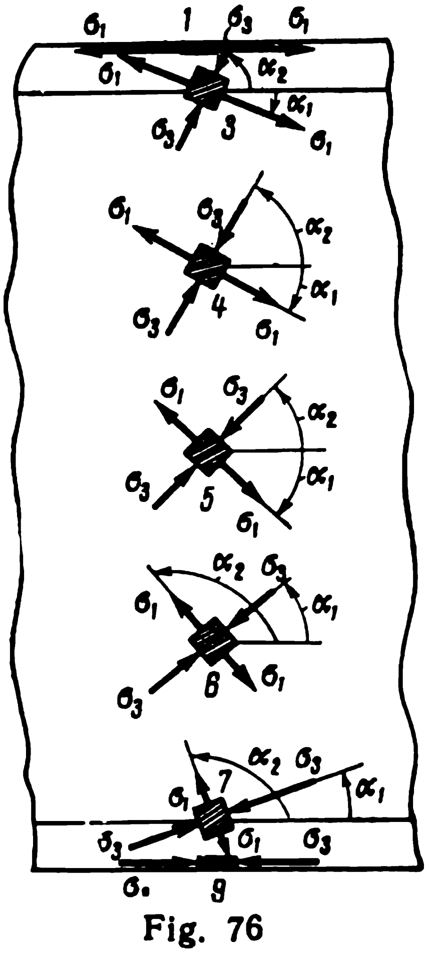


Fig. 76

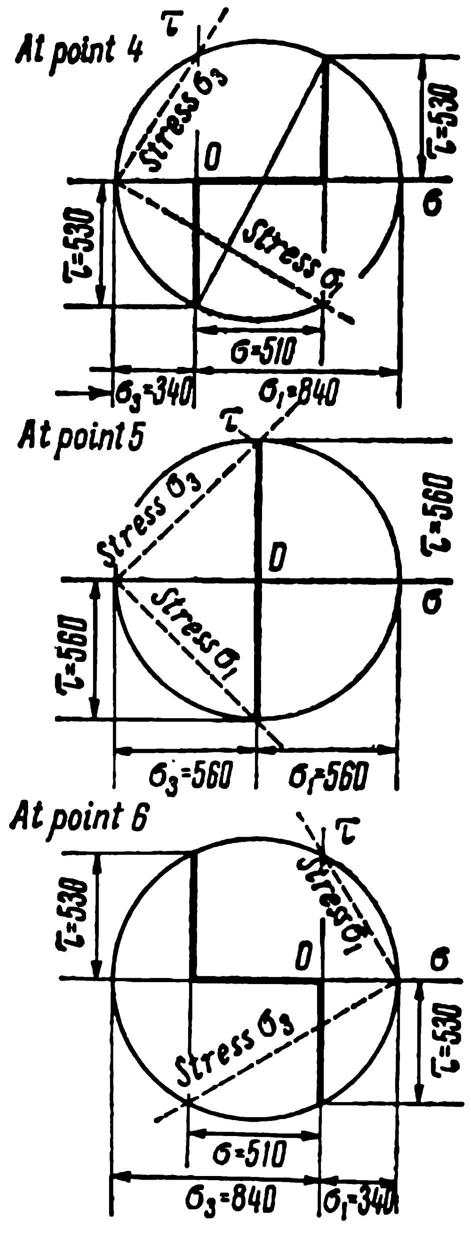
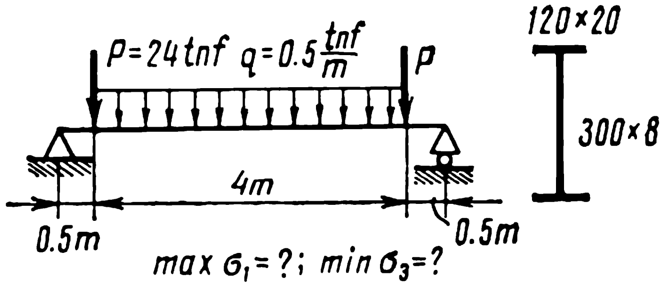


Fig. 77

Problems 368 through 373. Determine the required cross sections of the beams and make an overall check for their strength.

Assume that $[\sigma] = 1600$ kgf/cm^2 and $[\tau] = 1000$ kgf/cm^2, and make use of the third strength theory in checking the beams on the basis of the principal stresses.

364



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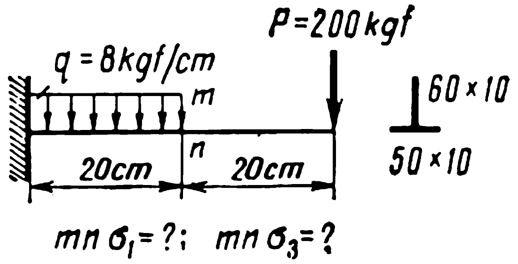

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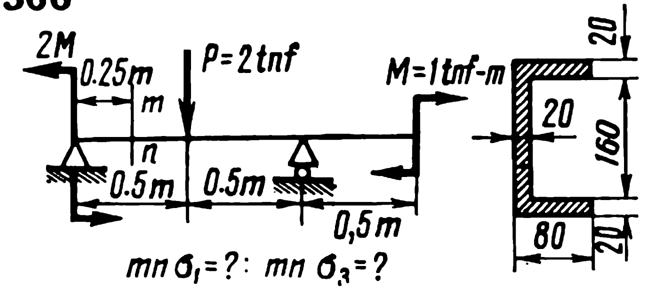

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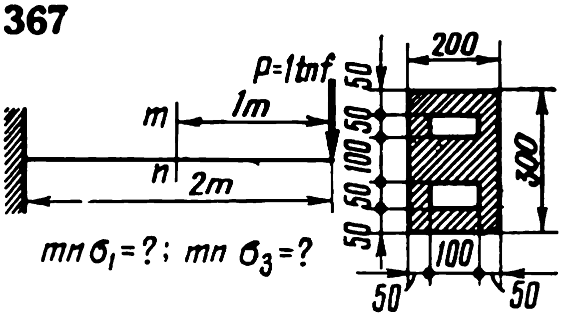

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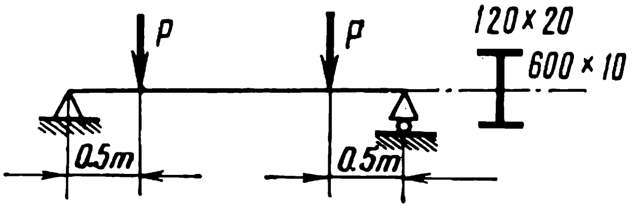

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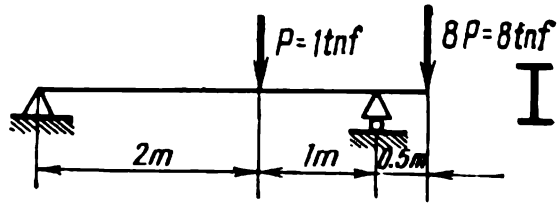

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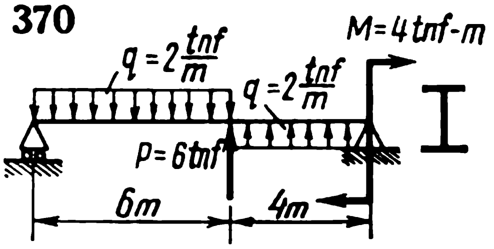

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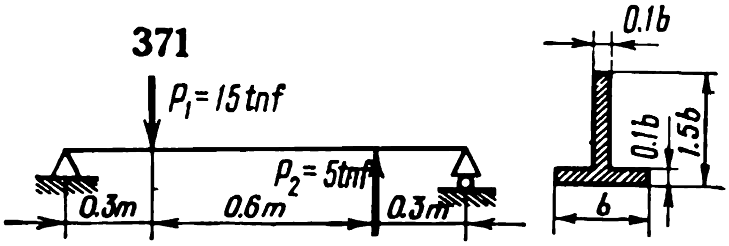

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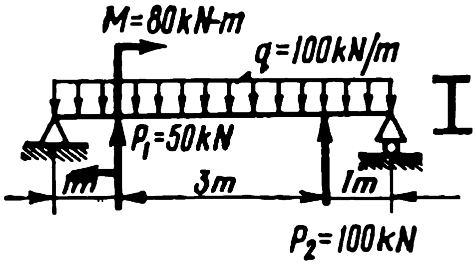

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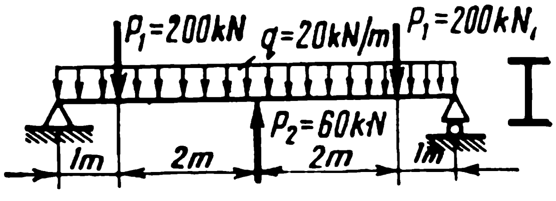


In Problems 372 and 373 assume that $[\sigma] = 160$ MN/m^2 and $[\tau] = 100$ MN/m^2. In Problem 368 determine the maximum permissible load and make an overall check for the beam strength.

8.5.

Fundamentals of Limit Design for Beams

The difference between strength analysis based on the permissible stress and on the limiting loads is, in the case of ductile materials, in the different stages of the strained state of the beam that are considered to be the dangerous state.

Limit design calculations are commonly carried out according to the normal stresses, not taking the strain hardening of the beam material into consideration. The calculations are based on idealized tension and compression diagrams for the beam material (Fig. 78).

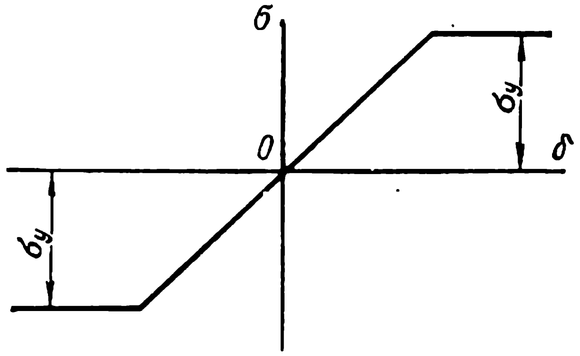


Fig. 78

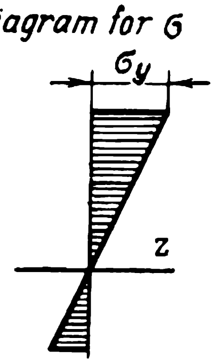


Fig. 79

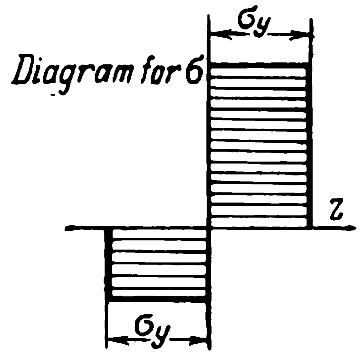


Fig. 80

In beam design based on allowable stresses the critical state of the beam is assumed to be one in which the normal stresses in the most highly stressed extreme fibres approach the yield point σ_y of the material (Fig. 79).

In accordance with the diagram for the stresses σ (Fig. 79) the bending moment at the critical state of the beam is

$$M_y = \sigma_y W \tag{111}$$

Introducing a factor of safety we obtain a formula for determining the permissible bending moment

$$M_{\max} = [\sigma] W \tag{112}$$

In calculations based on limit design the dangerous state of the beam is assumed to be one in which all the points of the dangerous section reach the yield point of the material (Fig. 80).

In accordance with the diagram for the stresses σ (Fig. 80), this state occurs when the bending moment reaches the following definite magnitude:

$$M'_y = 2\sigma_y S \tag{113}$$

in which S is the static moment of one half of the cross-sectional area of the beam about the centroidal axis z.

Since a further increase of the bending moment in this section is impossible, a "plastic hinge" is said to have been set up in this section and the beam becomes a geometrically variable system.

The design formula for determining the maximum permissible bending moment can be obtained by introducing the factor of safety. Thus

$$M'_{max} = 2\,[\sigma]\,S \tag{114}$$

Comparing formulas (112) and (114), we see that with the same factor of safety the allowable bending moment M'_{max} obtained by limit design calculations is η times greater than that obtained by calculations based on the permissible stress, where

$$\eta = \frac{M'_{max}}{M_{max}} = \frac{2S}{W} \tag{115}$$

The quantity η depends only on the geometry of the cross section of the beam.

Example 41. Find η for the shapes of cross sections illustrated in Fig. 81.

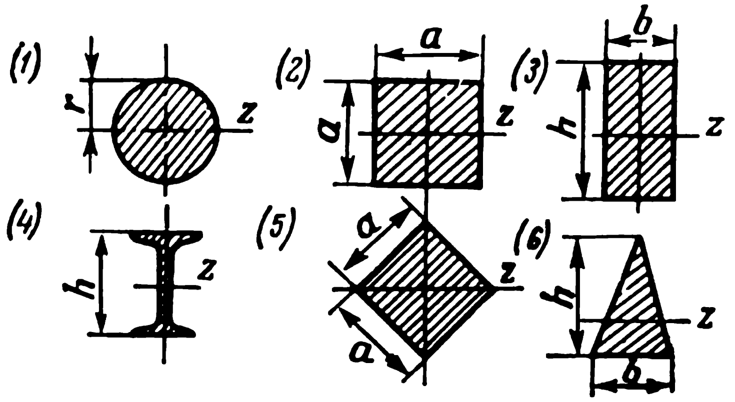


Fig. 81

Solution.

(1) $W = \frac{\pi r^3}{4}$; $S = \frac{2}{3} r^3$; $\eta = \frac{4r^3 \times 4}{3\pi r^3} = \frac{16}{3\pi} \cong 1.697$

(2) $W = \frac{a^3}{6}$; $S = \frac{a^3}{8}$; $\eta = \frac{3}{2} = 1.5$

(3) $W = \frac{bh^2}{6}$; $S = \frac{bh^2}{8}$; $\eta = 1.5$

(4) Since $W = \frac{2I}{h}$, then $\eta = \frac{2S}{W} = \frac{S}{I} h$.

From the data for standard rolled steel shapes, for example for I-beam No. 20a

$$I = 2030\ \text{cm}^4; \quad S = 114\ \text{cm}^3; \quad h = 20\ \text{cm}; \quad \eta = \frac{114 \times 20}{2030} \cong 1.12$$

$$(5)\quad W = \frac{2a^4}{12\,a\sqrt{2}} = \frac{a^3}{6\sqrt{2}}; \quad S = \frac{a^2}{2} \times \frac{1}{3} \times \frac{a\sqrt{2}}{2} = \frac{\sqrt{2}}{12}a^3;$$

$$\eta = \frac{\sqrt{2}\,a^3 \times 6\sqrt{2}}{6a^3} = 2$$

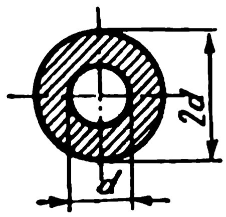
$$(6)\quad W = \frac{bh^3}{36} \times \frac{3}{2h} = \frac{bh^2}{24}; \quad S = \frac{bh}{4}\left(\frac{2}{3}h - \frac{2}{3}\frac{h}{\sqrt{2}}\right)$$

$$= \frac{bh^2}{12}(2-\sqrt{2}); \quad \eta = \frac{bh^2(2-\sqrt{2})\,24}{6bh^2} \cong 2.344$$

Problems 374 through 379. Find the ratio $\eta = \frac{M'_{max}}{M_{max}}$ of the maximum bending moments M'_{max} and M_{max} calculated on the basis of the limit design and permissible stress for the following cross sections of beams.

Problems 380 through 385. Determine the required dimensions or permissible loads on the basis of the permissible stresses and of limit design.

374



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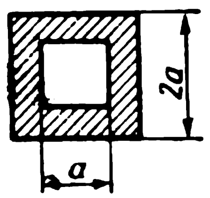

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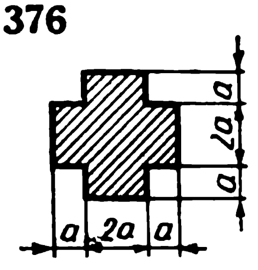

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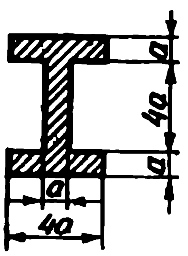

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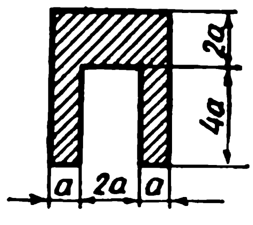

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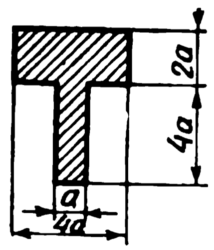
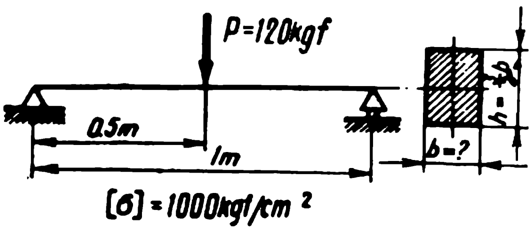


Notation: b, d, h, P_{max} and q_{max} are the required dimensions of the cross sections of beams and permissible loads for calculations based on the permissible stress; b', d', h', P'_{max}, q'_{max} are the same for calculations based on limit design.

380



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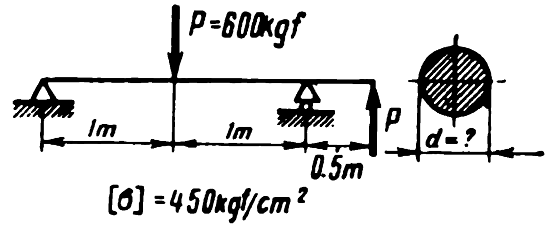

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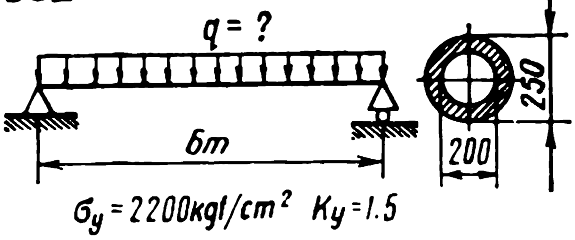

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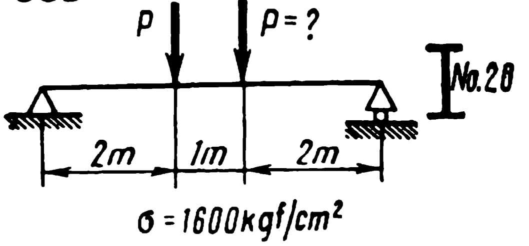

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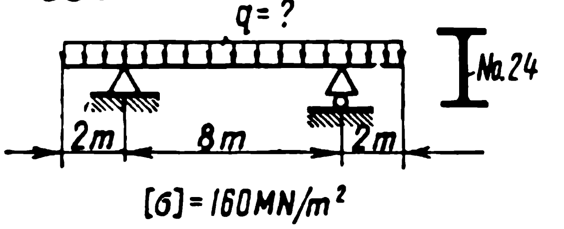

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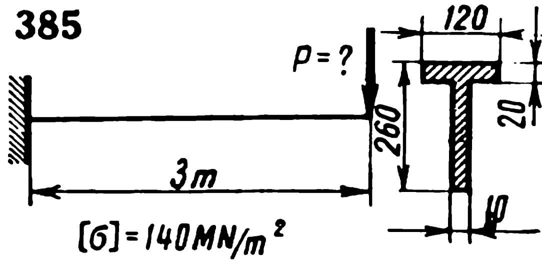


8.6.

Displacements in Bending

Displacements of cross sections of beams comprise:

(1) Linear displacements of the centres of gravity of the cross sections in a direction perpendicular to the beam geometric axis x of the beam. These are called *deflections* (f_x).

(2) Angular displacements of the cross sections about the neutral z-axis which are called *angles of rotation of the section* (θ_x).

The deflection f_x is considered positive if its direction coincides with the positive direction of the y-axis which is perpendicular to the geometric x-axis of the beam.

The angle of turn θ_x is considered positive if the cross section of the beam turns counterclockwise about the neutral z-axis.

The maximum and minimum deflections f_{max}, f_{min} and the angles of rotation θ_{max}, θ_{min} are assessed by their absolute values.

Since the transverse (shearing) force has an appreciable effect on the deformation of only short beams, f and θ are commonly determined as being due only to the bending moment.

Quantities f and θ can be determined by using the method of initial parameters, by the graphical, grapho-analytical and other methods.

METHOD OF INITIAL PARAMETERS

The initial parameters are the deflection f_0 and angle of rotation θ_0 of the cross section of the beam with the origin of coordinates located at the centre of gravity of the section.

It is practicable to have the origin of coordinates located at the centre of gravity of an end cross section of the beam.

The equations for f_x and θ_x in an arbitrary section of the beam at a distance x from the origin of coordinates (if the intensities of distributed loads q_x are power functions) can be written in the following form:

$$EIf_x = EIf_0 + EI\theta_0 \frac{x}{1!} + \sum M \frac{(x-a_m)^2}{2!} + \sum P \frac{(x-a_p)^3}{3!}$$
$$+ \sum q_{a_q} \frac{(x-a_q)^4}{4!} - \sum q_{b_q} \frac{(x-b_q)^4}{4!}$$
$$+ \sum q'_{a_q} \frac{(x-a_q)^5}{5!} - \sum q'_{b_q} \frac{(x-b_q)^5}{5!} + \ldots, \quad (116)$$

$$EI\theta_x = EI\theta_0 + \sum M \frac{(x-a_m)}{1!} + \sum P \frac{(x-a_p)^2}{2!}$$
$$+ \sum q_{a_q} \frac{(x-a_q)^3}{3!} - \sum q_{b_q} \frac{(x-b_q)^3}{3!}$$
$$+ \sum q'_{a_q} \frac{(x-a_q)^4}{4!} - \sum q'_{b_q} \frac{(x-b_q)^4}{4!} + \ldots \quad (117)$$

in which E = Young's modulus of elasticity of the beam material
I = moment of inertia of the beam cross section about neutral axis z
M = moments of external couples of forces
a_m = distances from the origin of coordinates to sections in which the couples are applied (Fig. 82*a*)

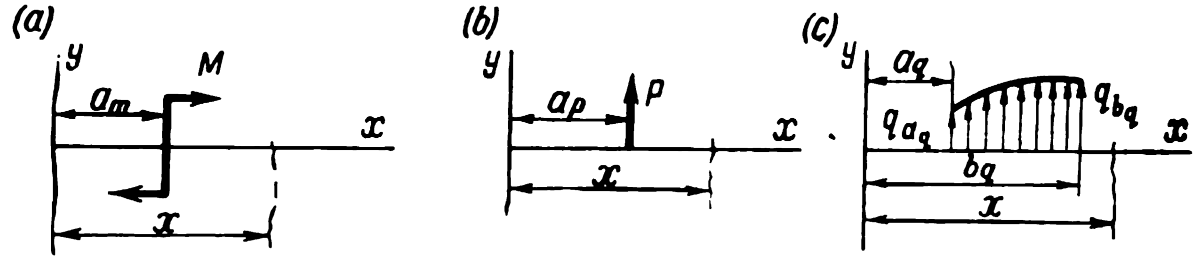


Fig. 82

P = concentrated transverse (shearing) forces (including reaction forces)
a_p = distances from the origin of coordinates along the beam axis to the points of application of forces P (Fig. 82*b*)

q_{a_q} and q'_{a_q} = magnitudes of q_x of the first, etc. derivatives, respectively, of q_x by x with $x = a_q$, i. e. for the cross sections from which the effect of the distributed forces begins (Fig. 82c)

q_{b_q} and q'_{b_q} = magnitudes of q_x of the first, etc. derivatives of q_x by x with $x = b_q$, i.e. for cross sections at which the effect of the distributed forces ends before the sections under consideration (Fig. 82c).

If the origin of coordinates is located at the centre of gravity of the right-hand end section ot the beam and the x-axis is directed to the left, the terms denoting the effect of the external couples in formulas (116) and (117) will be negative for the indicated direction of the loads in these formulas and the direction of rotation of the beam determined from formula (117) will be opposite to the previously assumed direction.

Of the two formulas only one (116) need be memorized; the other (117) can be obtained by differentiation.

The two initial parameters f_0 and θ_0 are determined from the following beam support conditions:

1. At a fixed end the deflection and angle of rotation of the section equal zero.
2. At hinged supports the deflections equal zero.

For symmetrical beams only one half of the beam is dealt with and the conditions of symmetry are made use of. These state that the angle

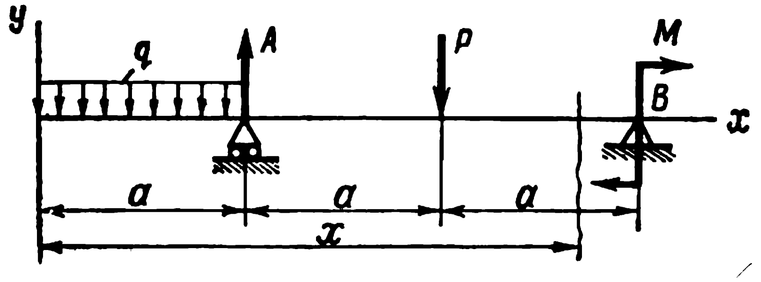


Fig. 83

of rotation of a section coinciding with the axis of symmetry of a beam, symmetrical about a vertical line, or the deflection of a section passing through the point of symmetry of a beam, symmetrical about a point, is equal to zero.

It is advisable to write only one equation (116) and one equation (117) for an arbitrary section of the last portion of the beam, taking the loads in the same order in which they occur beginning with the origin of the coordinates. Each of the equations is valid for determining f_x and θ_x in any portion of the beam, if in them the number of terms corresponding to the loads in the previous portions is marked off.

Example 42. Given: q, a, $P = 4qa$, $M = qa^2$ and EI (Fig. 83). Determine f_x and θ_x.

Solution. From the condition of statics $\sum M_B = 0$ we can write

$$-qa\times\frac{5}{2}a+A\times 2a-4qa\times a+qa^2=0;\quad A=\frac{11}{4}qa$$

From equation (116) for the last right-hand portion of the beam

$$EIf_x=\underbrace{\underbrace{\underbrace{EIf_0+EI\theta_0x-q\frac{x^4}{24}}_{\mathrm{I}}+\frac{11}{4}qa\frac{(x-a)^3}{6}+q\frac{(x-a)^4}{24}}_{\mathrm{II}}-4qa\frac{(x-2a)^2}{6}}_{\mathrm{III}}$$

For portion I: $0\leqslant x\leqslant a$.
For portion II: $a\leqslant x\leqslant 2a$.
For portion III: $2a\leqslant x\leqslant 3a$.
From the conditions of the beam supports:
for the left-hand support

$$EIf_a=EIf_0+EI\theta_0a-\frac{qa^4}{24}=0$$

for the right-hand support

$$EIf_{3a}=EIf_0+EI\theta_03a+\frac{88}{24}qa^4-\frac{2}{3}qa^4-\frac{81}{24}qa^4+\frac{16}{24}qa^4=0$$

i.e.

$$\left.\begin{aligned}&EIf_0+EI\theta_0a-\frac{qa^4}{24}=0;\\&EIf_0+3EI\theta_0a+\frac{7}{24}qa^4=0\end{aligned}\right\}$$

whence

$$2EI\theta_0a+\frac{8}{24}qa^4=0\quad\text{and}\quad\theta_0=-\frac{1}{6}\frac{qa^3}{EI}$$

After substituting

$$EIf_0-\frac{1}{6}qa^4-\frac{1}{24}qa^4=0$$

Hence

$$f_0=\frac{5}{24}\frac{qa^4}{EI}$$

The equations determining f_x and θ_x in any section of any portion thus become

$$EIf_x=\frac{5}{24}qa^4-\frac{1}{6}qa^3x-\frac{1}{24}qx^4+\frac{11}{24}qa(x-a)^3+\frac{1}{24}q(x-a)^4$$

$$0\leqslant x\leqslant a\,|\qquad\qquad a\leqslant x\leqslant 2a\,|$$

$$-\frac{2}{3}qa(x-2a)^3$$

$$2a\leqslant x\leqslant 3a\,|$$

$$EI\theta_x = -\frac{1}{6}qa^3 - \frac{1}{6}qx^3 \Big|_{0\leqslant x\leqslant a} + \frac{11}{8}qa(x-a)^2 + \frac{1}{6}q(x-a)^3 \Big|_{a\leqslant x\leqslant 2a} - 2qa(x-2a)^2 \Big|_{2a\leqslant x\leqslant 3a}$$

For example,
the angle of rotation of the section directly over the left-hand support is

$$\theta_a = \frac{1}{EI}\left(-\frac{1}{6}qa^3 - \frac{1}{6}qa^3\right) = -\frac{qa^3}{3EI}$$

the angle of rotation of the section directly over the right-hand support is

$$\theta_{3a} = \frac{qa^3}{EI}\left(-\frac{1}{6} - \frac{27}{6} + \frac{44}{8} + \frac{8}{6} - 2\right) = \frac{1}{6}\frac{qa^3}{EI}$$

and the deflection of the beam directly under the concentrated force is

$$f_{2a} = \frac{qa^4}{EI}\left(\frac{5}{24} - \frac{2}{6} - \frac{16}{24} + \frac{11}{24} + \frac{1}{24}\right) = -\frac{7}{24}\frac{qa^4}{EI}$$

Fig. 84 shows the approximate shape of the elastic line, or curve, of the beam (dashed line) and the calculated deflections and angles of rotation of the sections.

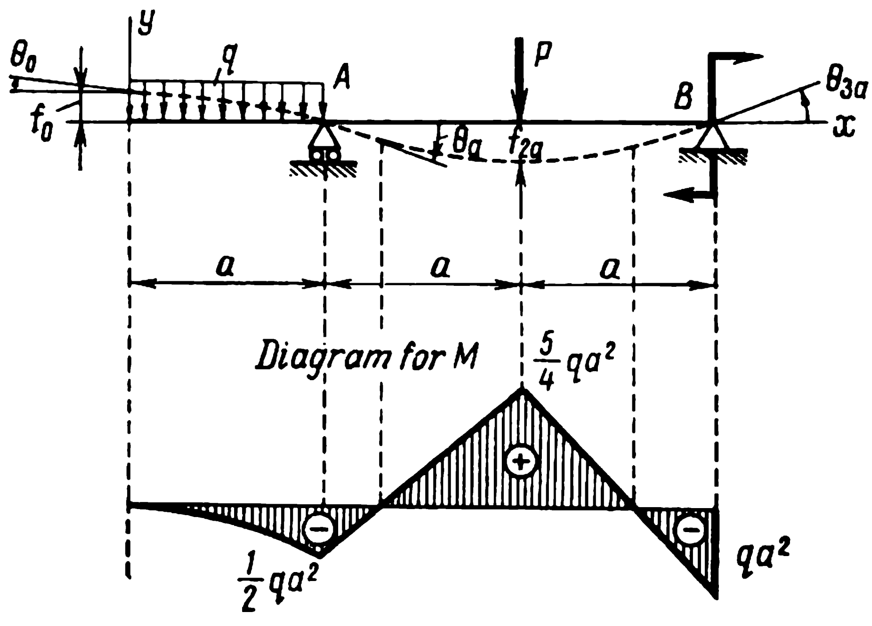


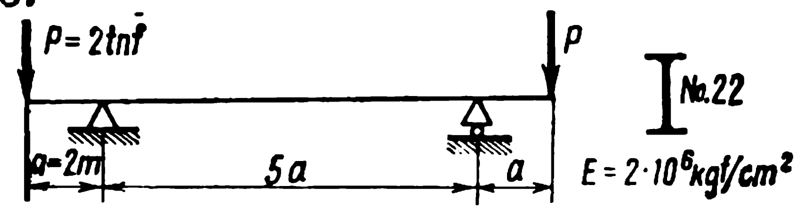
Fig. 84

It should be borne in mind that at sections of the beam in which the bending moment equals zero the elastic line must have an inflection point. At portions of the beam with a positive bending moment the

convexity of the elastic line faces downward, and at those with a negative bending moment, it faces upward (as in Fig. 84).

Problems 386 and 387. In Problem 386 determine deflections f and angles of rotation θ of sections of beams by integrating the differential equations of the elastic line, using the data and drawings of the following problems:

387



(1) in Problems 222 through 226 determine the deflections and angles of rotation of the free ends;

(2) in Problems 228, 229 and 231 determine the deflections at the middle of the spans and angles of rotation at the supports;

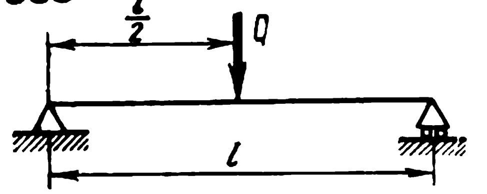
(3) in Problems 227 and 230 determine the maximum deflections and angles of rotation (absolute values) over the supports.

The loads, lengths and rigidities of the cross sections of the beams are assumed to be known.

In Problem 387 determine by how many per cent and in what direction the deflection at the middle of the span of the beam shown in the figure differs, when found by the approximate equation of the elastic line, from the deflection found by exact calculations based on the equation of a circular arc.

Problems 388 through 391. Determine by how many per cent the maximum bending moment and deflection for beams illustrated in Figs. 388 through 391 are greater than for similar beams on two supports subject to the same load Q but which is uniformly distributed throughout the beam length.

388



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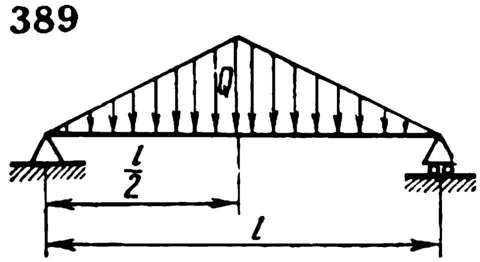

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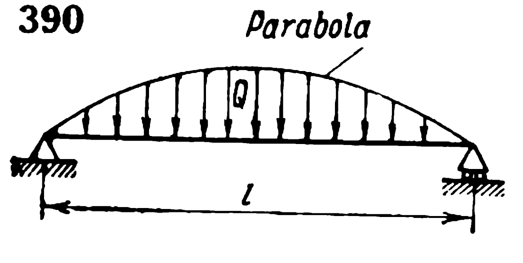

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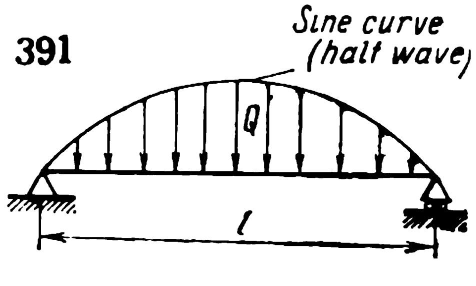

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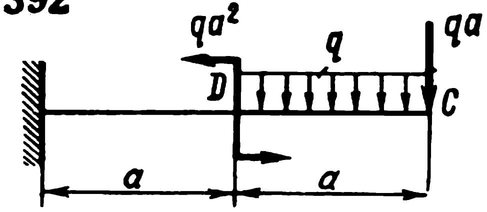

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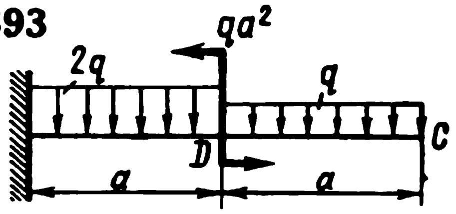

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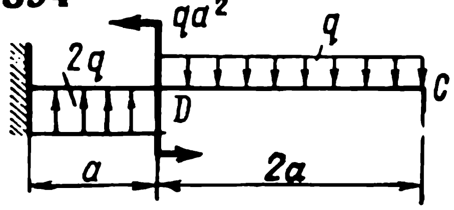

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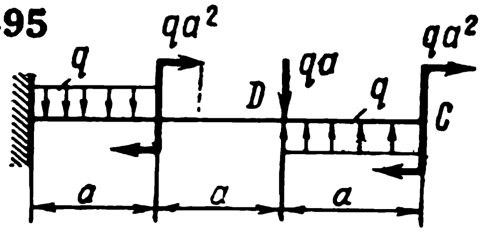

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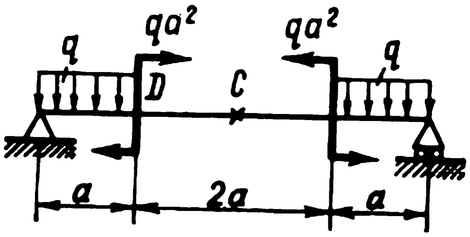

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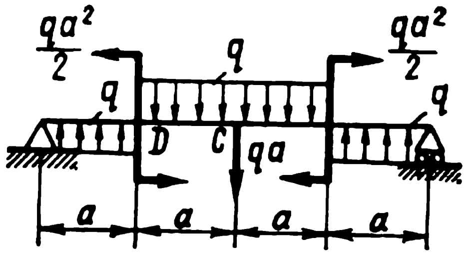

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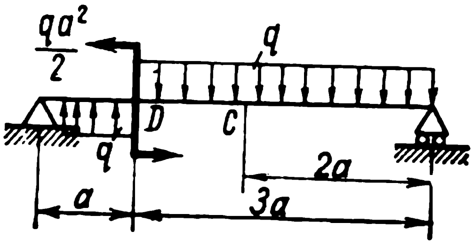

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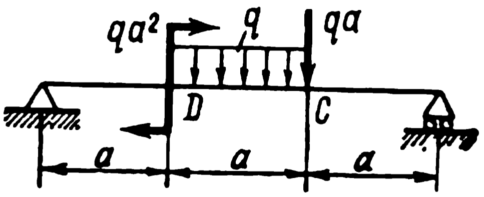

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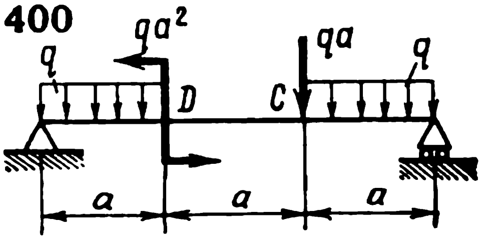

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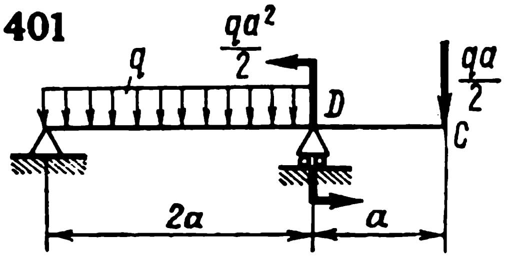

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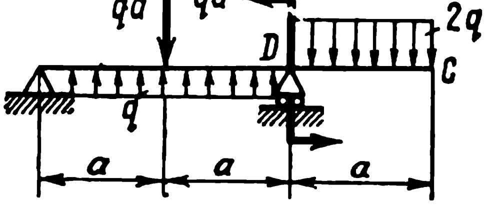

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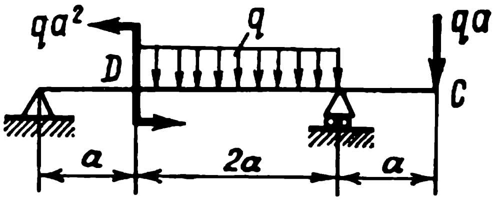

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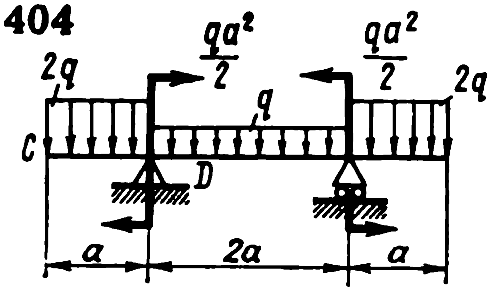

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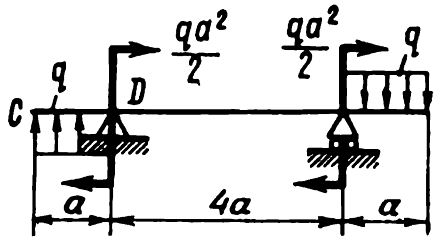

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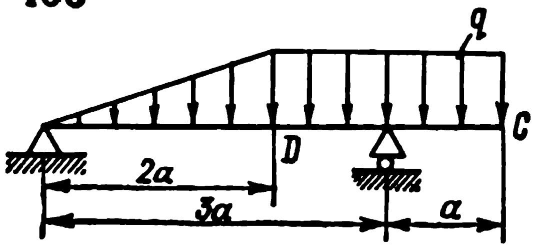

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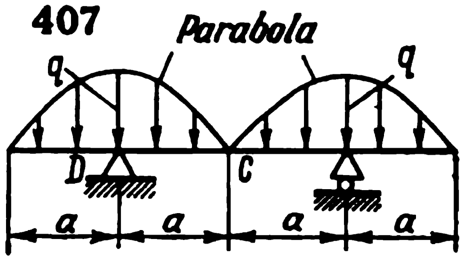

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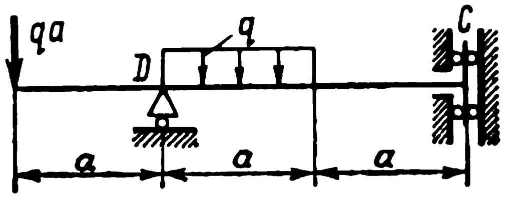

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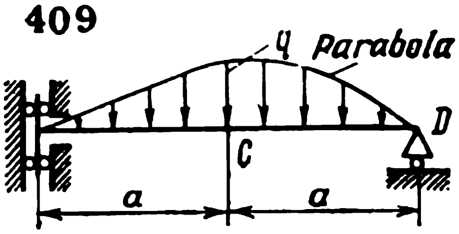


Problems 392 through 409. Determine deflections f_c of sections C and angles of rotation θ_D of sections D of the beams shown in the figures.

Assume q, a, E and I to be known.

GRAPHO-ANALYTICAL METHOD

First the diagram of the bending moment for the given beam is plotted. Then we assume that this diagram is a fictitious distributed load of a fictitious beam. The angle of rotation θ_x of any section of the given beam is determined as the ratio of transverse (shearing) force Q_{f_x} in the same section of the fictitious beam to the rigidity of the section of the given beam, i.e.

$$\theta_x = \frac{Q_{f_x}}{EI} \tag{118}$$

The deflection f_x of any section of the given beam is determined as the ratio of the bending moment M_{f_x} in the same section of the fictitious beam to rigidity EI of the given beam, i.e.

$$f_x = \frac{M_{f_x}}{EI} \tag{119}$$

According to formulas (118) and (119), the strain conditions at the supported points and at the boundaries of the given beam become conditions for Q_f and M_f of the fictitious beam.

To obtain a fictitious beam corresponding to the given beam, the following rules should be observed:

(1) A support at the end of the given beam remains a support at the end of the fictitious beam.

(2) A support which is not at the end of the given beam becomes an unsupported hinge joint of the fictitious beam.

(3) A fixed end of the given beam becomes a free end of the fictitious beam.

(4) A free end of the given beam becomes a fixed end of the fictitious beam.

(5) An unsupported hinge joint of the given beam becomes a support of the fictitious beam.

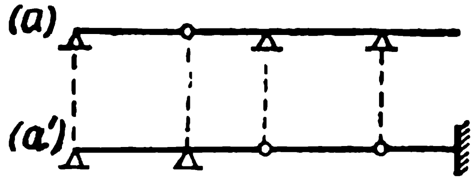


Fig. 85

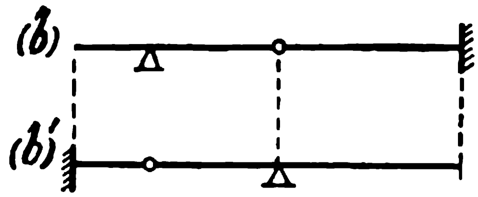


Fig. 86

The use of the rules for plotting fictitious beams is illustrated in Figs. 85 and 86 in which *a* and *b* are the given beams, *a'* and *b'* are the respective fictitious beams.

The grapho-analytical method is convenient for determining displacements of various sections of a beam when it is easy to find the areas and centres of gravity of the parts of the moment diagram due to the given load.

To conform to the adopted sign conventions for deflection and angle of rotation of the section it is necessary to assume that the positive bending moment diagram for the given beam to be the fictitious load acting upwards on the fictitious beam and the negative bending moment diagram for the given beam to be the fictitious load acting downwards on the fictitious beam.

Example 43. Given: P, l, E and I (Fig. 87).

Find θ_C, θ_A, f_A.

Solution. First we construct the bending moment diagram for the given beam ABC. It is a triangle vertex downward with the height Pl at section B.

Next we assume this diagram to be the fictitious load for the fictitious beam $A'B'C'$.

From the sum of moments with respect to the unsupported hinge joint B' (of forces to the right of it) we find the reaction

$$C' = \frac{1}{l} Pl \frac{l}{2} \times \frac{l}{3} = \frac{Pl^2}{6}$$

Hence

$$Q_{fC'} = -C' = -\frac{Pl^2}{6}$$

Since

$$Q_{f_{A'}} = -\frac{Pl^2}{6} + Pll = \frac{5}{6}Pl^2$$

and

$$M_{f_{A'}} = \frac{Pl^2}{6}2l - Pl^2l = -\frac{2}{3}Pl^3$$

then

$$\theta_C = \frac{Q_{fC'}}{EI} = -\frac{Pl^2}{6EI}; \quad \theta_A = \frac{Q_{f_{A'}}}{EI} = \frac{5}{6}\frac{Pl^2}{EI};$$

$$f_A = \frac{M_{f_{A'}}}{EI} = -\frac{2}{3}\frac{Pl^3}{EI}$$

Example 44. Given: M, a, E and I (Fig. 88). Find f_{max}.

Solution. The fictitious beam is a beam on two supports with a fictitious uniformly distributed load of intensity M on the second portion.

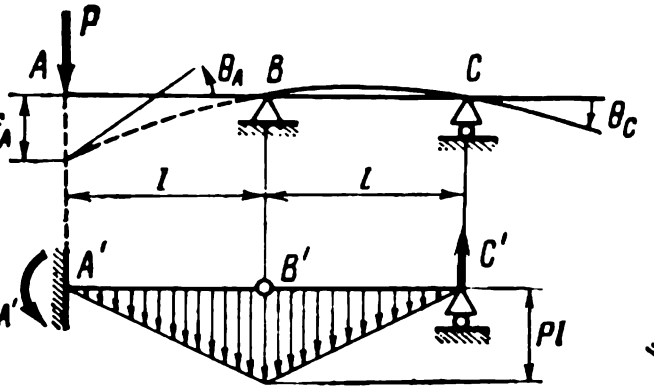


Fig. 87

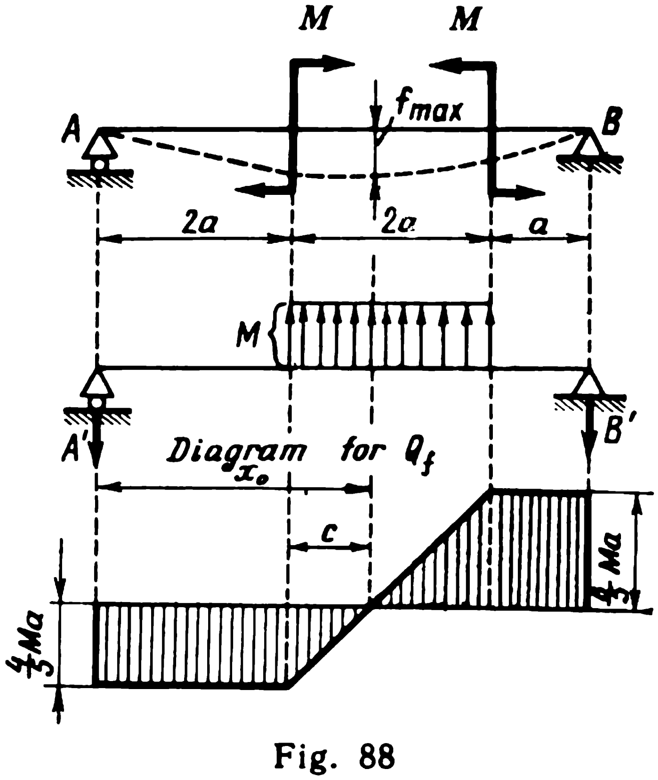


Fig. 88

From the sum of moments of forces with respect to supports A' and B' we find the fictitious reactions

$$B' = \frac{6}{5}Ma \text{ and } A' = \frac{4}{5}Ma$$

The diagram for Q_f is shown in Fig. 88. At the section in which $Q_f = 0$, $M_f = M_{f_{max}}$.

Since

$$|Q_f| = A' - Mc = \frac{4}{5}Ma - Mc = 0$$

then

$$c=\frac{4}{5}a$$

Hence

$$x_0=2a+c=\frac{14}{5}a$$

Next we find $M_{f_{\max}}$:

$$M_{f_{\max}}=-A'x_0+M\frac{c^2}{2}=-\frac{4}{5}Ma\frac{14}{5}a+\frac{M}{2}\times\frac{16}{25}a^2$$

$$=-\frac{48}{25}Ma^2$$

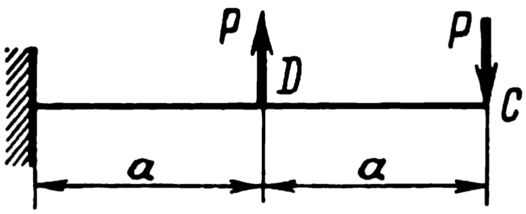
Then the maximum deflection is

$$f_{\max}=\frac{M_{f_{\max}}}{EI}=-\frac{48}{25}\frac{Ma^2}{EI}$$

Problems 410 through 421. Find deflections f_c of sections C and angles of rotation θ_D of sections D of the beams.

Assume that P, q, M, a, E and I are known both in this and in the subsequent paragraphs.

410



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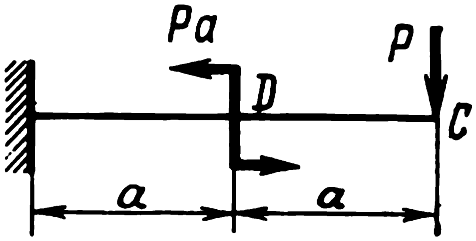

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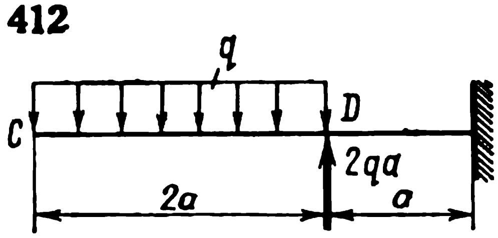

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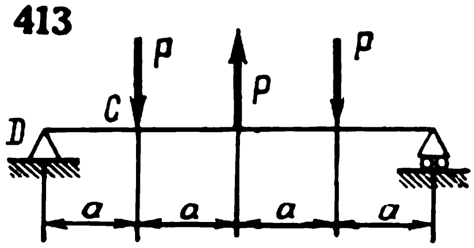

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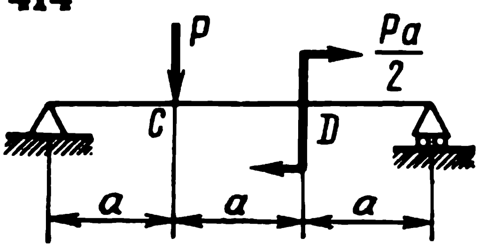

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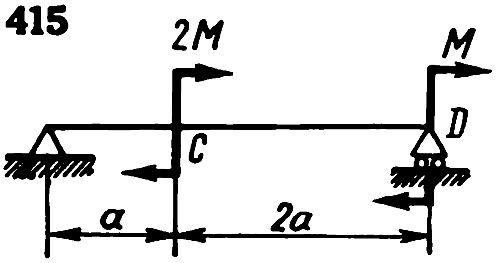

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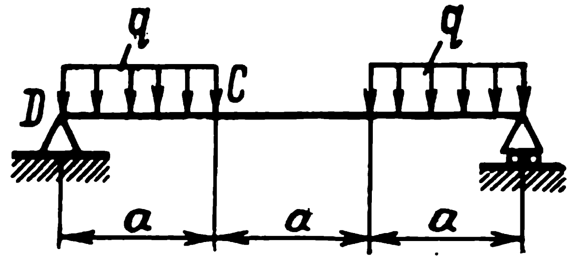

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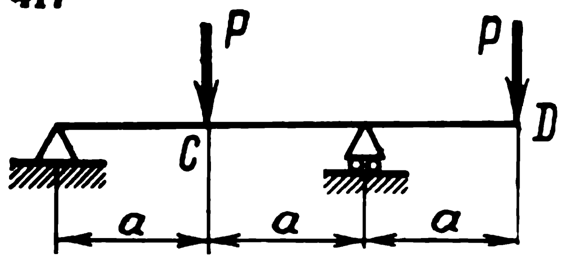

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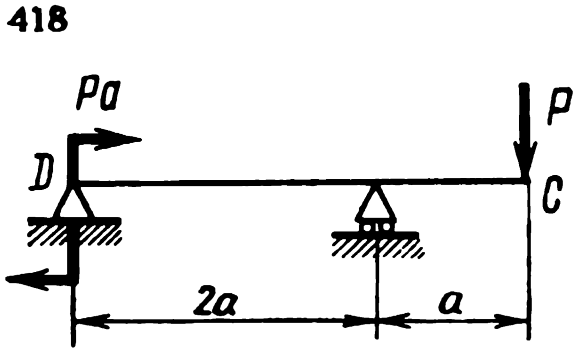

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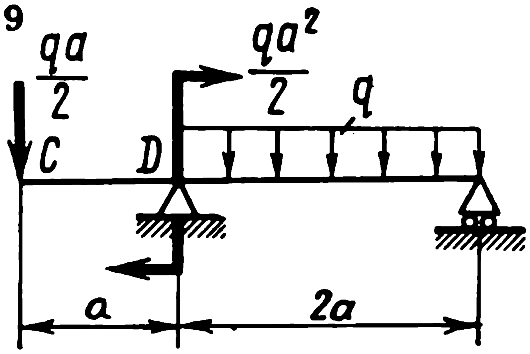

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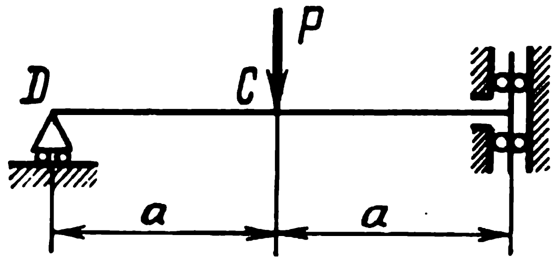

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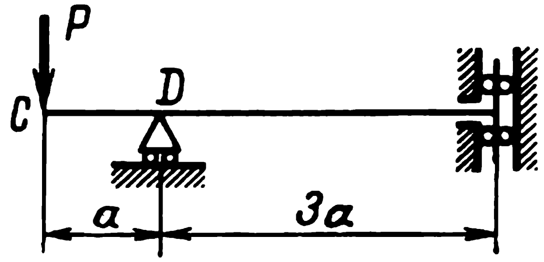
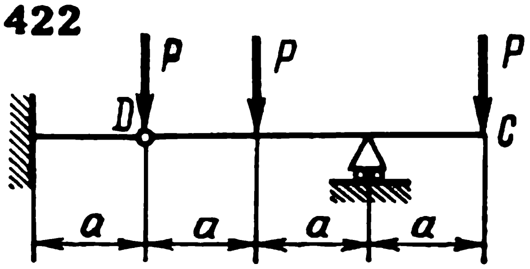


Problems 422 through 425. Find deflections f_c of sections C and angles of rotation θ_{D_1} and θ_{D_2} of the sections to the left and to the right of hinge D of the beams.

422



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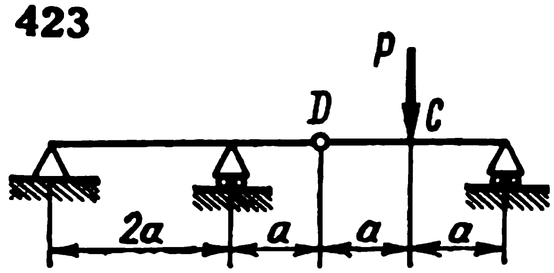

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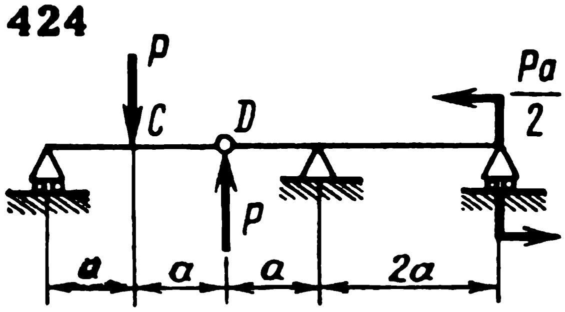

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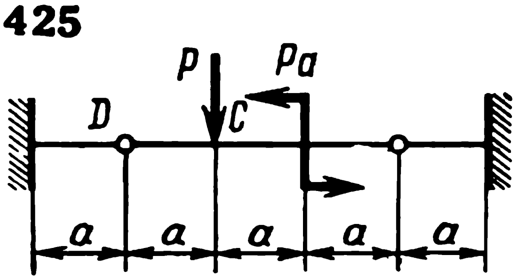
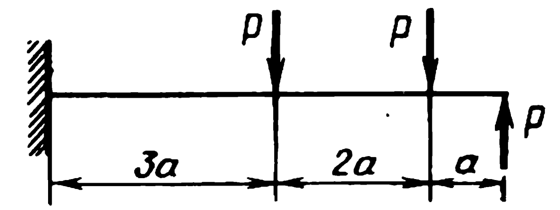


Problems 426 through 433. Find the maximum deflections $|f|_{max}$ and angles of rotation $|\theta|_{max}$ (in absolute value) of the beam sections. Plot the elastic line of the beams.

426



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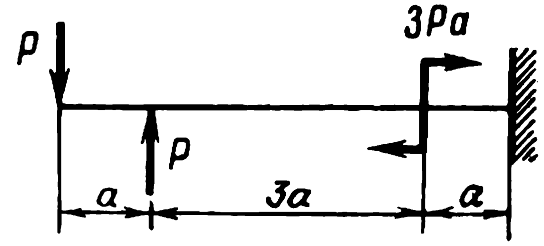

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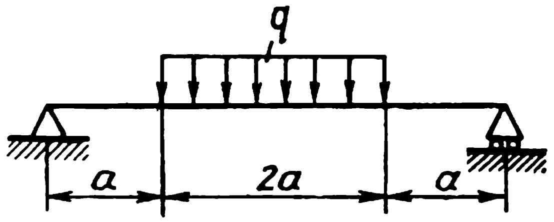

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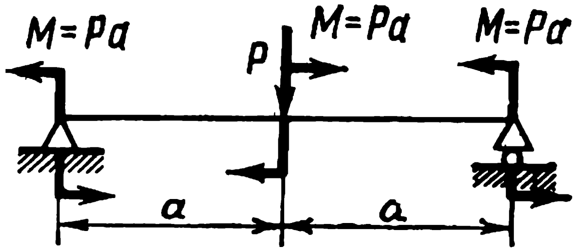

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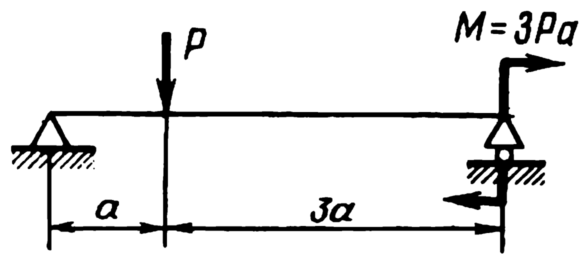

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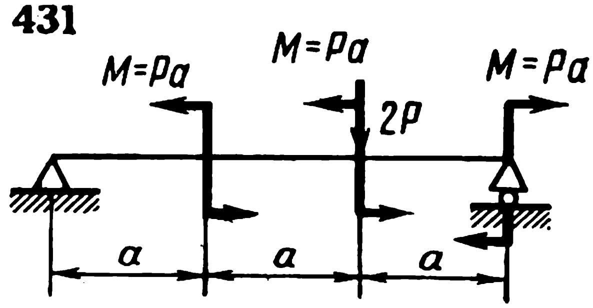

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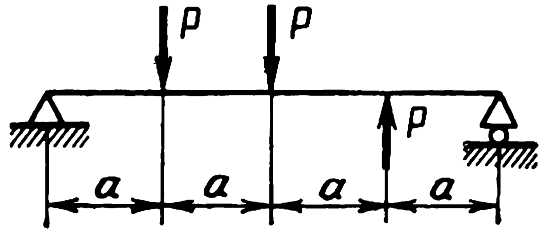

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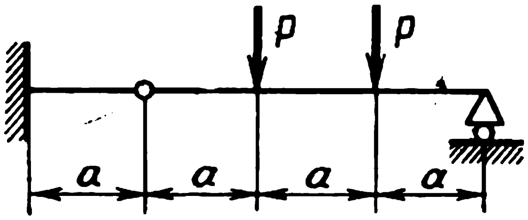
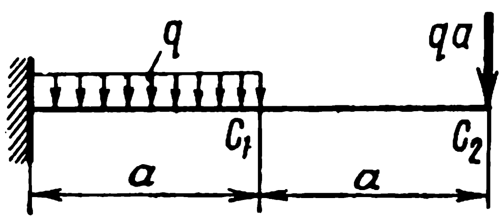


Problems 434 through 443. Find deflections f, f_1 and f_2 of sections C, C_1 and C_2 of the beams, using the tabulated values for the deflections and angles of rotation of the simplest types of beams (see Fig. 99).

434



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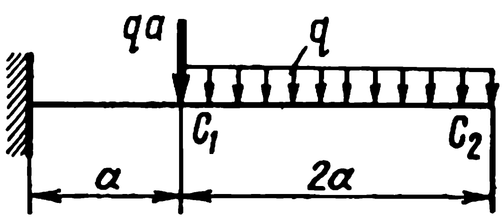

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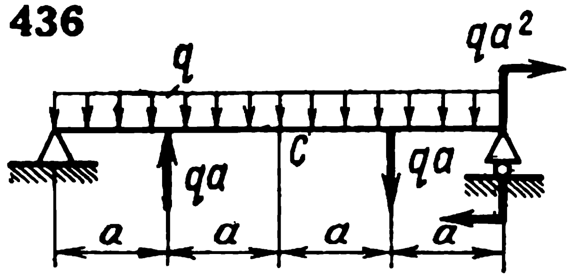

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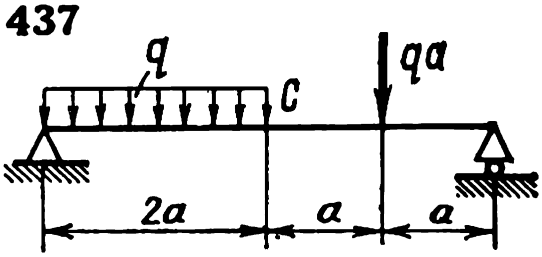

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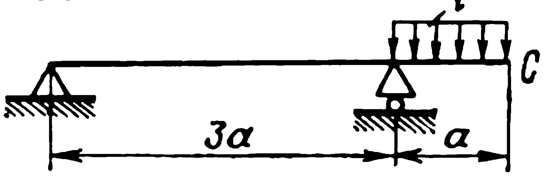

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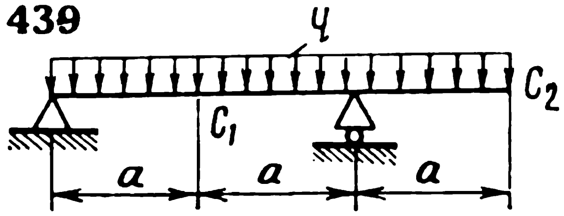

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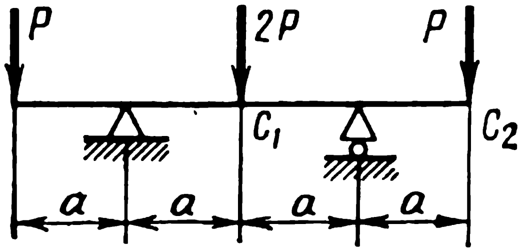

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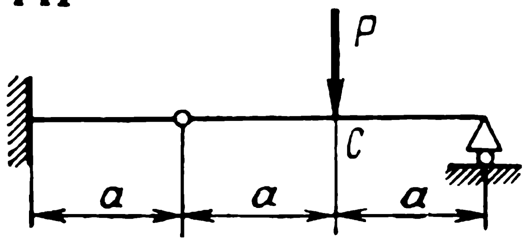

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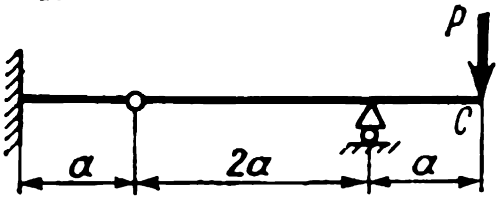

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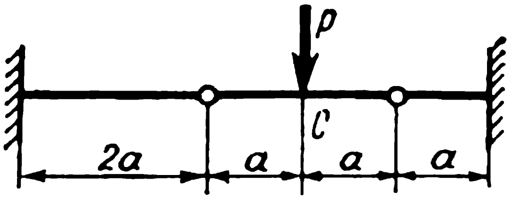
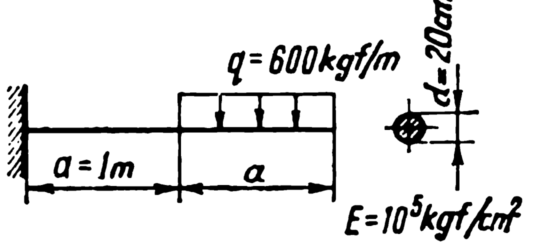


Problems 444 through 448. Using any method, find the maximum (absolute) values for the deflections $|f|_{max}$ of the beams.

444



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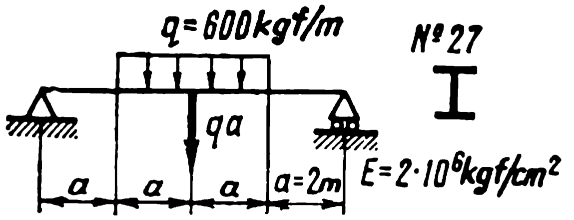

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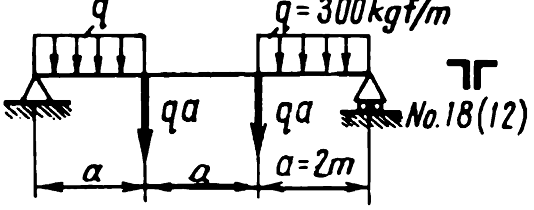

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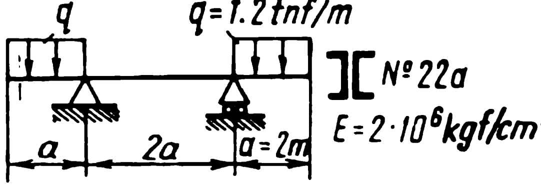

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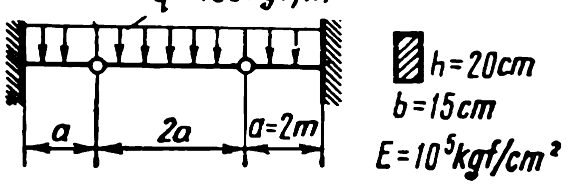
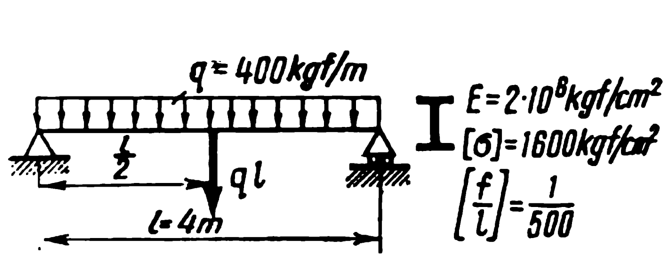


Problems 449 through 453. Determine the dimensions of the cross sections of the beams which satisfy the conditions of strength and rigidity.

449



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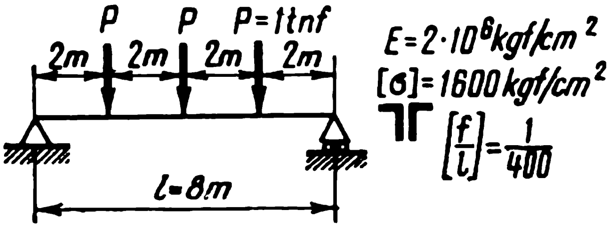

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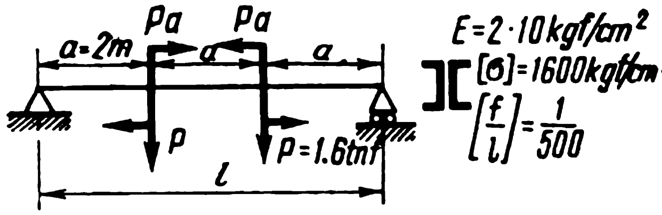

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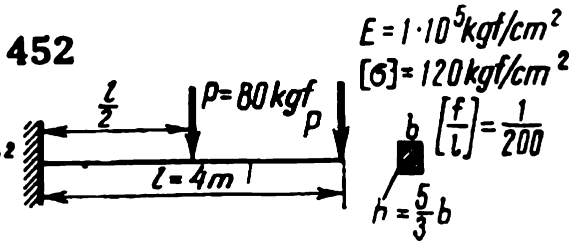

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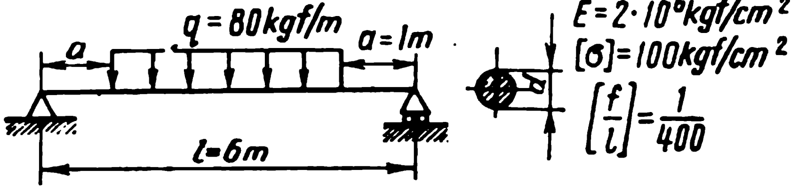
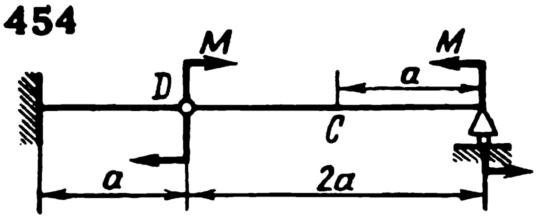


Problems 454 through 457. Find the ratio of deflections of sections C of the beams depending on the point of application of moment M (either to the left or to the right of hinge D).

454



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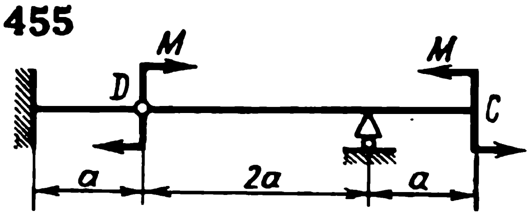

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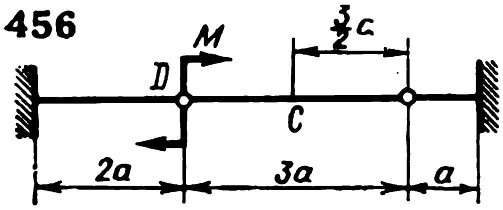

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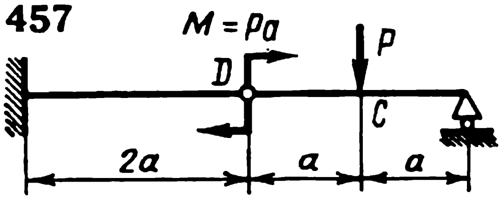


GRAPHICAL METHOD

If a graphical diagram of a fictitious bending moment M_{f_x} and a fictitious transverse (shearing) force Q_{f_x} is drawn for a fictitious beam, the funicular curve (approximate funicular polygon) will represent the elastic line of the given beam, while the shearing force line will represent the variation in the angle of rotation of the section.

The vertical distances y_x between the funicular curve and its closing string will be proportional to deflections f_x of the given beam, and the vertical distances y_x between the line of the fictitious shearing force and the line of its zero values will be proportional to the angles of rotation of the sections θ_x.

Since

$$f_x = \frac{M_{f_x}}{EI} = y_x \frac{H(\xi\eta)}{EI} \tag{120}$$

and

$$\theta_x = \frac{Q_{f_x}}{EI} = y'_x \frac{(\eta)}{EI} \tag{121}$$

in which H is the pole distance of the vector diagram, $\frac{1}{\xi}$ is the linear scale of the beam, $\frac{1}{\eta}$ is the scale of vectors determining the areas of parts of the bending moment diagram for the given beam and EI is the rigidity of the cross section of the latter beam, then the scale of the deflections will be $\frac{EI}{H(\eta\xi)}$ and the scale of the angle of rotation will be $\frac{EI}{\eta}$. Quantities H, ξ and η should be selected so as to ensure a sufficiently accurate measurement of y_x and y'_x on the drawing.

Commonly only the elastic line of the beam is plotted graphically. In the graphical method the fictitious load may be applied more conveniently to the given beam instead of the fictitious one. Then, in order to draw the closing string (or strings) of the funicular polygon, the following rules should be complied with, depending on the kinds of supports of the beam:

(1) at a fixed end the closing string is tangent to the funicular polygon;

(2) at a hinged support the closing string intersects the funicular polygon;

(3) at an unsupported hinge joint the closing string is a broken line (the closing strings intersect at one point in approaching the unsupported hinge joint from the left and from the right).

Fig. 89 illustrates the use of these rules for drawing the closing string (or strings) in which (a) is the given beam; (b) is the funicular curve and (c) is the closing string.

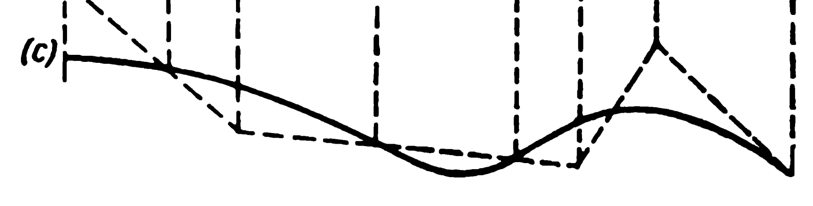


Fig. 89

Examples 45 and 46 illustrate the graphical construction of the elastic line of beams.

Example 45.

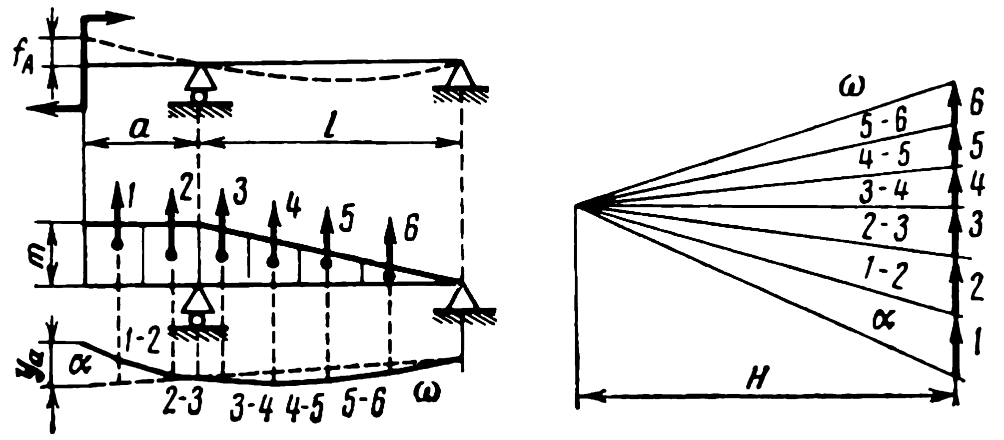


Fig. 90

Example 46.

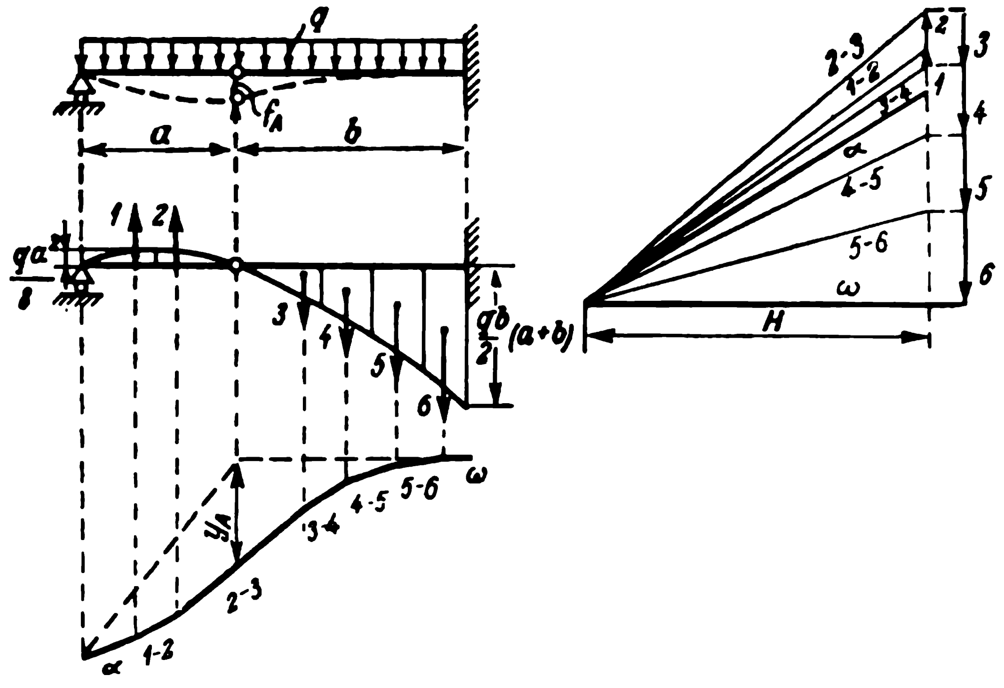


Fig. 91

In certain cases of beam design the condition of strength is supplemented by the condition of sufficient rigidity. The latter is expressed by the ratio of the absolute value of the maximum deflection $|f|_{max}$ to the beam span l which must not exceed a given value $\frac{1}{n}$. Thus

$$\frac{|f|_{max}}{l} \leqslant \frac{1}{n} \tag{122}$$

Depending on the purpose of the beam and its material, the permissible value of $\frac{1}{n}$ varies. For example, for steel beams this value may be in the range:

$$\frac{1}{n}=\frac{1}{200} \text{ to } \frac{1}{1000}$$

8.7.

Beams of Variable Cross Section

If the cross section of a beam varies gradually and only slightly over its length, the stresses can be found from the formulas for beams of a constant cross section.

Of a more rational design are beams of constant strength, sometimes called fish-bellied beams, in which the normal stresses are the same in the extreme fibres of each cross section.

DESIGNING BEAMS OF CONSTANT STRENGTH

For beams subject to pure bending a beam of constant cross section is, at the same time, a beam of constant strength. In the general case, however, in bending, such a beam is of variable cross section satisfying the equation

$$W_x=\frac{|M_x|}{[\sigma]} \tag{123}$$

in which M_x = bending moment in an arbitrary cross section
W_x = section modulus of this cross section.

If in some cross section of the beam the bending moment equals zero or is very small, whereas the shearing force is not zero or is large, the shape of the beam of constant strength determined from equation (123) is modified by the condition of strength (104) based on the shearing stresses.

Given below are examples of the design of beams of constant strength.

Example 47. Given: P, l, h = const, $[\sigma]$ and $[\tau]$ (Fig. 92). Determine b_x.

Solution. The bending moment in an arbitrary cross section is

$$M_x=\frac{Px}{2}$$

The section modulus of this cross section equals

$$W_x=\frac{b_xh^2}{6}$$

According to equation (123)

$$\frac{b_x h^2}{6} = \frac{Px}{2[\sigma]}$$

whence

$$b_x = \frac{3Px}{h^2[\sigma]}; \quad b_{x=0} = 0; \quad b_{x=\frac{l}{2}} = \frac{3}{2}\frac{Pl}{h^2[\sigma]}$$

Since in the cross sections directly over the supports where $M = 0$, $Q = \frac{P}{2}$. Hence the cross section cannot be of zero width. The width b_0 can be found from the condition of strength based on the shearing stresses:

$$\tau_{max} = \frac{3}{2}\frac{Q}{F} = \frac{3}{4}\frac{P}{b_0 h} \leqslant [\tau]$$

from which

$$b_0 \geqslant \frac{3}{4}\frac{P}{h[\tau]}$$

The shape of the beam of constant strength is shown in Fig. 92.

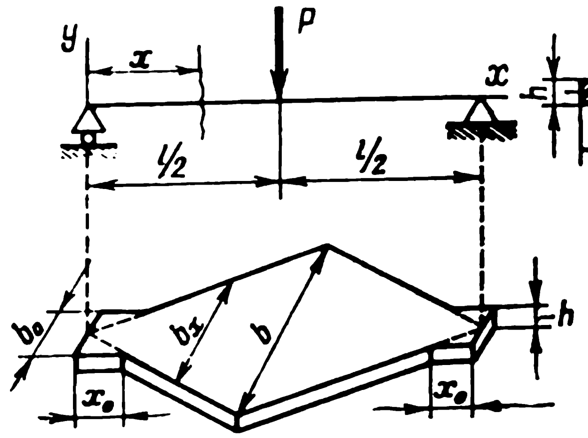


Fig. 92

The length x_0 of the ends of the beam having a constant width b_0 is found by the condition

$$b_0 = \frac{3}{4}\frac{P}{h[\tau]} = \frac{3Px_0}{h^2[\sigma]}$$

from which

$$x_0 = \frac{h}{4}\frac{[\sigma]}{[\tau]}$$

Since $[\tau] = (0.5 \text{ to } 0.6)\,[\sigma]$, then $x_0 \cong (0.5 \text{ to } 0.42)\,h$.

Example 48. Given: P, l, $[\sigma]$ and $[\tau]$ (Fig. 93). Determine d_x.

Solution. $M_x = -Px$; $W_x = \frac{\pi d_x^3}{32}$.

From equation (118) we find

$$\frac{\pi d_x^3}{32} = \frac{Px}{[\sigma]}$$

Hence

$$d_x = 2\sqrt[3]{\frac{4Px}{\pi[\sigma]}}; \quad d_{x=0} = 0; \quad d_{x=l} = 2\sqrt[3]{\frac{4Pl}{\pi[\sigma]}}$$

Since at $x = 0$, $M = 0$ and $Q = -P$, the shape of the beam should be modified according to the condition of strength based on the shear-

ing stresses. Thus

$$\tau_{max} = \frac{4}{3}\frac{|Q|}{F} = \frac{4P}{3\frac{\pi d_0^2}{4}} \leqslant [\tau]$$

from which

$$d_0 \geqslant 4\sqrt{\frac{P}{3\pi[\tau]}}$$

The required shape of the beam of constant strength is shown in Fig. 93*a*.

The length x_0 of the end of the beam having a constant diameter d_0 can be found by the condition

$$d_0 = 4\sqrt{\frac{P}{3\pi[\tau]}} = 2\sqrt[3]{\frac{4Px_0}{\pi[\sigma]}}$$

from which

$$x_0 = \frac{a_0}{6}\frac{[\sigma]}{[\tau]} \cong (0.33 \text{ to } 0.26)\,d_0$$

If a beam of constant strength is of very complex shape, a stepped beam can be made with a shape that is circumscribed about the beam

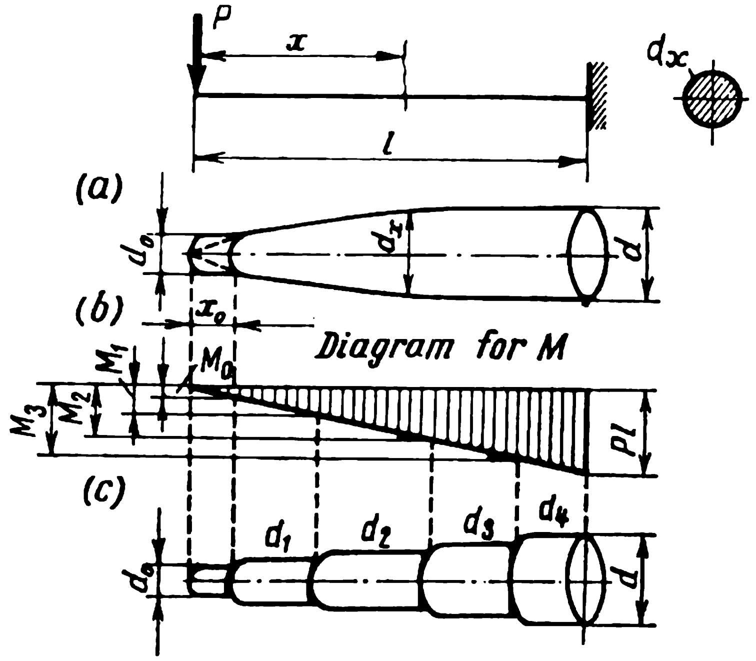


Fig. 93

of constant strength. The diameters d_0 and d being known, arbitrary diameters are selected between them, for example

$$d_0 < d_1 < d_2 < d_3 < d$$

Then we determine

$$W_0 = \frac{\pi d_0^3}{32}; \quad W_1 = \frac{\pi d_1^3}{32}; \quad W_2 = \frac{\pi d_2^3}{32}; \quad W_3 = \frac{\pi d_3^3}{32}$$

and

$$M_0 = W_0\,[\sigma];\quad M_1 = W_1\,[\sigma];\quad M_2 = W_2\,[\sigma];\quad M_3 = W_3\,[\sigma]$$

at which d_0, d_1, d_2 and d_3 provide for sufficient strength.

Finally, by laying off M_0, M_1, M_2 and M_3 on the bending moment diagram (Fig. 93*b*) we can find the length of the respective steps of the beam (Fig. 93*c*).

DETERMINING THE DISPLACEMENTS OF BEAMS OF VARIABLE CROSS SECTION

Displacements of variable-section beams can be determined by the use of the analytical, grapho-analytical or graphical method.

In using the *analytical method*, a differential elastic line equation is set up for each portion of the beam:

$$Ey'' = \frac{M_x}{I_x} \tag{124}$$

in which I_x is the variable moment of inertia of the cross-sectional areas of the beam with respect to the neutral axis.

The functions for the angle of rotation θ_x and deflection f_x are obtained by double integration. Thus

$$E\theta_x = Ey' = \int \frac{M_x}{I_x}\,dx + C_1 \tag{125}$$

and

$$Ef_x = Ey = \int \left[\int \frac{M_x}{I_x}\,dx\right] dx + C_1x + C_2 \tag{126}$$

The constants C_1 and C_2 of integration are determined from the conditions of support of the beam. For several portions use is also made of boundary conditions, viz. the conditions of equality of the angles of rotation and of the deflections in approaching the section boundaries from the left and from the right. The portions should be demarcated both by the changes in loading and by the different laws of variation of the cross sections of the beam.

Use can be made of the following equation in place of equation (124):

$$EI_0y'' = M_x\frac{I_0}{I_x} = M_r \tag{127}$$

in which I_0 = moment of inertia of a constant cross section to which the beam is conventionally reduced

$M_r = M_x\frac{I_0}{I_x}$ = reduced bending moment.

Example 49. Given: P, l, h = const (rectangular beam of constant strength with constant height h and variable width b_x—see Example 47) and E (Fig. 94).

Find θ_{max} and f_{max}.
Solution. In accordance with Example 47

$$M_x = \frac{P}{2}x \quad \text{and} \quad I_x = \frac{b_x h^3}{12} = \frac{Phx}{4[\sigma]}$$

Therefore equation (124) becomes

$$Ey'' = \frac{M_x}{I_x} = \frac{2[\sigma]}{h} \quad \text{or} \quad \frac{Eh}{2[\sigma]}y'' = 1$$

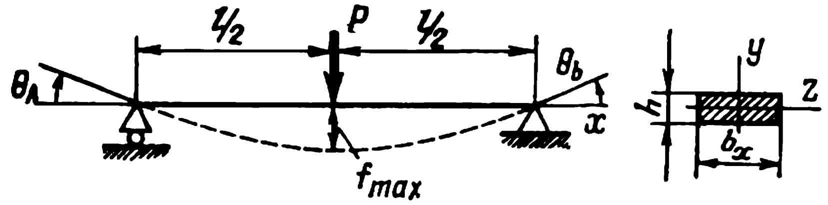


Fig. 94

We integrate this equation twice and obtain

$$\frac{Eh}{2[\sigma]}y' = x + C_1 \quad \text{and} \quad \frac{Eh}{2[\sigma]}y = \frac{x^2}{2} + C_1 x + C_2$$

Since at $x = 0$, $y = 0$ and at $x = \frac{l}{2}$, $y' = 0$, then $C_2 = 0$ and $C_1 = -\frac{l}{2}$.

Therefore

$$\theta_{max} = \theta_A = -\theta_B = (y')_{x=0} = \frac{2C_1[\sigma]}{Eh} = -\frac{l[\sigma]}{Eh}$$

and

$$f_{max} = (y)_{x=\frac{l}{2}} = -\frac{l^2[\sigma]}{4Eh}$$

It follows from Example 47 that

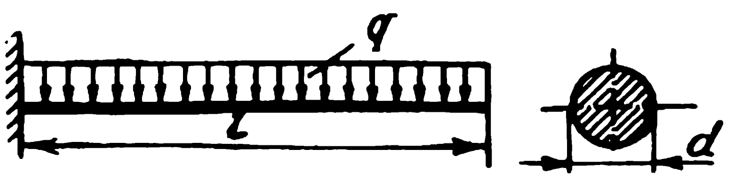
$$[\sigma] = \frac{3}{2}\frac{Pl}{h^2 b}$$

hence

$$\theta_{max} = -\frac{3}{2}\frac{Pl^2}{Eh^3 b} \quad \text{and} \quad f_{max} = -\frac{3}{8}\frac{Pl^3}{Eh^3 b}$$

Problems 458 through 466. Find the shapes of beams of constant strength and the maximum deflections (absolute values).

458



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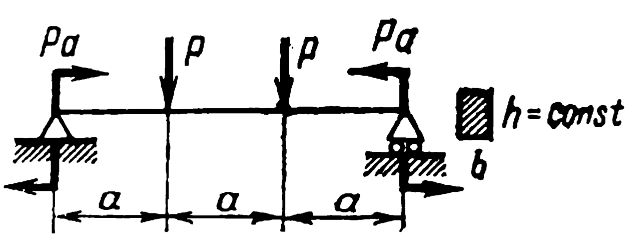

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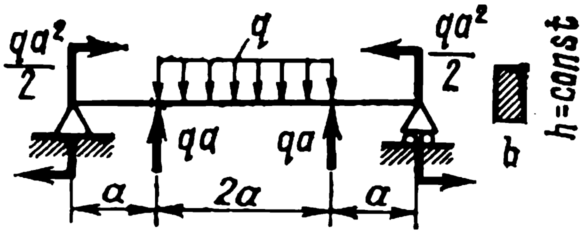

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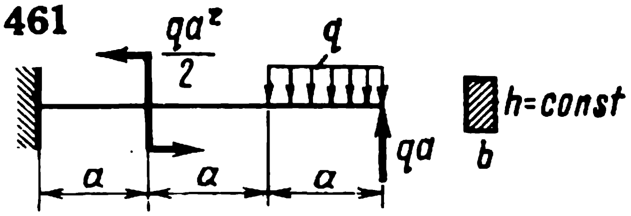

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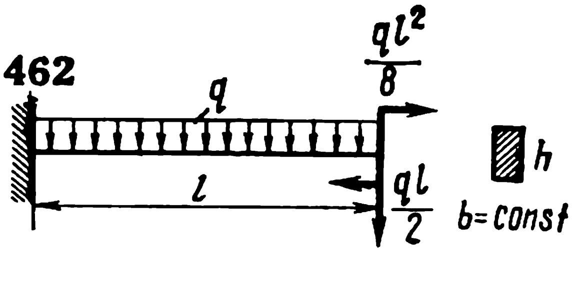

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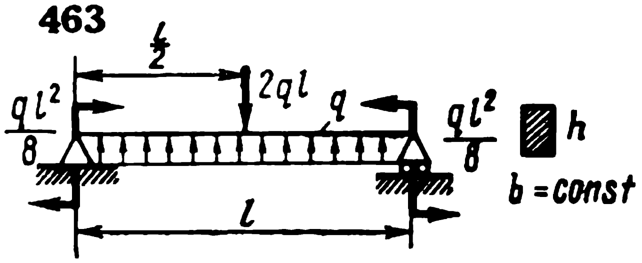

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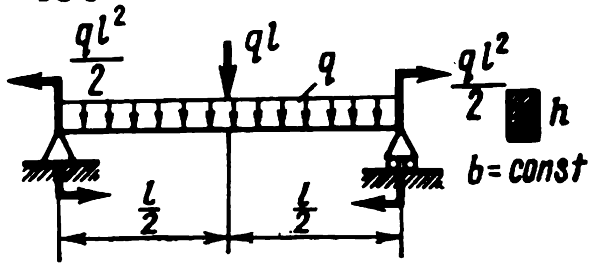

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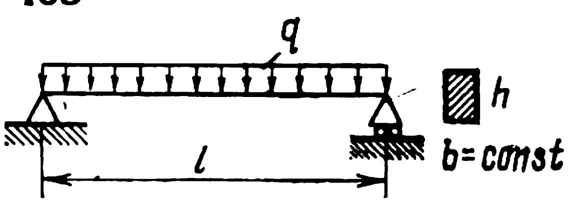

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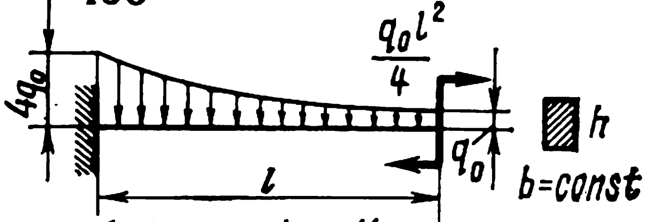
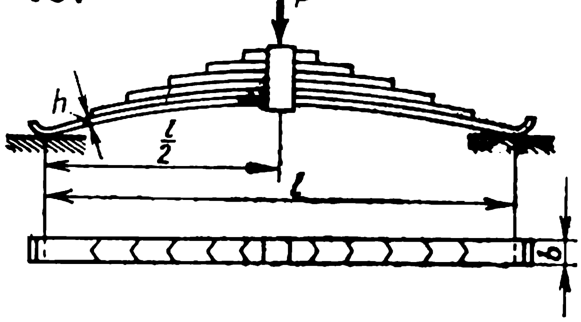


Parabolic load with apex at free end

Problem 467. Let $l = 1$ m, $b = 60$ mm, $h = 5$ mm, $n = 10$ leaves, $[\sigma] = 1600$ kgf/cm² and $E = 2 \times 10^6$ kgf/cm².

Find P_{max} and $|f|_{max}$. Friction between the spring leaves should be neglected.

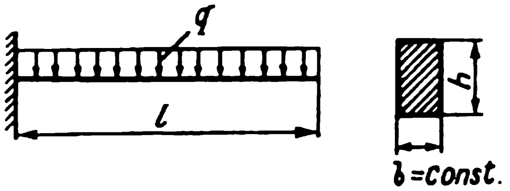
467



Problems 468 and 469. Determine the shapes of the beams of constant strength; the effect of the shearing stresses on the shapes; deflections of the free ends of the beams and the angles of rotation of the end

cross sections (explain the results obtained). Assume q, l, h_{max}, E and I_{max} to be known.

468



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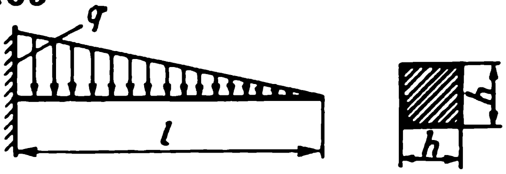
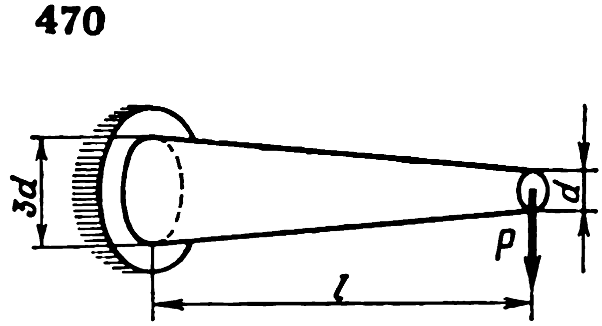


Problems 470 through 475. Determine in which cross sections of the beams the maximum normal stresses will be developed, and by how many times σ_{max}, $|f|_{max}$ and $|\theta|_{max}$ are greater than in similar beams of a constant (maximum) cross section.

470



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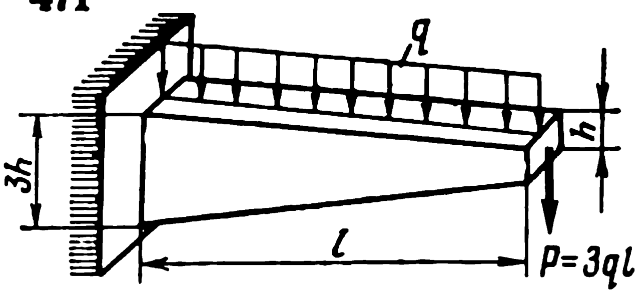

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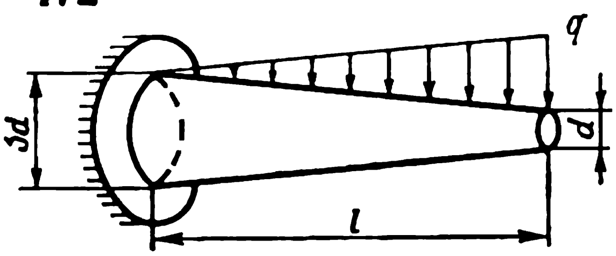

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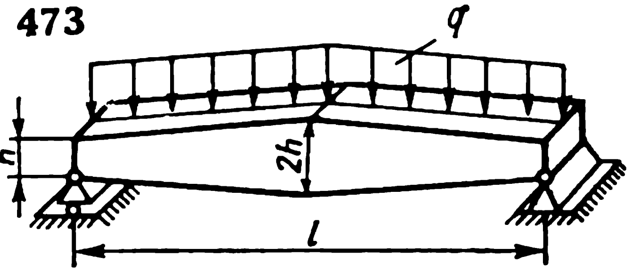

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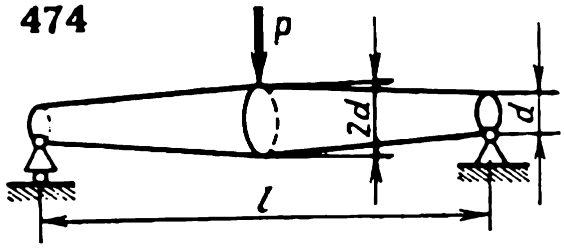

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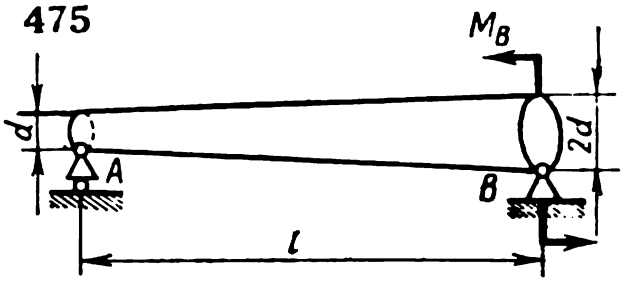
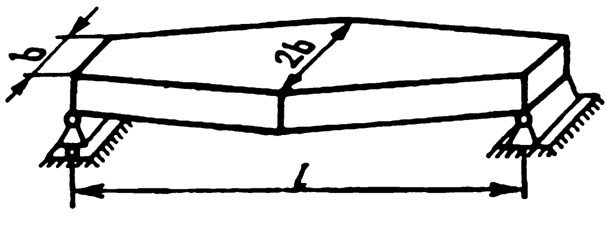


Problems 476 and 477. Determine the maximum normal stresses σ_{max} due to the dead weight of the beams and find the position of the dangerous cross section.

476



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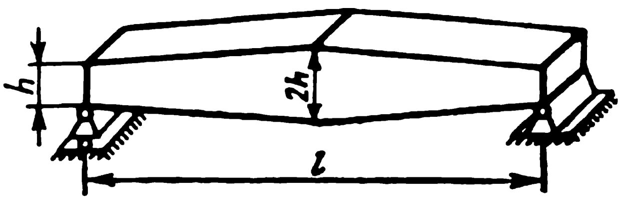
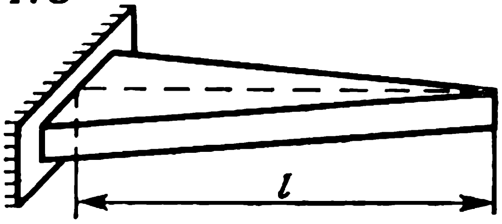


Assume M_{max} and W_{max} to be known.

Problems 478 through 481. Determine the deflections of the free ends of the beams due to their dead weight.

The weight of the beams Q, l, E and I_{max} are assumed to be known.

478



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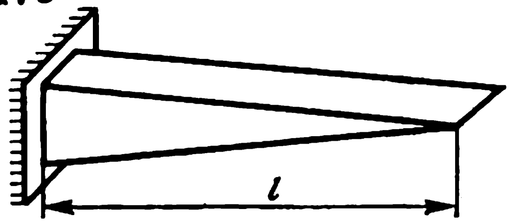

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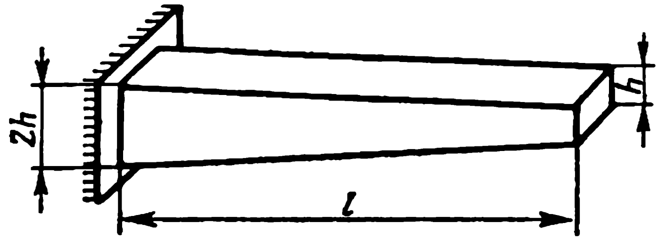

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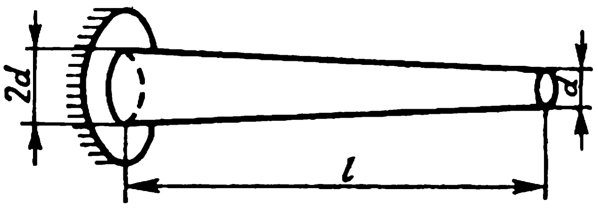


In the *grapho-analytical method* of determining strains (deformations) in a beam of variable cross section, the reduced bending moment M_r (and not the true bending moment M_x) is assumed to be the fictitious load of a fictitious beam. Thus

$$M_r = M_x \frac{I_0}{I_x}$$

which is in agreement with equation (127).

The angle of rotation and the deflection of an arbitrary cross section are found by the formulas

$$\theta_x = \frac{Q_{f_x}}{EI_0} \tag{128}$$

and

$$f_x = \frac{M_{f_x}}{EI_0} \tag{129}$$

in which Q_{f_x} and M_{f_x} = fictitious transverse (shearing) force and fictitious bending moment of the fictitious beam

EI_0 = rigidity of the constant cross section to which that of the beam is conventionally reduced.

Assuming I_0 to be the moment of inertia of the cross section at which M_{max} is applied, the reduced bending moment of beams of constant strength and with a section of constant height is

$$M_r = M_{max} = \text{const}$$

Example 50. Given: P, a, $M = 2Pa$, E and I (Fig. 95).

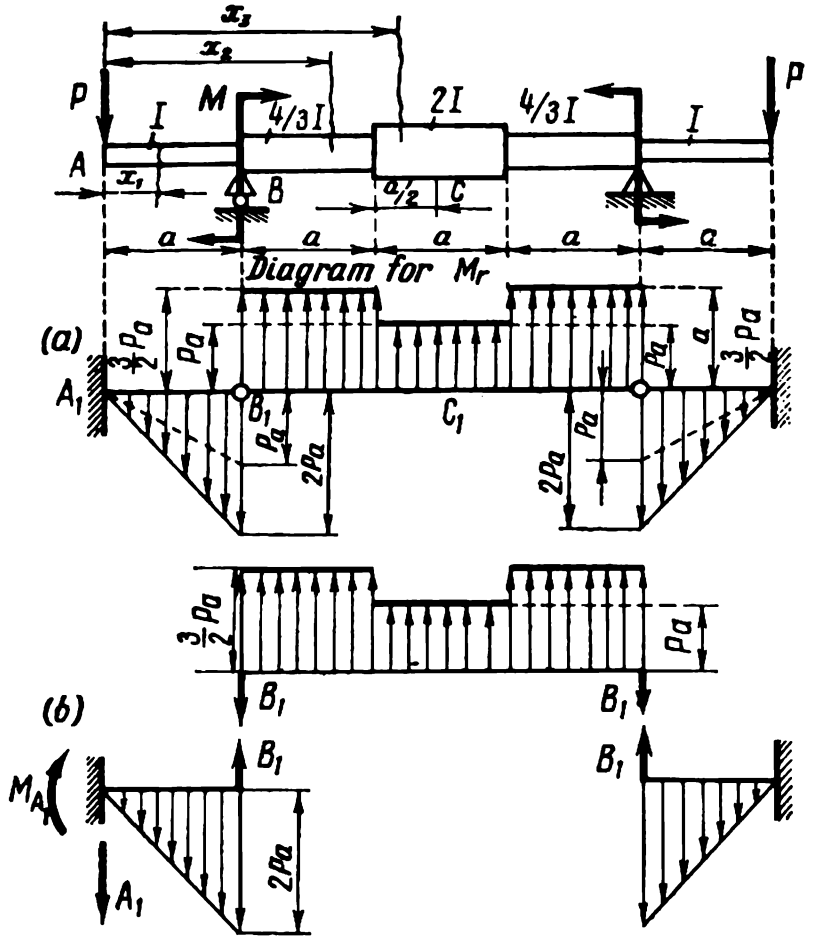


Fig. 95

Determine θ_A, θ_B, f_A and f_C.

Solution. The bending moments in the portions of the beam are: $M_{x_1} = -P_x$; $M_{x_2} = M_{x_3} = Pa$.

We assume that $I_0 = 2I$ and find the reduced bending moments in the portions

$$M_{r_1} = M_{x_1}\frac{2I}{I} = -2Px_1;$$

$$M_{r_2} = M_{x_2}\frac{2I}{\frac{4}{3}I} = \frac{3}{2}Pa; \qquad M_{r_3} = M_{x_3} = Pa$$

Next we take the diagram of the reduced moment as the fictitious load of a fictitious beam (Fig. 95*a*).

Then we cut the fictitious beam into two main beams (cantilevers) and one simply supported beam (Fig. 95*b*).

Since

$$B_1 = \frac{3}{2}Pa \times a + Pa \times \frac{a}{2} = 2Pa^2;$$

$$Q_{f_A} = -2Pa^2 + 2Pa \times \frac{a}{2} = -Pa^2; \quad Q_{f_{B_1}} = -B_1 = -2Pa^2$$

$$M_{f_{A_1}}=2Pa^2\times a-Pa^2\times\frac{2}{3}\,a=\frac{4}{3}\,Pa^3;$$

$$M_{f_{C_1}}=-2Pa^2\times\frac{3}{2}\,a+\frac{3}{2}\,Pa^2\times a+\frac{1}{2}\,Pa^2\times\frac{1}{4}\,a=-\frac{11}{8}\,Pa^3$$

then

$$\theta_A=\frac{Q_{f_{A_1}}}{2EI}=-\frac{Pa^2}{2EI};\quad \theta_B=\frac{Q_{f_{B_1}}}{2EI}=-\frac{Pa^2}{EI};$$

$$f_A=\frac{M_{f_{A_1}}}{2EI}=\frac{2Pa^3}{3EI};\quad f_C=\frac{M_{f_{C_1}}}{EI}=-\frac{11}{16}\,\frac{Pa^3}{EI}$$

Example 51. Given: q, l, E, h = const; and a rectangular beam of constant strength, constant height h and variable width b_x (Fig. 96). Find θ_A and f_A.

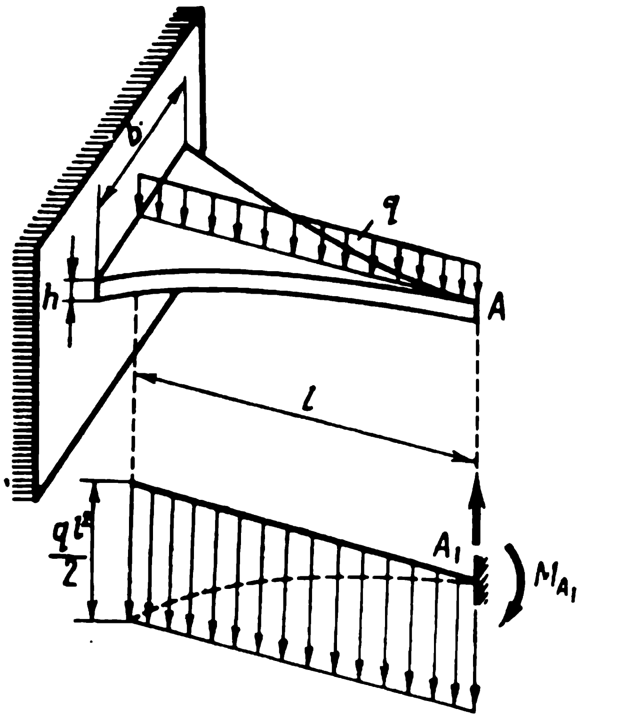


Fig. 96

Solution. We assume that

$$I_0=\frac{bh^3}{12}$$

then

$$M_r=M_{\max}=-\frac{ql^2}{2};$$

$$Q_{f_{A_1}}=-\frac{ql^2}{2}\times l=-\frac{ql^3}{2};$$

$$M_{f_{A_1}}=-\frac{ql^2}{2}\times l\times\frac{l}{2}=-\frac{ql^4}{4}$$

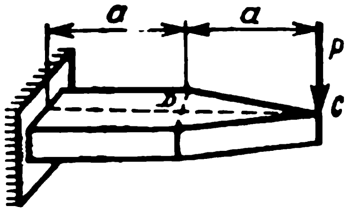
The sought-for displacements are

$$\theta_A=\frac{Q_{f_{A_1}}}{EI_0}=-\frac{ql^3}{2EI_0}\quad\text{and}$$

$$f_A=\frac{M_{f_{A_1}}}{EI_0}=-\frac{ql^4}{4EI_0}$$

Problems 482 through 485. Determine the deflections f_C of cross sections C and angles of rotation θ_D of cross sections D of the beams. Assume that P, a, E and $I_{\max}$ are known.

482



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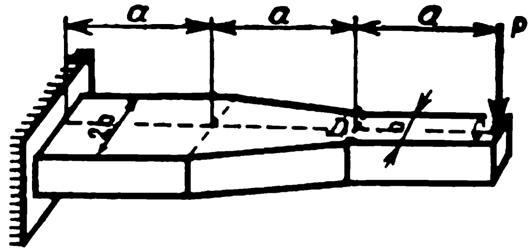

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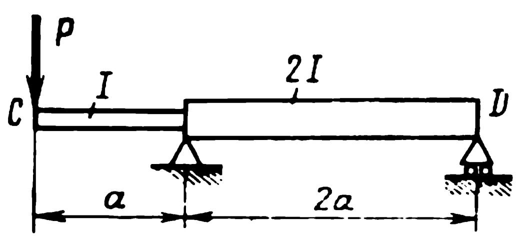

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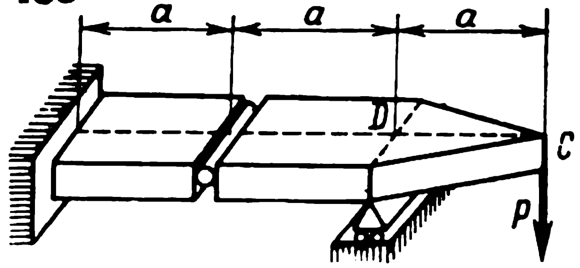
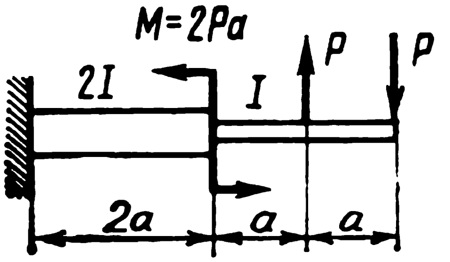


Problems 486 through 491. Determine the maximum deflections and angles of rotation of sections of the beams (absolute values). Assume that P, M, a, E, and I_{max} are known.

486



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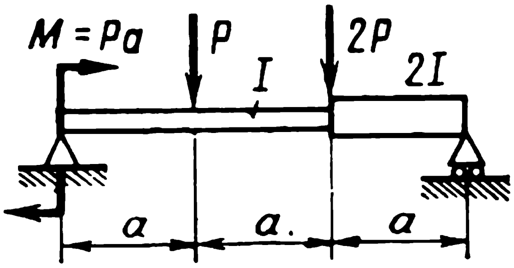

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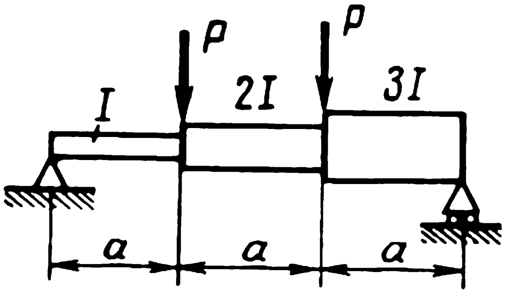

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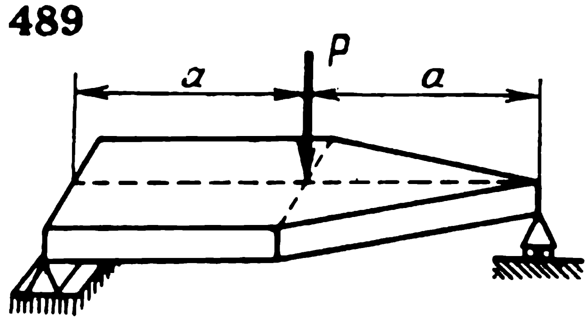

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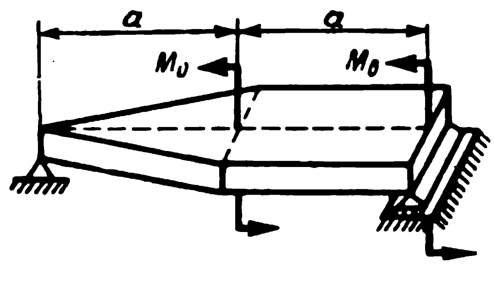

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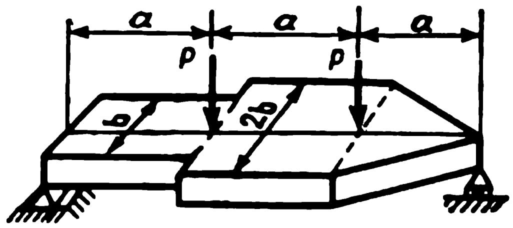


The *graphical method* of determining the displacements of beams of variable cross section is used in the same way as for beams of constant cross section.

The difference is that the reduced bending moment M_r is assumed to be the fictitious load of the fictitious or given beam of variable cross section instead of the true bending moment M_x. The rules for drawing the closing string (or strings) of the funicular polygon are valid here as well.

The scale of deflections is the quantity $\frac{EI_0}{H(\xi\eta)}$ in which I_0 is the moment of inertia of the constant cross section to which that of the beam is conventionally reduced.

The true bending moment can also be taken as the fictitious load, provided the pole distance of the vector diagram is variable and equal to

$$H_x = H_0 \frac{I_x}{I_0} \tag{130}$$

in which H_0 is the arbitrary (in magnitude) pole distance taken for the selected cross section with the moment of inertia I_0.

Example 52. Figure 97*a*, *b* and *c* illustrates a graphical construction of the elastic line of a stepped beam using the reduced bending moment for which I_0 is assumed to be equal to I.

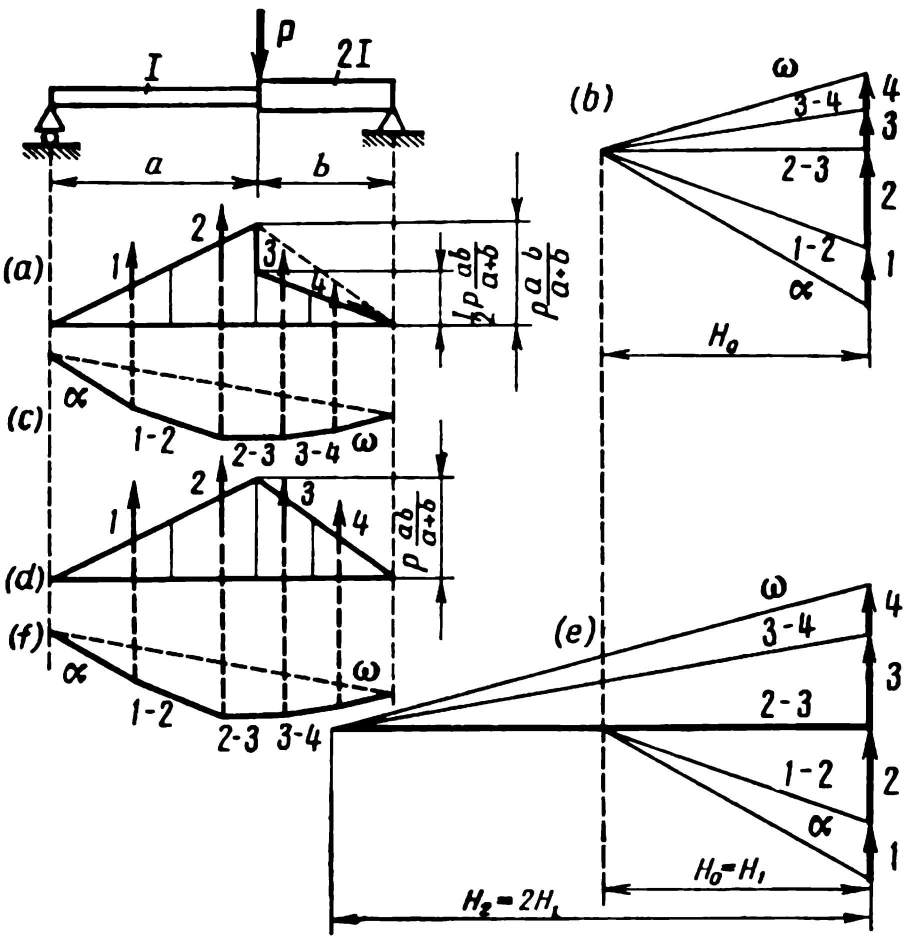


Fig. 97

Fig. 97*d*, *e* and *f* illustrates a graphical construction of the elastic line of the same beam using the true bending moment and a variable pole distance of the vector diagram, assuming that

$$H_0 = H_1 \quad \text{and} \quad H_2 = H_0 \frac{2I}{I} = 2H_1$$

8.8.

Statically Indeterminate Beams

Statically indeterminate beams are ones for which it is impossible to determine all the components of the reactions at the supports by the use of only the conditions of statics (equilibrium).

The degree of static indeterminancy of a beam is determined by the number of redundant unknowns which cannot be determined by the use of the conditions of statics alone.

METHODS OF ANALYSIS OF STATICALLY INDETERMINATE BEAMS

Method of initial parameters. General integrals are written for the differential equations of the elastic line of the given statically indeterminate beam in terms of the initial parameters. The initial parameters and components of the reactions are determined on the basis of the support conditions of the beam and from the conditions of statics.

Example 53. Given: q, l, E and I (Fig. 98).

Find A, B, M_B, f_x, θ_x, Q_x and M_x.

Solution. Since at the left-hand support $f_0 = 0$, in accordance with equations (116) and (117) the functions for the deflection and angle of rotation can be written as

$$EIf_x = EI\theta_0 x + A\frac{x^3}{6} - \frac{qx^5}{l \times 120};$$

$$EI\theta_x = EI\theta_0 + A\frac{x^2}{2} - \frac{q}{l} \times \frac{x^4}{24}$$

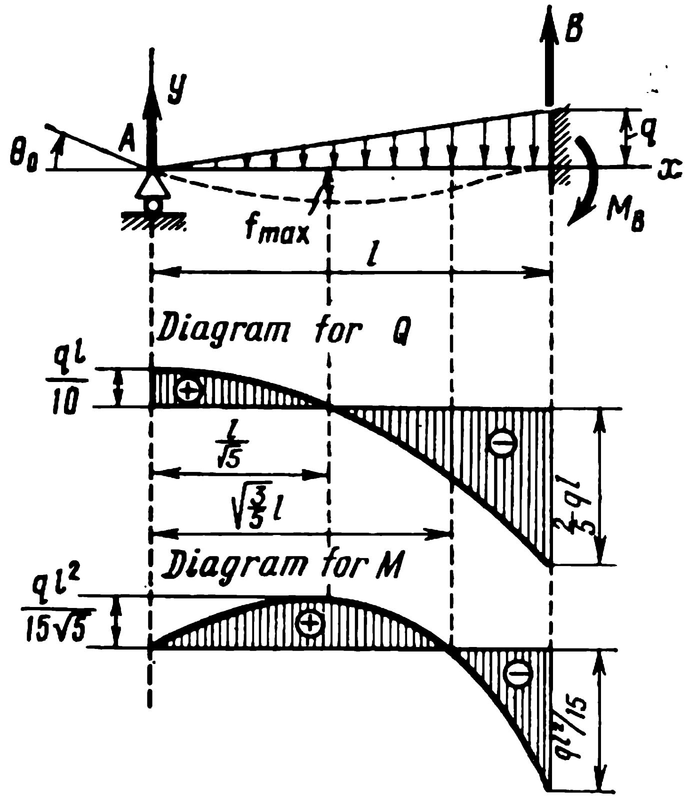


Fig. 98

According to the conditions for supporting the beam $f_l = 0$ and $\theta_l = 0$, i.e.

$$EIf_l = EI\theta_0 l + A\frac{l^3}{6} - \frac{ql^4}{120} = 0;$$

$$EI\theta_l = EI\theta_0 + A\frac{l^2}{2} - \frac{ql^3}{24} = 0$$

From which

$$A\frac{l^2}{3}=\frac{ql^3}{30}; \quad A=\frac{ql}{10} \quad \text{and} \quad \theta_0=\frac{1}{EI}\left(\frac{ql^3}{24}-\frac{ql^3}{20}\right)=-\frac{ql^3}{120EI}$$

Thus

$$EIf_x=-\frac{ql^3}{120}x+\frac{ql}{60}x^3-\frac{q}{l}\frac{x^5}{120}$$

and

$$EI\theta_x=-\frac{ql^3}{120}+\frac{ql}{20}x^2-\frac{q}{l}\times\frac{x^4}{24}$$

Assuming that $\theta_x=0$, from the equation

$$x^4-\frac{6l^2}{5}x^2+\frac{l^4}{5}=0$$

we obtain $x=\frac{l}{\sqrt{5}}\cong 0.447l$ at which the deflection reaches its maximum value

$$f_{\max}=f_{x=\frac{l}{\sqrt{5}}}=-\frac{2}{375}\frac{ql^4}{EI}$$

The shearing force and the bending moment are expressed by the functions

$$Q_x=\frac{ql}{10}-\frac{q}{l}\times\frac{x^2}{2} \quad \text{and} \quad M_x=\frac{ql}{10}x-\frac{q}{l}\times\frac{x^3}{6}$$

Assuming that $M_x=0$, we find $x=l\sqrt{\frac{3}{5}}\cong 0.775l$ at which the elastic line has an inflection point.

Assuming that $Q_x=0$, we find $x=\frac{l}{\sqrt{5}}\cong 0.447l$ at which the bending moment reaches its maximum value

$$M_{\max}=M_{x=\frac{l}{\sqrt{5}}}=\frac{ql^2}{15\sqrt{5}};$$

$$Q_{x=0}=\frac{ql}{10}=A; \quad Q_{x=l}=-\frac{2}{5}ql=-B; \quad M_{x=0}=0;$$

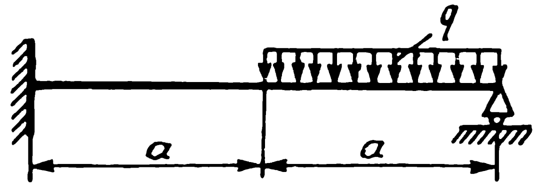
$$M_{x=l}=-\frac{ql^2}{15}=M_B$$

Then these data are used for plotting the diagrams for Q and M and the elastic line of the beam.

Problems 492 through 495. Analyse the statically indeterminate beams.

Plot the diagrams of the transverse (shearing) force Q and bending moment M in Problems 492 and 493; determine the angle of rotation of the cross section directly over the support in Problem 494 and determine the deflection f of cross section B in Problem 495.

492



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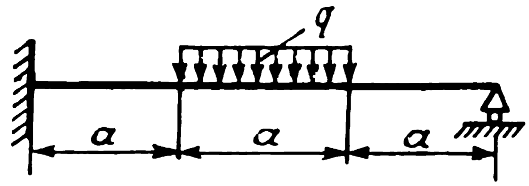

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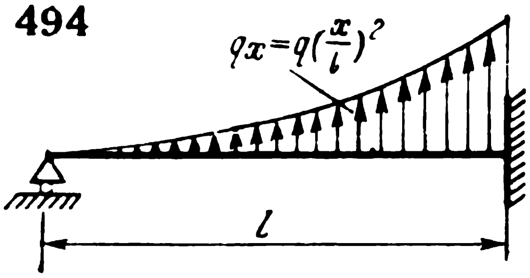

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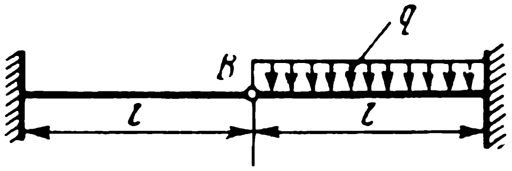


Method of comparing displacements. The statically indeterminate beam is relieved of redundant constraints until it becomes statically determinate, without changing the geometry of the beam. Such a beam is said to be basic. Then the basic beam is first loaded only with the

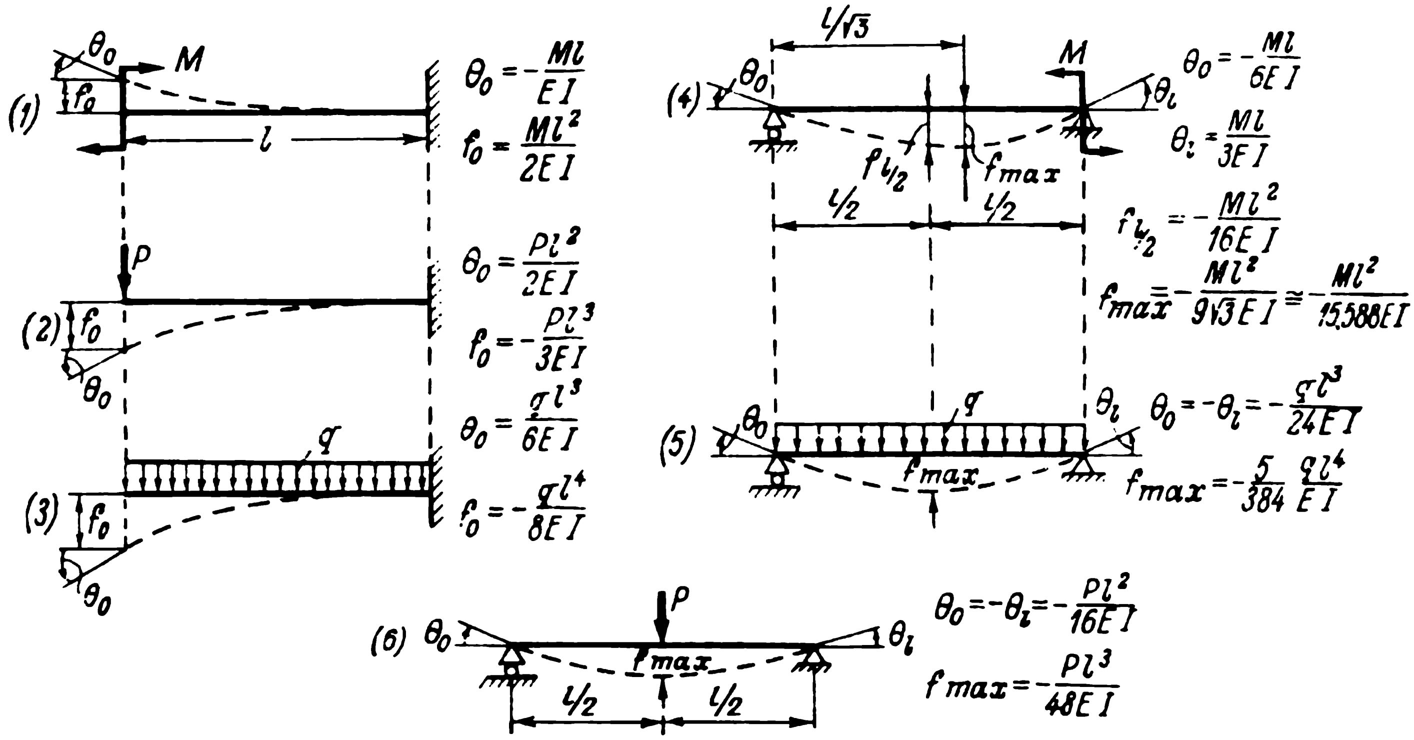


Fig. 99

given forces and, at the relieved cross sections, displacements corresponding to the dropped redundant unknowns are determined (deflection for a force; angle of rotation for a moment). Next the same relieved (auxiliary) beam is loaded only with the redundant unknowns and the displacements corresponding to these redundant unknowns are determined for the relieved cross sections of the beam.

The values of the redundant unknowns should be selected so that the total displacements found at the relieved cross sections of the basic

and auxiliary beams correspond to the displacements at the same sections of the given statically indeterminate beam.

This method can be advantageously employed in cases when both the basic and auxiliary beams can be reduced to compiled tables of beams for which the values of the required displacements are known.

It is useful to know the values of certain displacements for at least the simplest beams for which data are listed in tables (Fig. 99).

Example 54. Given: M, l, E and I (Fig. 100).

Find A, B, M_B, θ_A and $f_{\frac{l}{2}}$.

Draw the diagrams for M and Q.

Solution. We shall assume reaction A to be the redundant unknown. The cantilever loaded with the given couple of moment M is the basic beam I.

The displacement (deflection) of the basic beam at the relieved cross section is (see the data for the corresponding beam in Fig. 99)

$$f_{A_1}=\frac{Ml^2}{2EI}$$

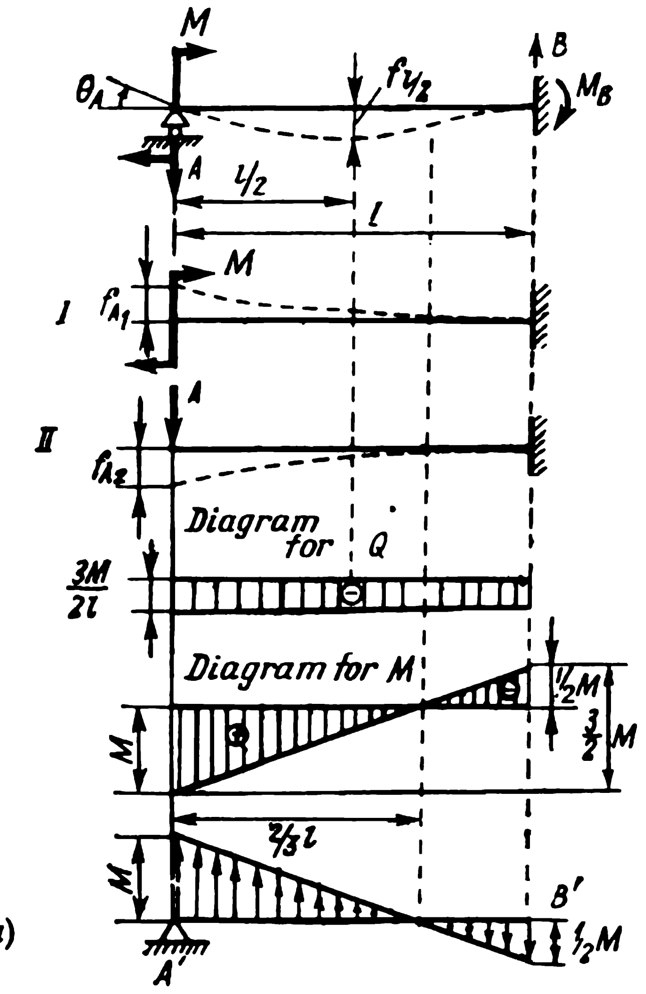


Fig. 100

The deflection of auxiliary beam II due to the unknown force A at the relieved cross section is (see beam 2 in Fig. 99).

$$f_{A_2}=-\frac{Al^3}{3EI}$$

Since the deflection of the given beam on support A is $f_A=0$, we obtain, by comparing the total displacements of beams I and II with that of the given beam,

$$f_A=f_{A_1}+f_{A_2}=0$$

or

$$\frac{Ml^2}{2EI}-\frac{Al^3}{3EI}=0$$

from which

$$A=\frac{3}{2}\frac{M}{l}$$

Since at any cross section of the given beam the transverse (shearing) force $Q = A$, the shearing force diagram is a rectangle of height $\frac{3M}{2l}$. The reaction at the fixed cross section is $B = -A$.

The bending moment diagram can be readily constructed by superposing the diagrams for the basic and auxiliary beams. For the basic beam the bending moment diagram is a positive rectangle of height M. For the auxiliary beam the bending moment diagram is a negative triangle of a zero height at the left-hand end and of height $3/2\,M$ at the right-hand end.

Superposing these diagrams we obtain the resultant bending moment diagram for the given beam. The reaction force moment at the fixed cross section is

$$M_B = -\frac{3}{2}M + M = -\frac{M}{2}$$

We use the grapho-analytical method to determine the displacements of the beam. The fictitious beam and fictitious load are shown in Fig. 100*a*.

Since

$$Q_{fA_1} = \frac{M}{2} \times \frac{l}{6} - M\frac{l}{3} = -\frac{Ml}{4}$$

and

$$M_{fl/2} = -\frac{Ml}{12}\left(\frac{l}{2} - \frac{l}{9}\right) + \frac{M}{4} \times \frac{l}{12} \times \frac{l}{18} = -\frac{Ml^2}{32}$$

then

$$\theta_A = \frac{Q_{fA_1}}{EI} = -\frac{Ml}{4EI} \quad \text{and} \quad f_{l/2} = \frac{M_{fl/2}}{EI} = -\frac{Ml^2}{32EI}$$

It is expedient to note that the fictitious beam, supported only at one end, must be in equilibrium under the fictitious load, i.e. the moment of the fictitious load about the left end of the fictitious beam must equal zero.

The redundant unknowns may be not only the reaction components at the supports of the given beam. They may also be the internal efforts Q and M at a cross section where, if cut, the beam is reduced to the simple beams for which tabular data is available.

Example 55. Given: q and a (Fig. 101).

Find A, B, M_A and M_B.

Solution. First we relieve the redundant constraints of the given twofold statically indeterminate beam by cutting it into two beams (*I* and *II*) at cross section C.

Next we assume that the transverse (shearing) force Q_C and bending moment M_C in this section are the redundant unknowns.

Then we reduce the conditions for comparing the displacements of beams *I* and *II* to the following system of two equations:

$$\left.\begin{aligned}\theta_{C_1}&=\theta_{C_2};\\ f_{C_1}&=f_{C_2}\end{aligned}\right\}$$

From the beams (1) and (2) with listed data (see Fig. 99), we obtain for beam *I*

$$\theta_{C_1}=\frac{M_Ca}{EI}-\frac{Q_Ca^2}{2EI}\quad\text{and}\quad f_{C_1}=\frac{M_Ca^2}{2EI}-\frac{Q_Ca^3}{3EI}$$

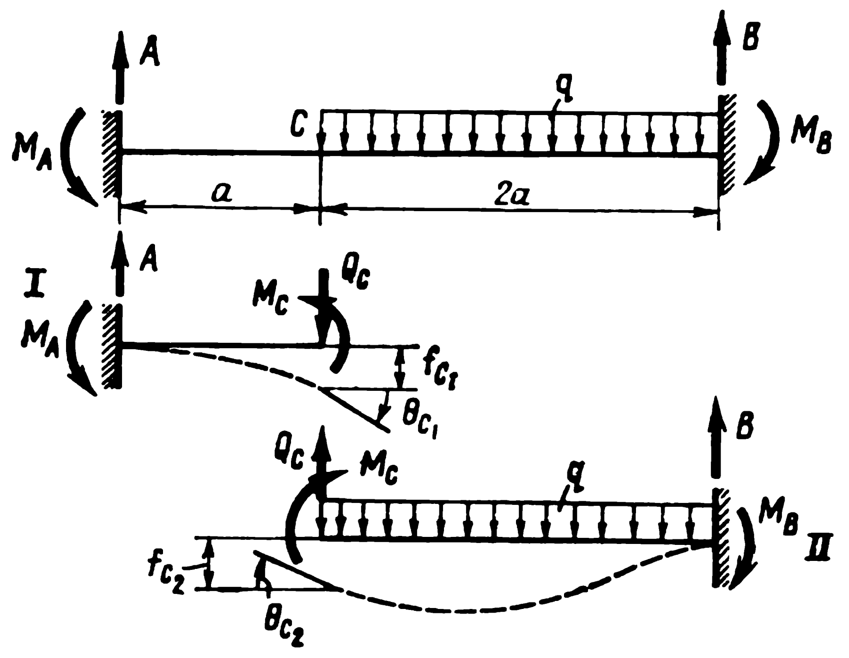


Fig. 101

From beams (1), (2) and (3) (see Fig. 99) we obtain for beam *II*

$$\theta_{C_2}=-\frac{M_C2a}{EI}-\frac{Q_C(2a)^2}{2EI}+\frac{q(2a)^3}{6EI}$$

and

$$f_{C_2}=\frac{M_C(2a)^2}{2EI}+\frac{Q_C(2a)^3}{3EI}-\frac{q(2a)^4}{8EI}$$

Substituting the values into the displacement comparison equations, we obtain

$$\left.\begin{aligned}3M_C+1.5Q_Ca&=\frac{4}{3}qa^2;\\ 1.5M_C+3Q_Ca&=2qa^2\end{aligned}\right\}$$

from which

$$Q_C=\frac{16}{27}qa\quad\text{and}\quad M_C=\frac{4}{27}qa^2$$

Hence, reaction $A = Q_C = \frac{16}{27} qa$ and the reaction force moment is

$$M_A = Q_C a - M_C = \frac{4qa^2}{9}$$

Reaction $B = 2qa - Q_C = \frac{38}{27} qa$ and the reaction force moment is

$$M_B = Q_C 2a - M_C - 2qa^2 = \frac{3}{2} qa^2$$

The method of comparing displacements can also be employed in the analysis of statically indeterminate beam and bar systems (i.e. systems, consisting of beams to which elastic bars are hinged) and frame systems (i.e. bar systems with rigid joints at which the angles between the bars do not change when the frame is deformed).

As a rule the tensile or compression strains of the members are neglected in the analysis of frame systems.

Example 56. Let AB be a uniformly loaded cantilever beam with known values of q, l, E_1 and I_1, AC is an elastic tie rod with known values of l_2, E_2 and F_2 (Fig. 102).

Determine the axial force N in the tie rod.

Solution. First we divide the given beam and bar system at hinge joint A into beam AB and elastic tie rod AC. The beam is subject to the uniformly distributed load

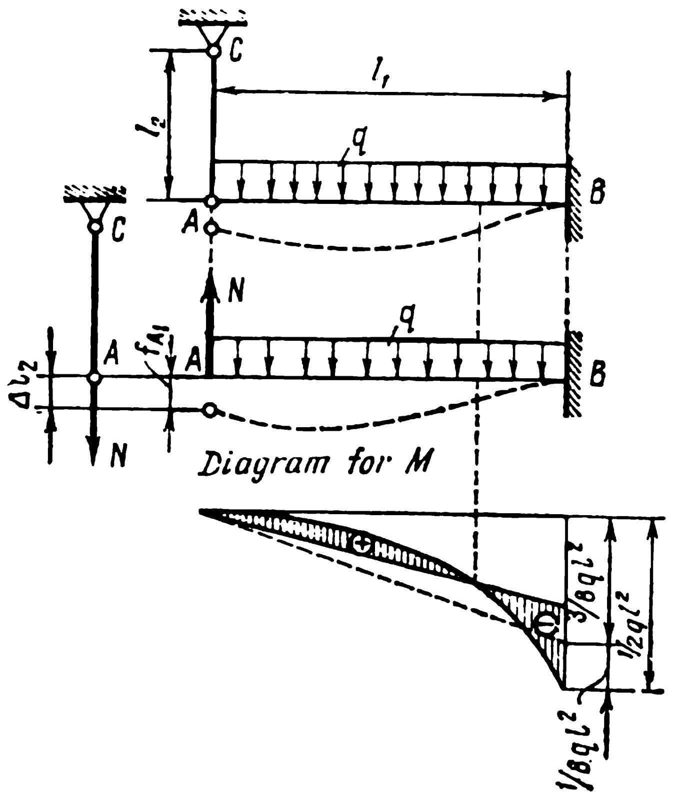


Fig. 102

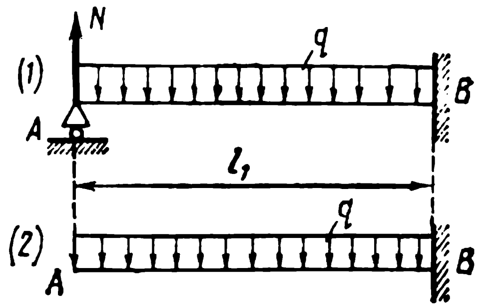


Fig. 103

q and the unknown force N. The tie rod is subject only to force N.

The equation for comparing the displacements of ends A of the beam and tie rod should be of the form $|f_{A1}| = \Delta l_2$.

From beams (2) and (3) with listed data (see Fig. 99) we obtain

$$|f_{A1}| = \frac{ql_1^4}{8E_1I_1} - \frac{Nl_1^3}{3E_1I_1}$$

According to Hooke's law

$$\Delta l_2 = \frac{N l_2}{E_2 F_2}$$

Substituting f_{A_1} and Δl_2 into the displacement comparison condition, we obtain

$$\frac{q l_1^4}{8 E_1 I_1} - \frac{N l_1^3}{3 E_1 I_1} = \frac{N l_2}{E_2 F_2}$$

from which

$$N = \frac{3}{8} q l_1 \frac{1}{1 + 3 \frac{l_2}{l_1^3} \frac{E_1 I_1}{E_2 F_2}}$$

For particular cases (Fig. 103):

(1) $E_2 F_2 = \infty$; $\quad N = \frac{3}{8} q l_1$

(2) $E_2 F_2 = 0$; $\quad N = 0$

Thus, depending on the rigidity of the elastic tie rod, the force N can vary in the range $0 \leqslant N \leqslant \frac{3}{8} q l_1$ and the bending moment M_B at the fixed end of the beam in the range

$$\frac{q l_1^2}{8} \leqslant M_B \leqslant \frac{q l_1^2}{2}$$

Figure 102 shows the diagram of the bending moment of the beam for an arbitrary M_B within the given range of its variation.

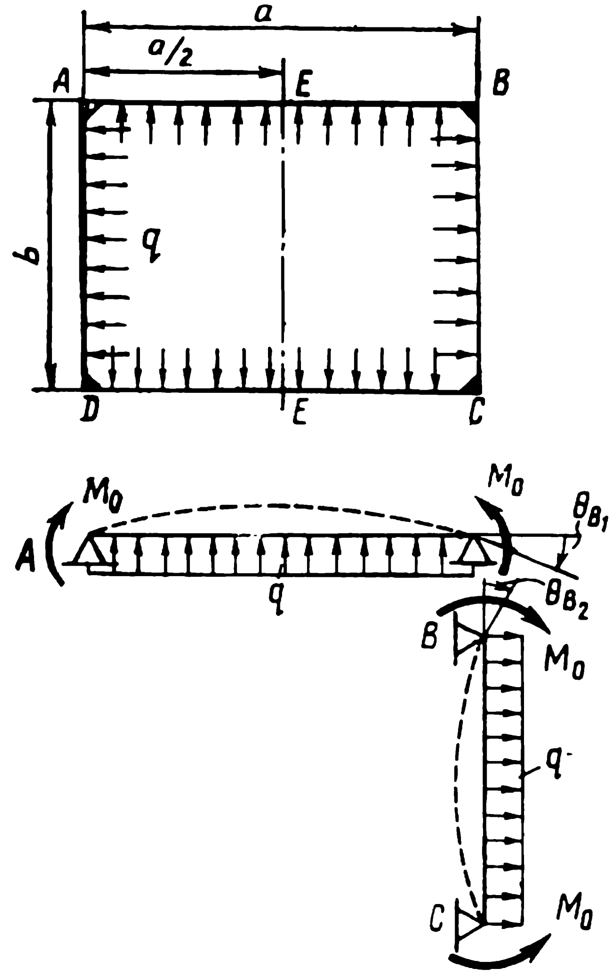


Fig. 104

Example 57. Let the structure of Fig. 104 be a rectangular frame with sides a and b, flexural rigidity EI. It is loaded with an internal uniformly distributed force q normal to the walls.

Determine the increase δ in the distance between the middle sections E-E.

Solution. First we cut the frame into beams at the joints. Since the frame is symmetrical, it will be sufficient to consider two beams, for example, AB and BC. Neglecting the elongation of the bars of the frame, each beam can be dealt with as one supported at the ends and loaded with a distributed load q and unknown moments M_0 at the cross sections directly over the supports. These moments represent the rigidity of the frame joints.

Since the angles of the frame must remain right angles, the condition for comparing the displacements is written in the following form: $\theta_{B_1} = \theta_{B_2}$.

From beams (4) and (5) with listed data (see Fig. 99)

$$\theta_{B_1} = \frac{qa^3}{24EI} - \frac{M_0 a}{2EI} \quad \text{and} \quad \theta_{B_2} = \frac{M_0 b}{2EI} - \frac{qb^3}{24EI}$$

From the condition for comparing displacements

$$\frac{qa^3}{12} + \frac{qb^3}{12} = M_0 (a + b)$$

whence

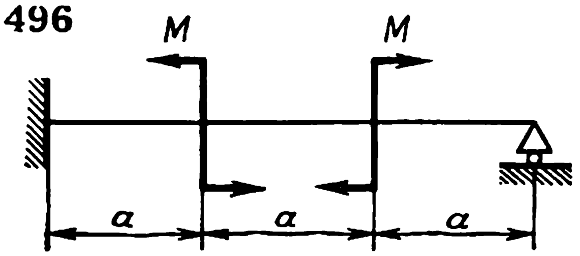
$$M_0 = \frac{q}{12}(a^2 - ab + b^2)$$

From the same beam data the increase in the distance between the middle points of sides a is

$$\delta = \frac{2}{EI}\left(\frac{5}{384} qa^4 - \frac{M_0 a^2}{16}\right) = \frac{qa^2}{192EI}(a^2 + 4ab - 4b^2)$$

Problems 496 through 499. Analyse the statically indeterminate beams and construct the diagrams for M and Q.

496



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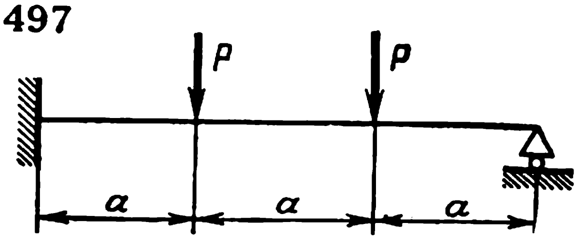

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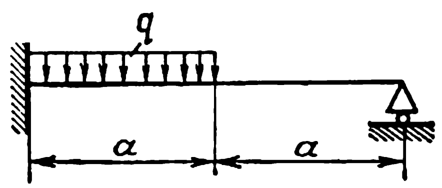

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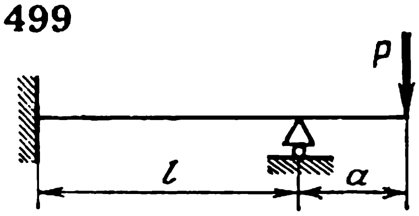


Problems 500 through 505. Analyse the statically indeterminate beams and determine the allowable loads.

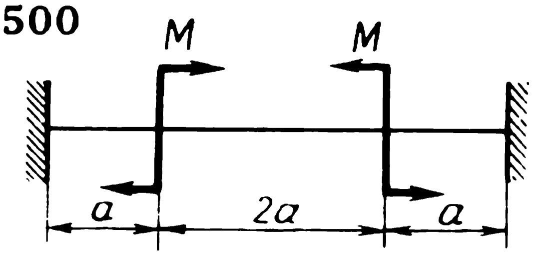
Assume the length equal to 3 m and $[\sigma] = 1600$ kgf/cm^2 for all the beams.

In Problems 500 and 501 the beams are round ($d = 10$ cm).

In Problems 502 and 503 the beams are annular ($D = 12$ cm and $d = 8$ cm).

In Problems 504 and 505 the beams are of square cross section (10 × 10 cm).

500



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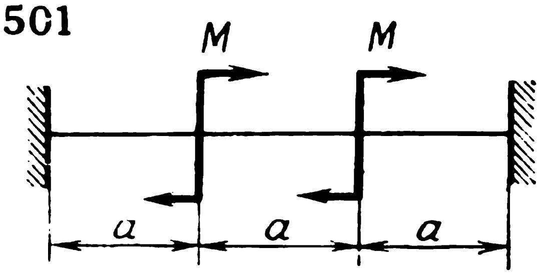

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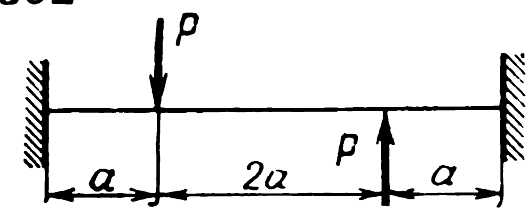

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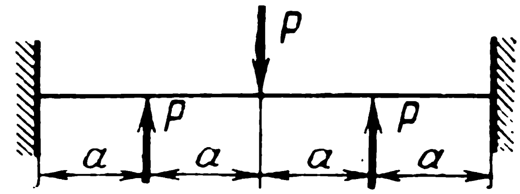

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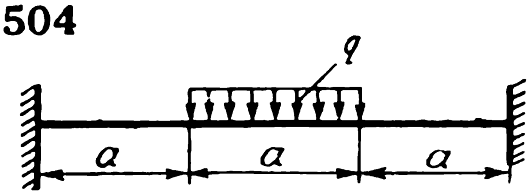

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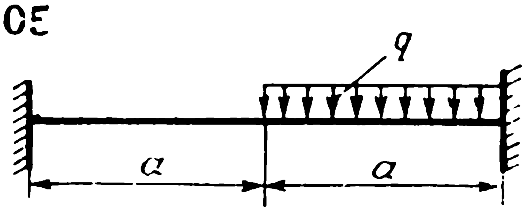
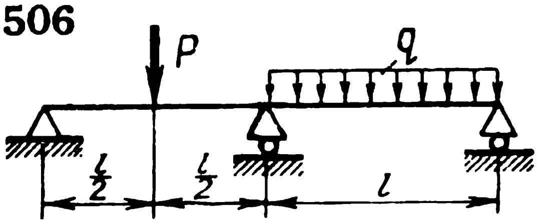


Problems 506 through 509. Select the required size Nos. of I-beams, if $P = 2$ tnf, $q = 2$ tnf/m, $M = 3$ tnf-m, $l = 2$ m and $[\sigma] = 1600$ kgf/cm².

506



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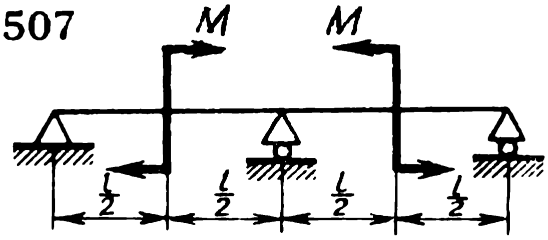

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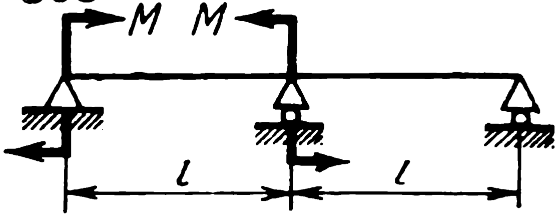

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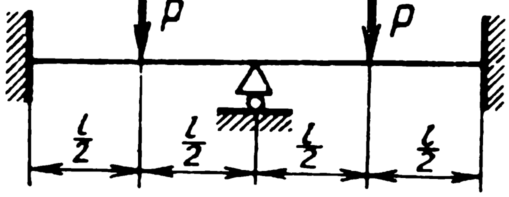
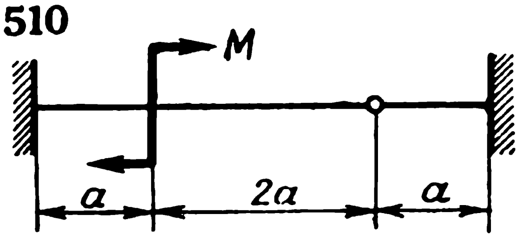


Problems 510 through 513. Determine the maximum bending moment M_{max}.

510



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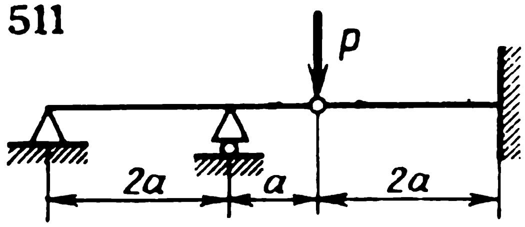

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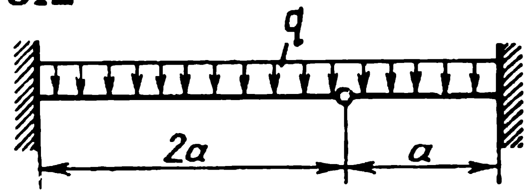
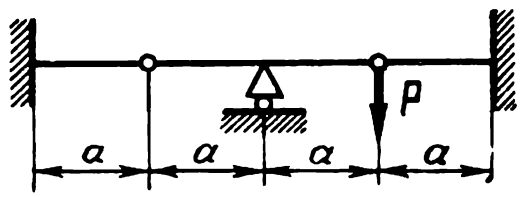
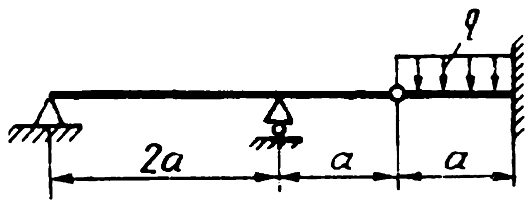


513



Problems 514 through 519. Determine the lowering y at the intermediate hinged joint or slide block.

514



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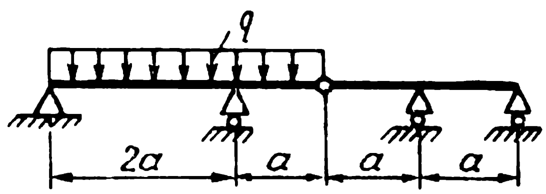

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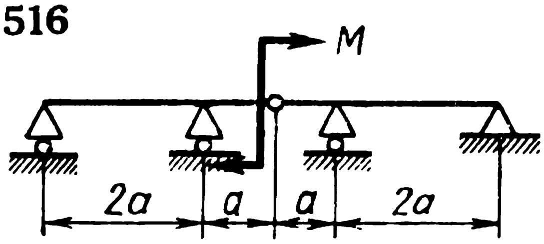

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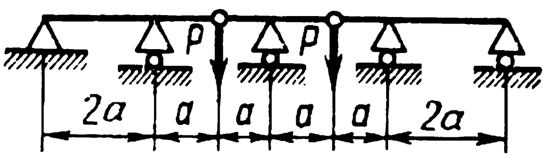

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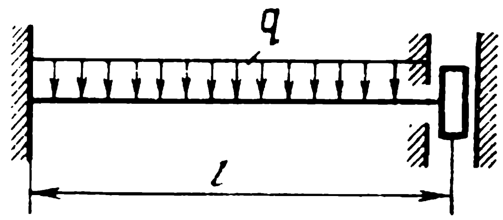

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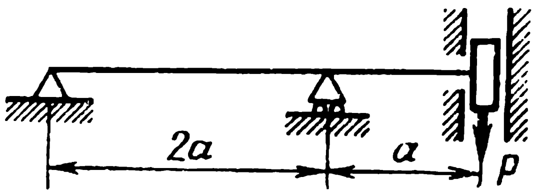
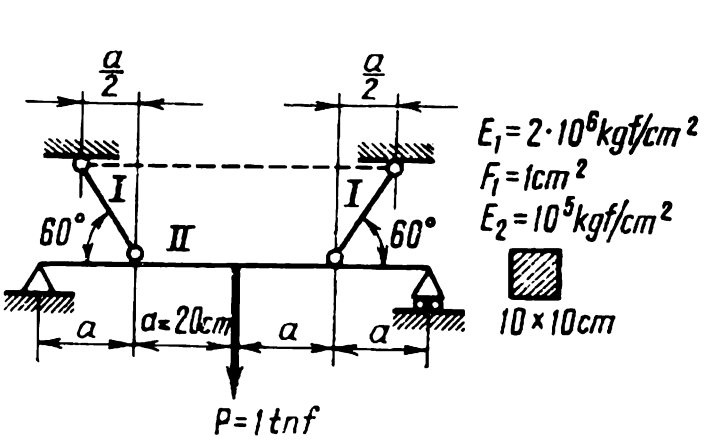


Problems 520 through 525. Determine the axial forces N in the tie rods.

520



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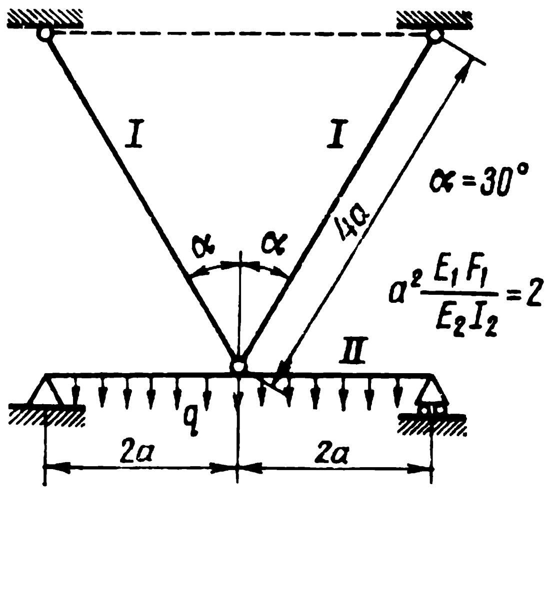

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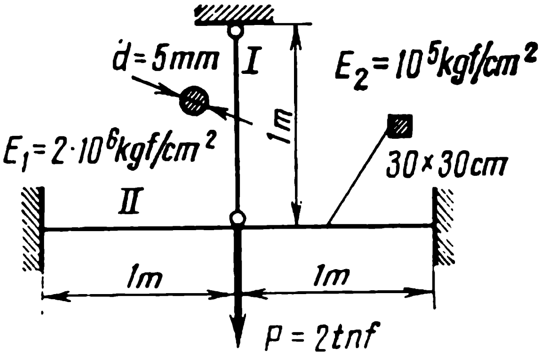

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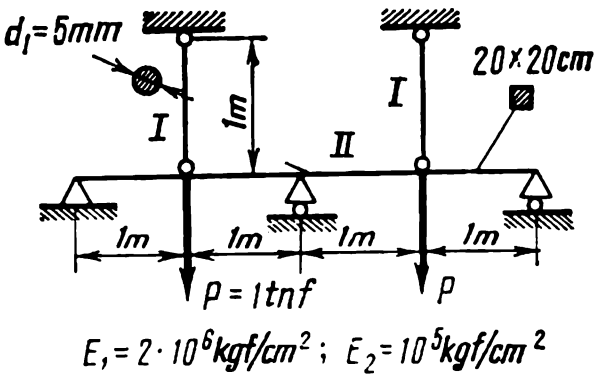

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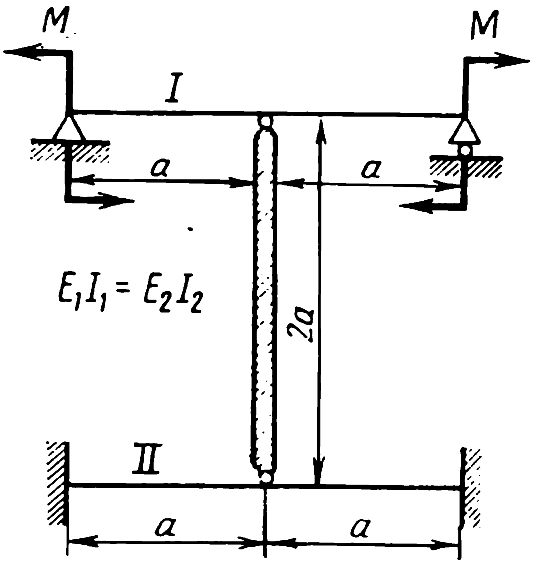

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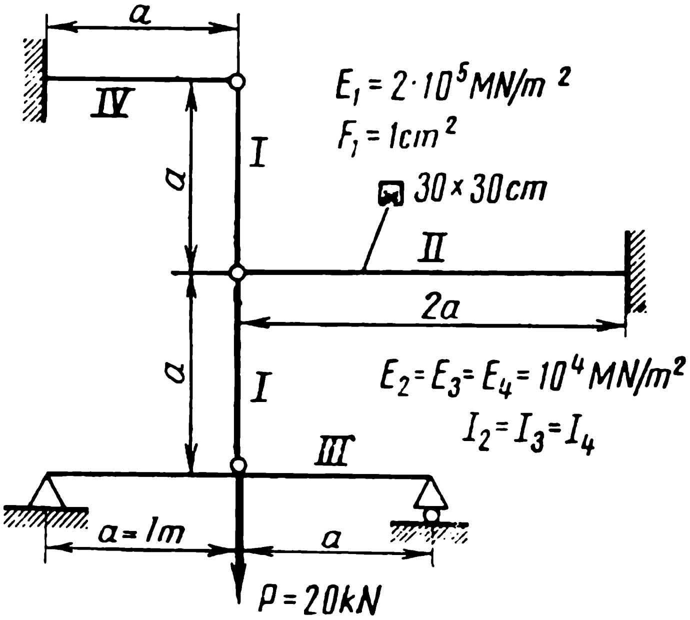
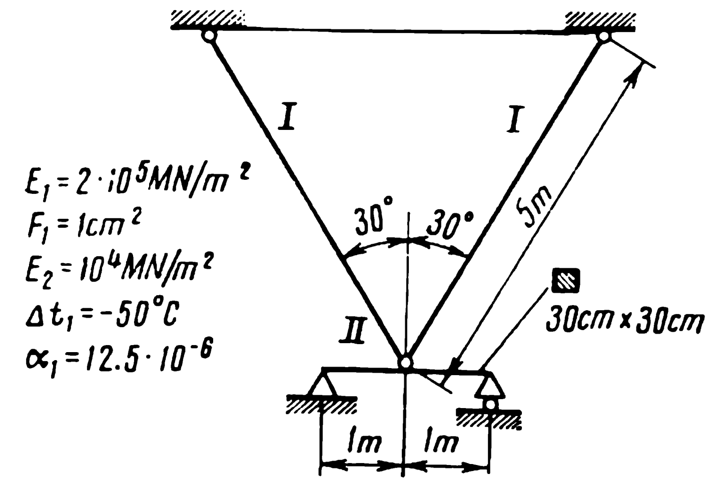


Problems 526 through 529. In the elements of the beam and bar systems determine:

(1) temperature stresses (Problems 526 and 529);

(2) assembly stresses (Problems 527 and 528).

526



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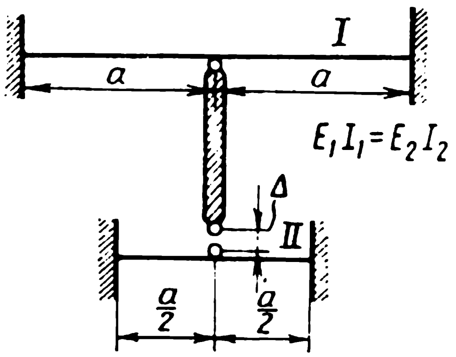

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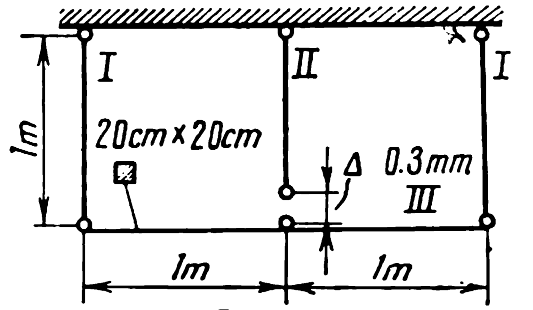
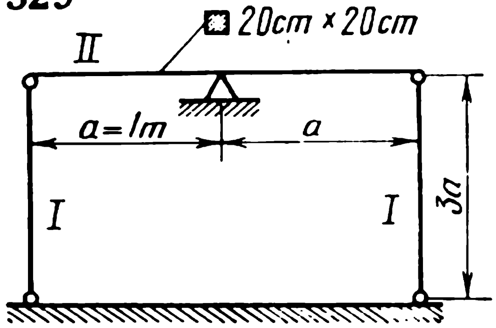


$E_1 = E_2 = 2 \cdot 10^5$ MN/m^2; $F_1 = F_2 = 1$ cm^2
$E_3 = 10^4$ MN/m^2

529



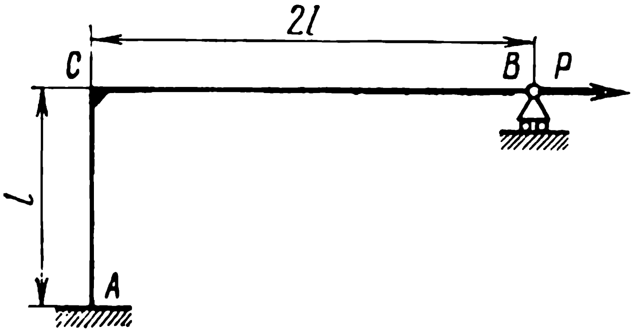
$E_1 = 2 \cdot 10^6$ kgf/cm^2; $F_1 = 2$ cm^2
$E_2 = 10^5$ kgf/cm^2; $\Delta t_1 = -50$°C;
$\alpha_1 = 12.5 \cdot 10^{-6}$

Problems 530 through 537. Analyse the statically indeterminate frame systems and determine the maximum bending moments M_{max}.

The tensile and compression strains of the frame bars may be neglected. The flexural rigidity of cross sections of all the bars of each frame is the same.

In Problems 534 and 535 the strains of the stretched and compressed elements should be taken into account.

530



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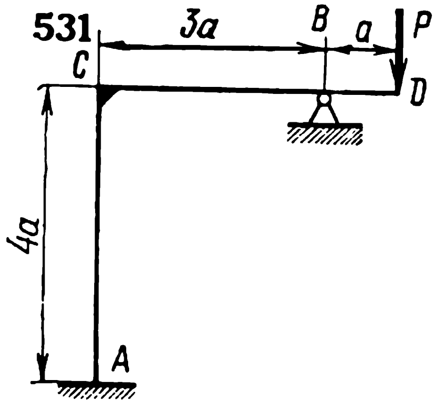

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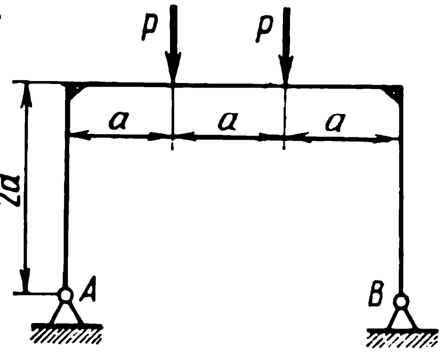

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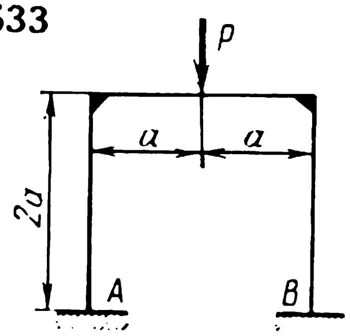

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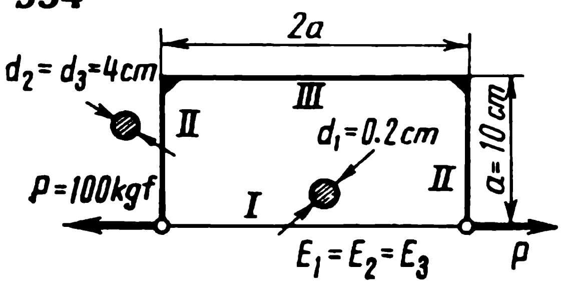

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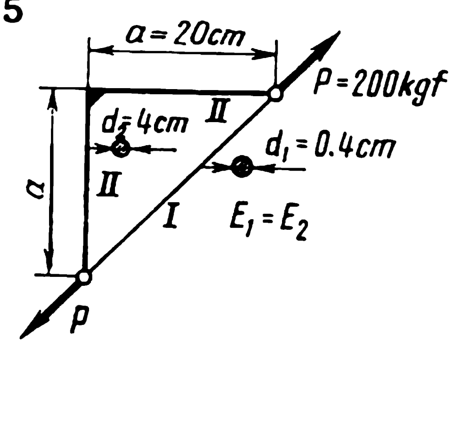

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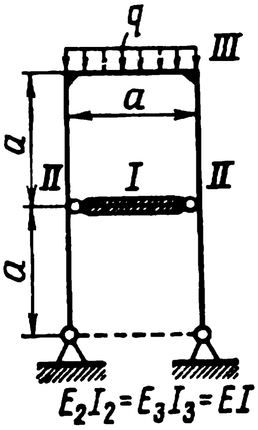

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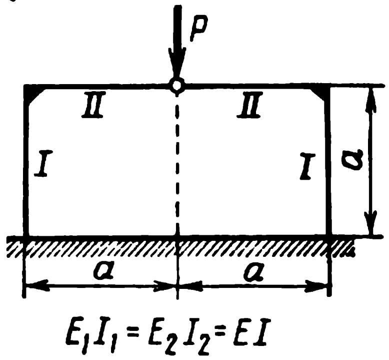
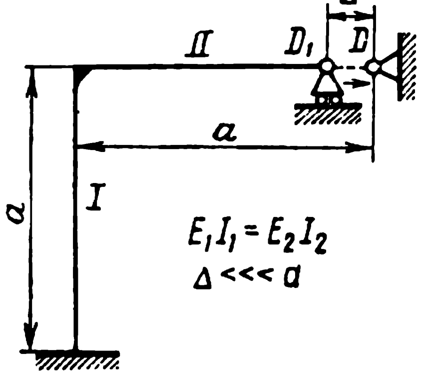


Problems 538 through 543. Determine the support reactions or forces N in tie rods of frame systems which are developed in assembly (538 and 539) or are due to temperature variations (540 through 543).

Assume that $E_{st} = 2 \times 10^6$ kgf/cm²; $E_{cu} = 10^6$ kgf/cm²

$$\alpha_{st} = 12 \times 10^{-6}; \quad \alpha_{cu} = 16 \times 10^{-6}$$

538



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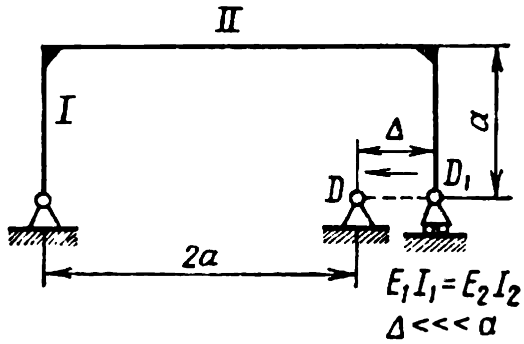

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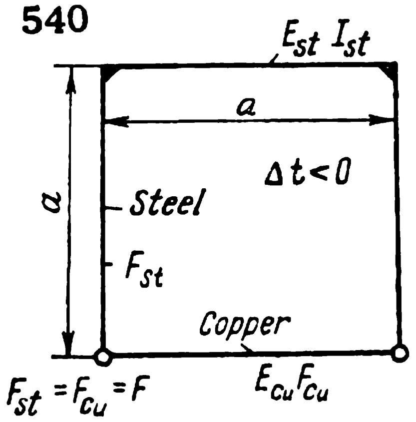

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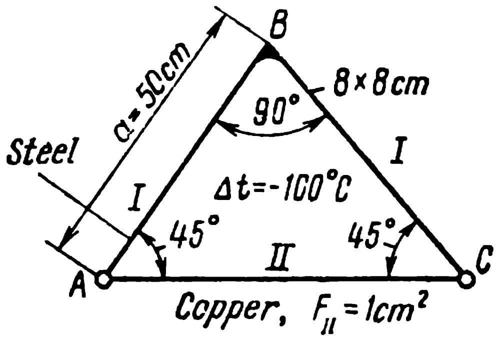

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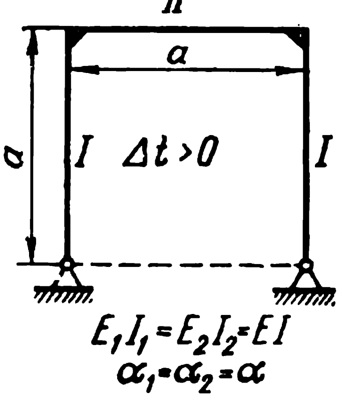

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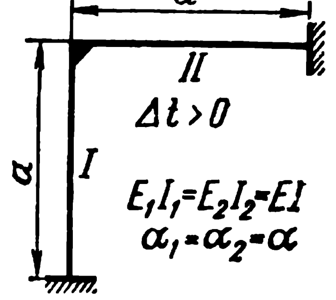


Three moment equation method. This method can be advantageously used in the analysis of multispan continuous beams, i.e. beams which have several spans without any hinged joints.

Let us consider the beam of Fig. 105 as an n-span continuous beam of a constant cross section of rigidity EI with all the supports at one level. Such a beam is n — 1-fold statically indeterminate.

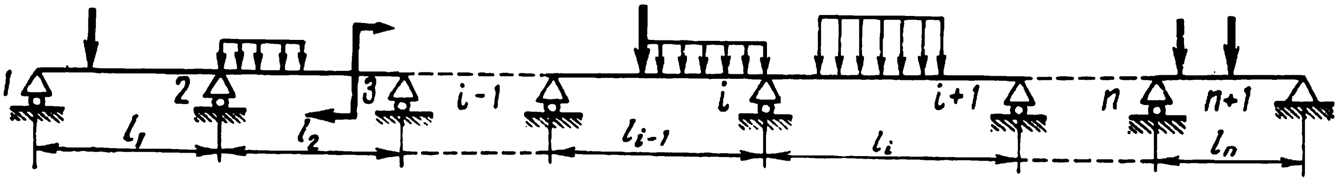


Fig. 105

We relieve the redundant constraints by cutting the beam at cross sections directly above the supports into n small beams simply supported at the ends. The bending moments at the cut-through cross sections above the supports will be the redundant unknowns. Then we comply

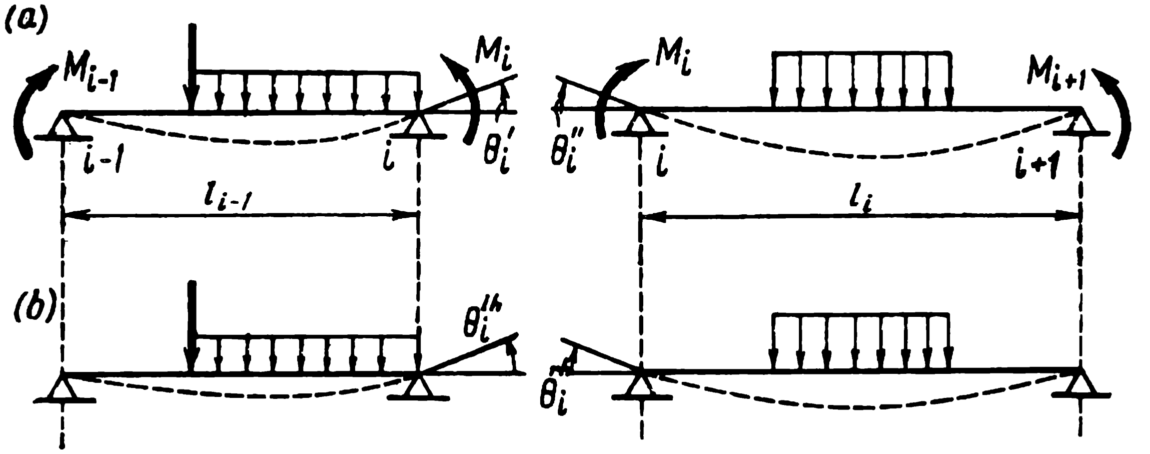


Fig. 106

with the condition of smooth conjugation of the elastic lines by equating the angles of rotation of the cross sections ($\theta_i' = \theta_i''$) at the intermediate i-th support for each pair of two adjacent small beams with spans l_{i-1} and l_i (Fig. 106*a*). As a result, a system of n — 1 equations of three moments of the following canonical type are obtained:

$$M_{i-1}l_{i-1} + 2M_i(l_{i-1} + l_i) + M_{i+1}l_i = -6EI(\theta_i^l + \theta_i^r) \qquad (131)$$

Here M_{i-1}, M_i and M_{i+1} are the three unknown moments at the sections directly over the supports $i - 1$, i and $i + 1$; θ_i^l and θ_i^r are the angles of rotation of the cross sections over the i-th support of the small left- and right-hand beams due only to the loads in their spans (Fig. 106*b*). The values of θ_i^l and θ_i^r can be found by the use of any expedient method, or taken from the data listed for simple beams, if possible.

Substituting 2, 3, 4, . . ., n for i in equation (131) will provide a system of $n - 1$ equations which, upon being solved, will give the values of all the unknown bending moments at the sections over the supports of the continuous beam.

The following considerations may prove useful in the practical application of the method of three moment equations:

1. Angles θ_i^l and θ_i^r are substituted into the equations with the sign "plus", if the cross section of the small beam turns in the direction indicated in Fig. 106*b*.

2. If the continuous beam ends in a loaded cantilever (Fig. 107*a*), the latter is not included in the three moment equation as a span. The cantilever is replaced with the moment due to the load applied

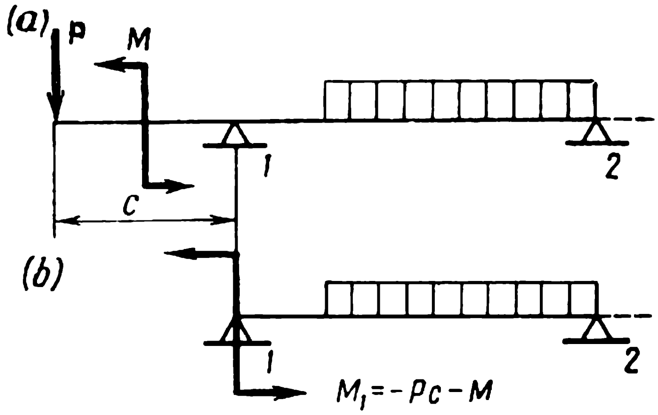


Fig. 107

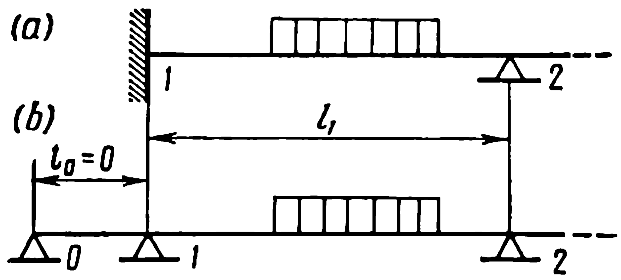


Fig. 108

to it, calculated with respect to the nearest support and applied to it with the respective sign (Fig. 107*b*). This moment should be included in the left-hand part of the three moment equation.

3. If the end cross section of the continuous beam is fixed (Fig. 108*a*), its angle of rotation is zero ($\theta_1 = 0$). In the final form this condition can be represented in the three moment equation by replacing the fixed end with a fictitious span of a length $l_0 = 0$ (Fig. 108*b*). Considering the two adjacent spans l_0 and l_1 the condition $\theta_1 = 0$ can be written in the three moment equation as

$$2M_1l_1+M_2l_1=-6EI\theta_1^r \qquad (132)$$

4. If an external concentrated force couple is applied at the section directly over an intermediate support of the continuous beam (Fig. 109*a*), it is better to include the moment M of this couple in the intraspan load.

The moment of the couple can be referred either to the left-hand span alone (Fig. 109*b*), or to the right-hand span alone (Fig. 109*c*) or to both the left- and right-hand spans simultaneously in which case it is divided arbitrarily (Fig. 109*d*). The moment of the couple should preferably be referred to the less loaded span.

After finding all the bending moments at the sections directly over the supports of the continuous beam by the use of the three moment equations, the reactions are calculated for each of the small beams supported at the ends due to the load in the span and the moments applied at the ends. For example, the reaction at the i-th support of the continuous beam will be the sum of the reactions at the i-th supports of two adjacent small beams. Each of them is determined by the conditions of statics. The general formula for the reaction at the i-th support will be of the form

$$A_i = A_i^0 + \frac{M_{i-1} - M_i}{l_{i-1}} + \frac{M_{i+1} - M_i}{l_i} \tag{133}$$

in which A_i^0 is the total reaction at the i-th supports of two adjacent small beams due only to the given loads in the spans;

$\frac{M_{i-1} - M_i}{l_{i-1}}$ is the reaction at the i-th support of the left-hand small beam due only to the moments M_{i-1} and M_i;

$\frac{M_{i+1} - M_i}{l_i}$ is the reaction at the i-th support of the right-hand small beam due only to the moments M_{i+1} and M_i.

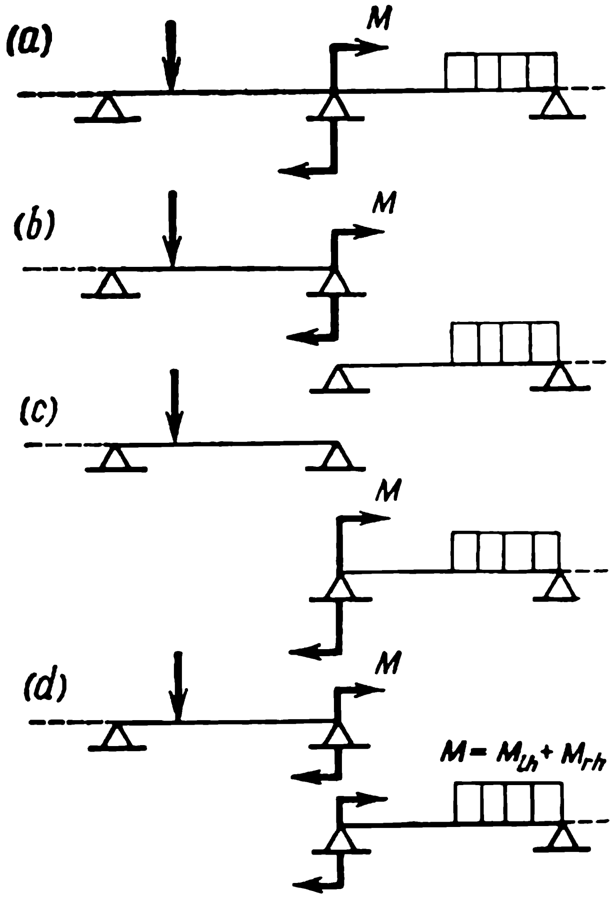


Fig. 109

In determining the reaction at a fixed end the fictitious zero span should be neglected.

In determining the reaction of the support which is followed by the loaded cantilever, the value A_i^0 includes the resultant of all the external forces acting on the cantilever.

The diagrams of transverse (shearing) forces and bending moments for a continuous beam can be constructed separately for each span, as for a small beam simply supported at the ends and subject to the given forces and support moments. These diagrams can also be drawn by the method of superposition of the diagrams of the moments at the sections over the supports and of the given forces in each span.

The angle of rotation and deflection in an arbitrary section of a continuous beam can be expediently determined by considering only one span as a small beam simply supported at the ends and subject to the given load within the span and the moments at the sections directly

over the supports. All the methods used in the analysis of statically determinate beams are applicable here.

Example 58. Let $P = 2$ tnf, $M = 4$ tnf-m, $q = 6$ tnf/m, $c = 1$ m, $l_1 = 3$ m, $l_2 = 2$ m, $[\sigma] = 1600$ kgf/cm² and $E = 2 \times 10^6$ kgf/cm² (Fig. 110*a*).

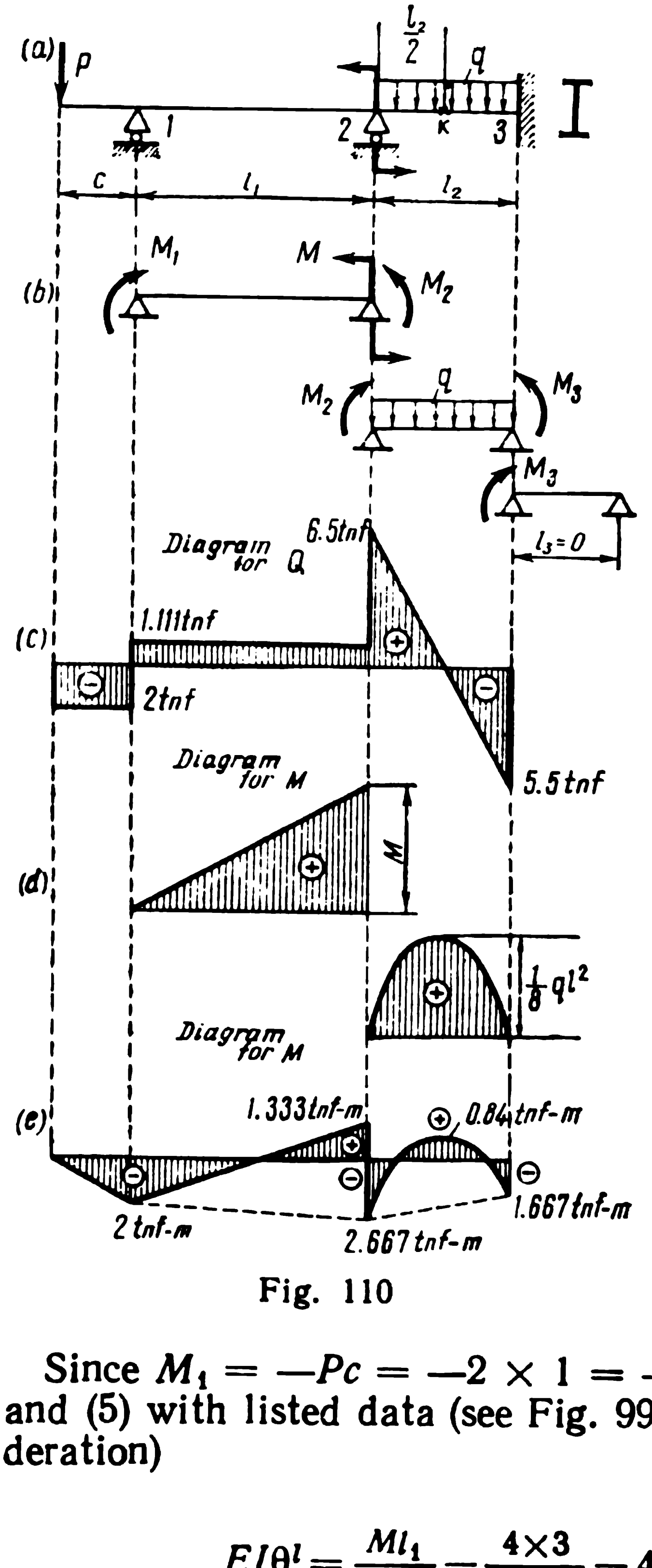


Fig. 110

Determine the size number of the I-beam and f_k (the deflection at point k).

Solution. First we divide the beam into two small beams with spans l_1 and l_2 (Fig. 110*b*). We load the small beam of span l_1 at the left-hand support with the moment M_1 due to force P acting on the cantilever, and at the right-hand support, with a force couple of moment M and the unknown bending moment M_2.

Next we load the small beam of span l_2 with a distributed load q, with the unknown bending moment M_2 at the left-hand support, and with the unknown bending moment M_3 at the right-hand support.

Then we introduce the fictitious span $l_3 = 0$ in place of the fixed end.

For the given two-fold statically indeterminate beam the two three-moment equations are of the form

$$\left.\begin{aligned} M_1l_1 + 2M_2(l_1+l_2) + M_3l_2 &= -6EI(\theta_1^l - \theta_1^r); \\ M_2l_2 + 2M_3l_2 &= -6EI\theta_2^l \end{aligned}\right\}$$

Since $M_1 = -Pc = -2 \times 1 = -2$ tnf-m and from the beams (4) and (5) with listed data (see Fig. 99) (taking the sign rule into consideration)

$$EI\theta_1^l = \frac{Ml_1}{3} = \frac{4\times3}{3} = 4 \text{ tnf-m}^2;$$

$$EI\theta_1^r = EI\theta_2^l = \frac{ql_2^3}{24} = \frac{6\times2^3}{24} = 2 \text{ tnf-m}^2$$

then the three moment equations will be of the form

$$\left.\begin{aligned} 10M_2 + 2M_3 &= -30; \\ M_2 + 2M_3 &= -6 \end{aligned}\right\}$$

from which $M_2 = -\frac{8}{3} \cong -2.667$ tnf-m; $M_3 = -\frac{5}{3} \cong -1.667$ tnf-m. Using formula (133) we determine the reactions A_1, A_2 and A_3 at the support cross sections

$$A_1 = A_1^0 + \frac{M_2 - M_1}{l_1} = P + \frac{M}{l_1} + \frac{M_2 - M_1}{l_1}$$

$$= 2 + \frac{4}{3} + \frac{-\frac{8}{3}+2}{3} = \frac{28}{9} \cong 3.111 \text{ tnf};$$

$$A_2 = A_2^0 + \frac{M_1 - M_2}{l_1} + \frac{M_3 - M_2}{l_2} = -\frac{M}{l_1} + \frac{ql_2}{2} + \frac{M_1 - M_2}{l_2}$$

$$+ \frac{M_3 - M_2}{l_2} = -\frac{4}{3} + 6 + \frac{-2+\frac{8}{3}}{3} + \frac{-\frac{5}{3}+\frac{8}{3}}{2} = \frac{97}{18} \cong 5.389 \text{ tnf};$$

$$A_3 = A_3^0 + \frac{M_2 - M_3}{l_2} = \frac{ql_2}{2} + \frac{M_2 - M_3}{l_2}$$

$$= 6 + \frac{-\frac{8}{3}+\frac{5}{3}}{2} = \frac{11}{2} = 5.5 \text{ tnf}$$

Then we check the calculated reactions by the sum of the projections on the vertical axis

$$A_1 + A_2 + A_3 - P - ql_2 = 0; \quad \frac{1}{18}(56 + 97 + 99) - 2 - 12 = 0$$

Now we can plot the diagram for the transverse (shearing) force, as shown in Fig. 110*c*.

Using the superposition method we construct the diagram for the bending moment. Owing to the loads in the span the diagram is triangular for the small left-hand beam and parabolic for the small right-hand beam (Fig. 110*d*).

The diagram for moments M_1, M_2 and M_3 is shown as a broken dotted line (Fig. 110*e*). We add to it the rectilinear diagram due to force P on the cantilever. We plot ordinates of the diagrams, from the dotted lines in the spans, for the given forces alone. Taking the signs of the moment and load diagrams into consideration, we obtain the resultant diagram of the bending moment.

Next we find x at which $M_{\max_{sp}}$ will be obtained in the right-hand span

$$x = \frac{Q}{q} = \frac{117}{18 \times 6} \cong 1.08 \text{ m}$$

Since

$$\Delta M=\frac{Qx}{2}=\frac{117}{18}\times\frac{1.08}{2}=3.51 \text{ tnf-m}$$

then

$$M_{\max_{sp}}=3.51-\frac{8}{3}\cong 0.84 \text{ tnf-m}$$

As is evident from the diagram

$$M_{\max}=\frac{8}{3} \text{ tnf-m}$$

According to the design formula

$$W=\frac{M_{\max}}{[\sigma]}=\frac{8\times 10^5}{3\times 16\times 10^2}\cong 167 \text{ cm}^3$$

For I-beam No. 18a (from data for rolled steel shapes)

$$W=159 \text{ cm}^3 \text{ and } \frac{\sigma_{\max}-[\sigma]}{[\sigma]}\times 100=\frac{167-159}{159}\times 100=4.8\%$$

(i.e. $\leqslant 5\%$ overstress, which is permissible).

We select I-beam No. 18a, for which

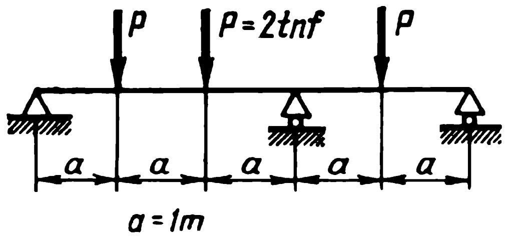
$$\sigma_{\max}=\frac{M_{\max}}{W}=\frac{8\times 10^5}{3\times 159}=1667 \text{ kgf/cm}^2 \text{ and } I=1430 \text{ cm}^4$$

For determining the deflection f_k make use of superposition and the listed data for the deflection of beams (5) and (4) (see Fig. 99).

$$f_k=-\frac{5}{384}\frac{ql_1^4}{EI}-\frac{M_2 l_2^2}{16EI}-\frac{M_3 l_2^2}{16EI}=\frac{10^9}{16\times 2\times 10^6\times 1.43\times 10^3}\left(-\frac{5}{24}\right.$$
$$\left.\times 6\times 16+\frac{8}{3}\times 4+\frac{5}{3}\times 4\right)\cong 0.058 \text{ cm}\cong 0.58 \text{ mm}$$

Problems 544 through 549. Analyse the statically indeterminate beams (544 and 545); select the required cross-sectional dimensions of the beams (546 through 549).

544



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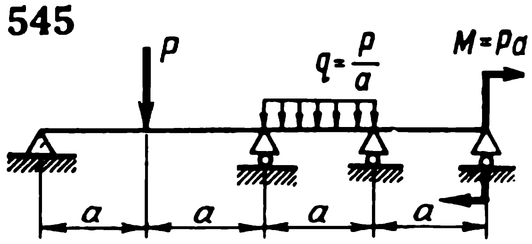

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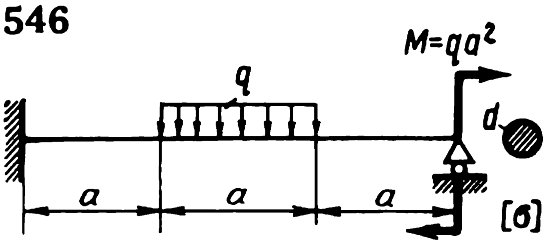

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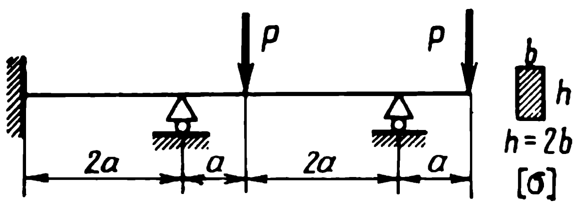

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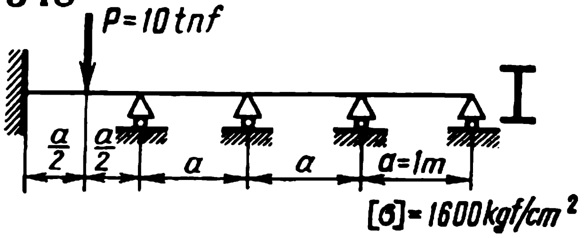

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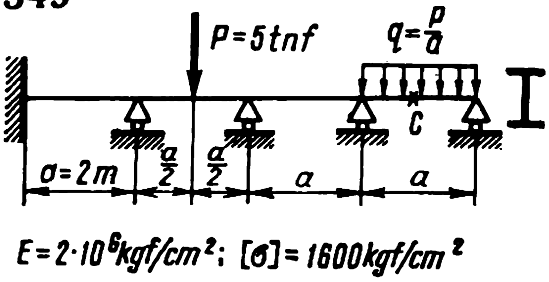


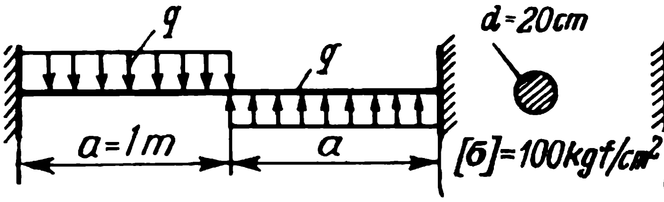
In Problem 549 also find the deflection at cross section C.

Problems 550 through 555. Determine the permissible loads for the beams.

In Problems 551, 552 and 553 assume that $[\sigma] = 1600$ kgf/cm²

In Problems 554 and 555 assume that $[\sigma] = 160$ MN/m².

550



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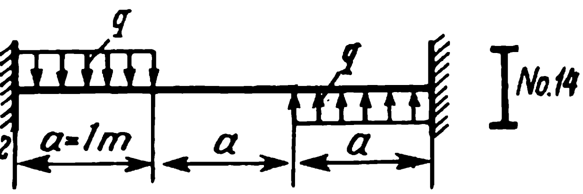

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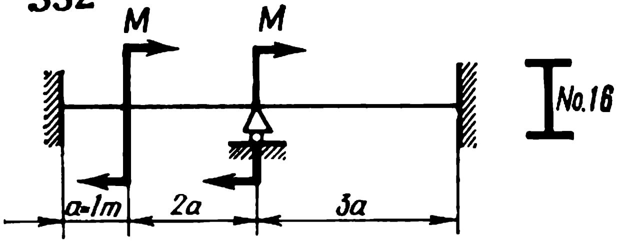

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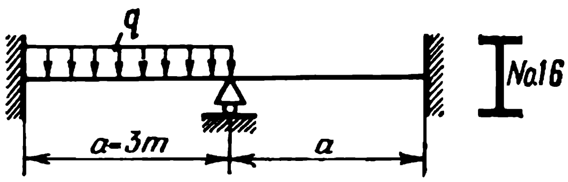

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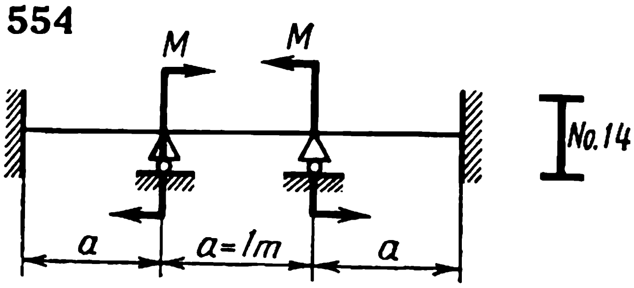

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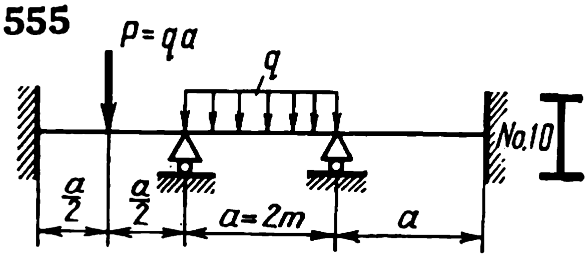
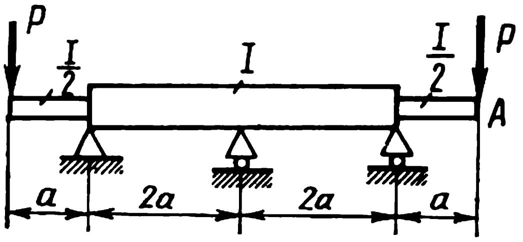


Problems 556 and 557. Determine the deflection of the stepped beams at cross sections A.

556



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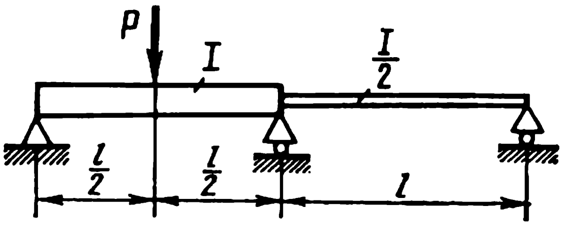
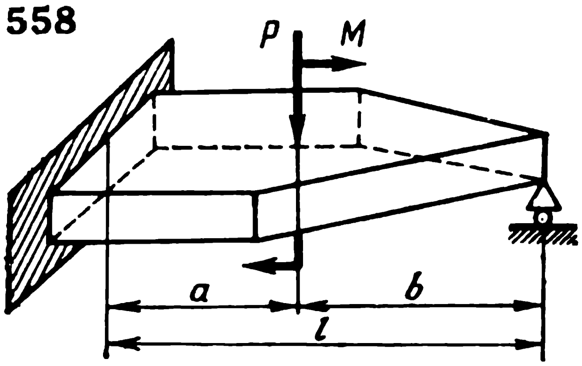


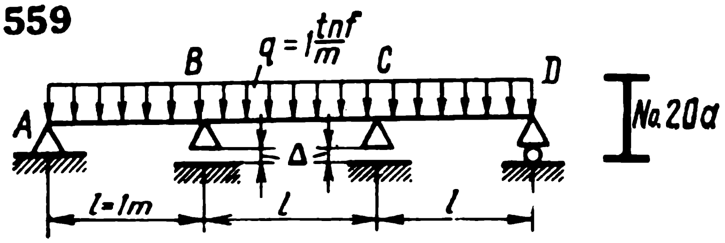
Problem 558. Find M, a and b for which the beam of variable cross section will be one of constant strength.

558



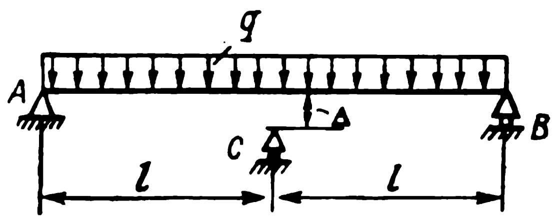
Problem 559. Find the maximum bending moment for the beam, if the middle supports settle by the amount $\Delta = 1$ mm ($E = 2 \times$ $\times\ 10^6$ kgf/cm^2).

559



Problems 560 and 561. Determine the clearances Δ at which the maximum normal stresses in beams of given rigidity EI of cross section are minimum.

560



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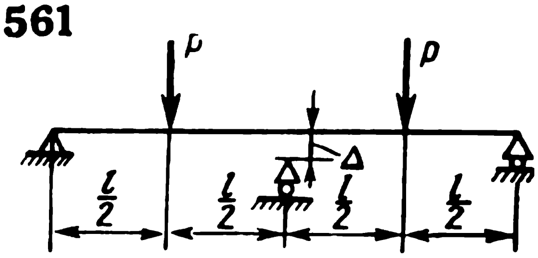


FUNDAMENTALS OF LIMIT DESIGN FOR STATICALLY INDETERMINATE BEAMS

It is assumed that the statically indeterminate beam is made of a ductile material, having an idealized tension and compression diagrams.

Since the supporting power of each span of the beam is exhausted (the limiting case is reached) when three plastic hinges occur in it (one in the span and two over the supports), all the spans can be considered separately and independently of one another. The method of equalizing the bending moments can be conveniently employed to analyse the beam.

For a beam of uniform cross section the permissible bending moment will also be constant and can be found from formula (114)

$$M'_{max} = 2\,[\sigma]\,S$$

in which S is the static moment of half the cross-sectional area about the centroidal axis.

The final (equalized) bending moment diagram is constructed for each span in such a way that the ordinates are equal to the values M'_{max} at the sections directly over the supports and in the span. The value M'_{max} is found from the loads in the spans on the basis of the

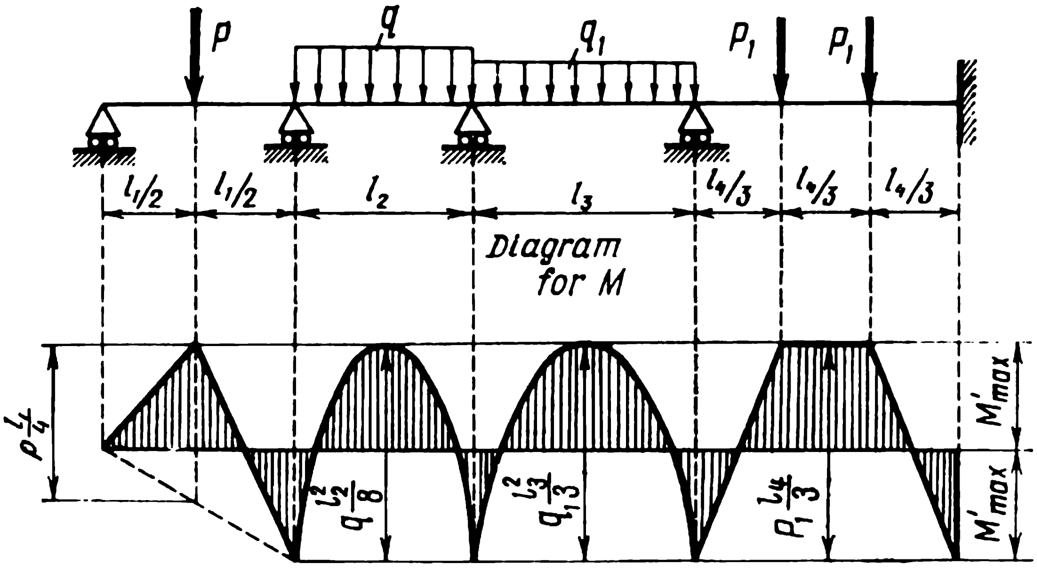


Fig. 111

geometry of the obtained equalized diagrams. The maximum permissible loads for each span are determined by the use of formula (114).

If the end of the beam is on a hinged support, two plastic hinges (in the span and at the section directly over the intermediate support) will be sufficient for the last span to reach a geometrically variable state. In this case the bending moment at the end of the beam equals zero and in the span and at the section over the intermediate support it equals M'_{max}.

Fig. 111 illustrates the construction of the diagram of equalized bending moments for a continuous beam. For a stepped beam with given values of S the equalized diagram of bending moments is constructed for each step in accordance with the respective value of M_{max}. Subsequent calculations are similar to those for beams of uniform cross section.

If the required cross section of a beam is to be selected for a given load, this load is used to determine the permissible bending moments M'_{max} at the sections of each span where the plastic hinges should occur. The largest of these moments is used to determine $S = \frac{M'_{max}}{2[\sigma]}$ and the dimensions of the cross section.

In designing a beam of variable cross section its dimensions are found for each span in the same way.

Example 59. Given: S, l_1, l_2 and $[\sigma]$ (Fig. 112).

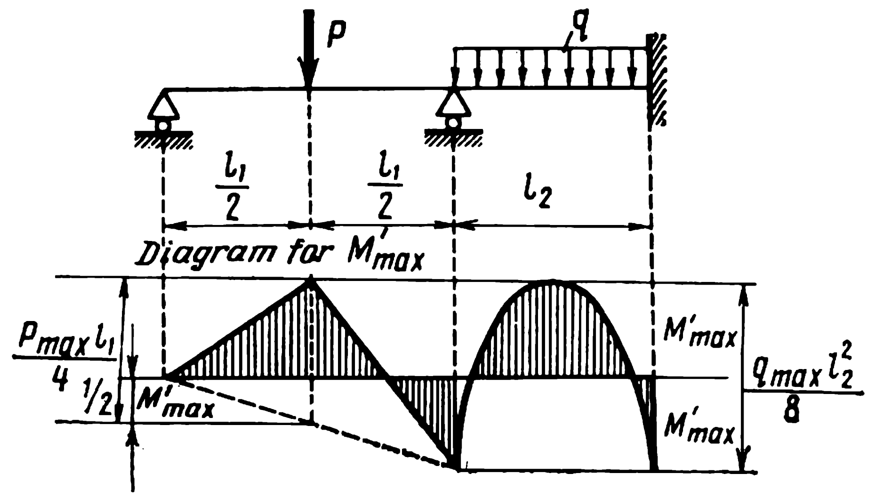


Fig. 112

Find $P'_{\max}$ and $q'_{\max}$.

Solution. First we construct the diagram for the "equalized" per missible bending moment

$$M'_{\max} = 2\,[\sigma]\,S$$

Since from this diagram

$$\frac{P'_{\max} l_1}{4} = \frac{3}{2} M'_{\max} = 3\,[\sigma]\,S$$

and

$$\frac{q'_{\max} l_2^2}{8} = 2\,M'_{\max} = 4\,[\sigma[\,S$$

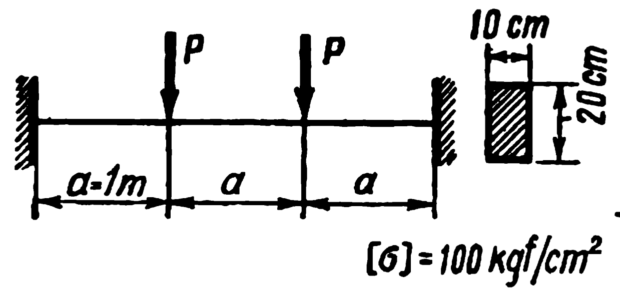
then the maximum permissible loads in the spans of the beam should be as follows:

$$P'_{\max} = 12\,[\sigma]\frac{S}{l_1} \quad \text{and} \quad q_{\max} = 32\,[\sigma]\,\frac{S}{l_2^2}$$

Problems 562 through 567. Determine the permissible forces for the beams and systems.

In Problems 564 through 567 assume that $[\sigma]$, a, W and $\eta = \frac{2S}{W}$ are known.

562



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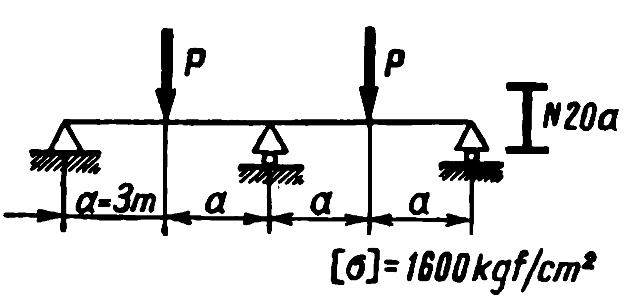

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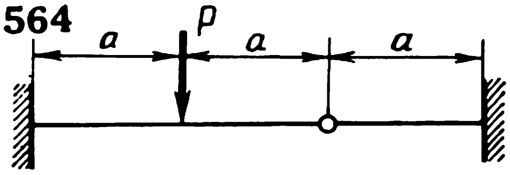

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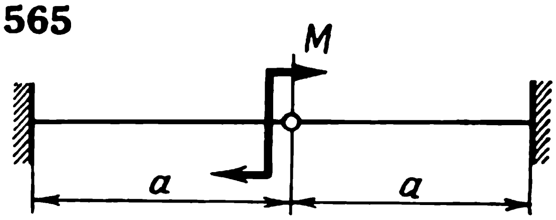

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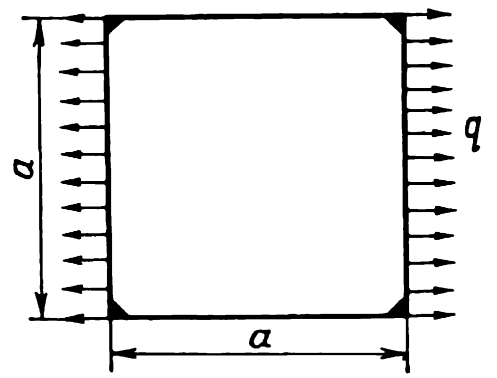

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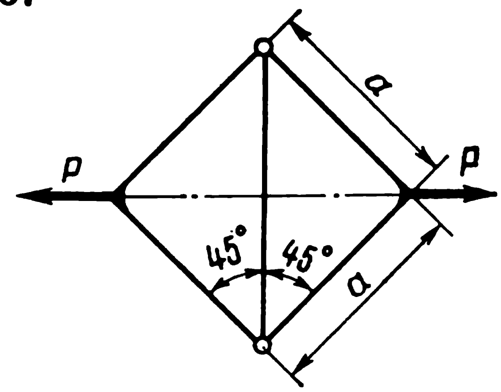
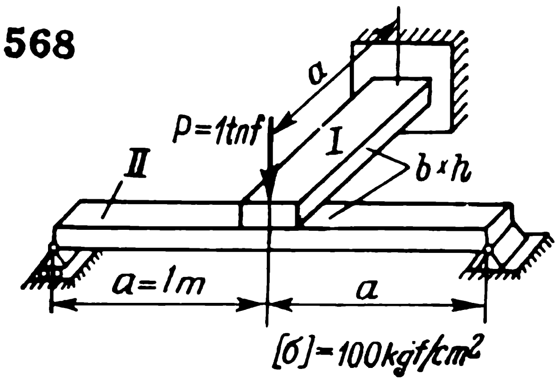


Problems 568 through 571. Determine the dimensions of the cross sections of the beams and systems.

Assume that the cross sections of the beams are rectangular ($b = 2h$) and that the beams and tie rods are of the same material.

568



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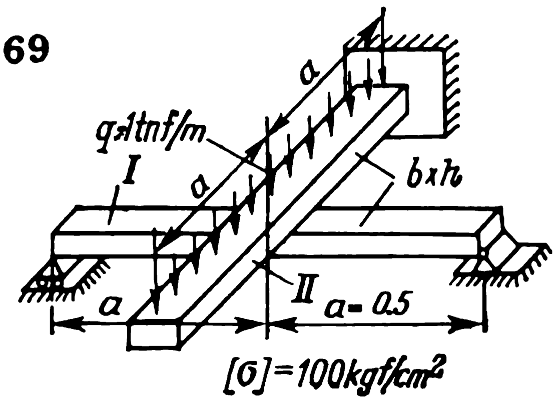

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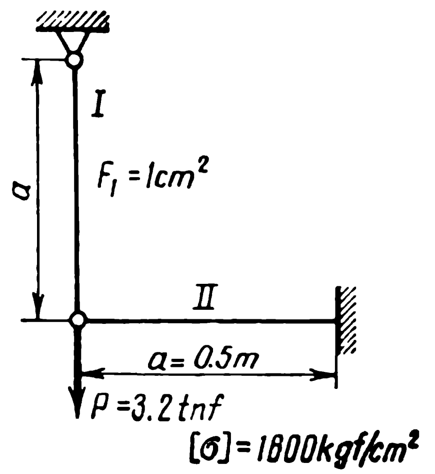

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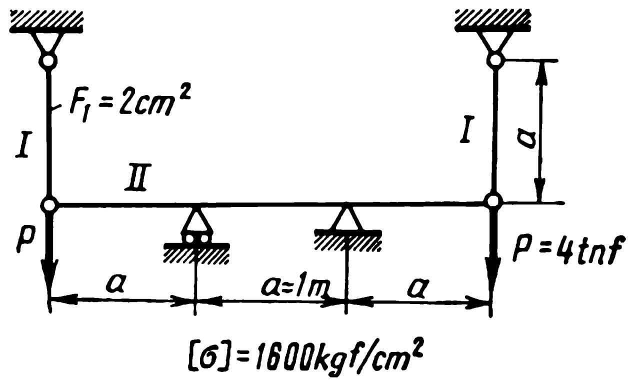
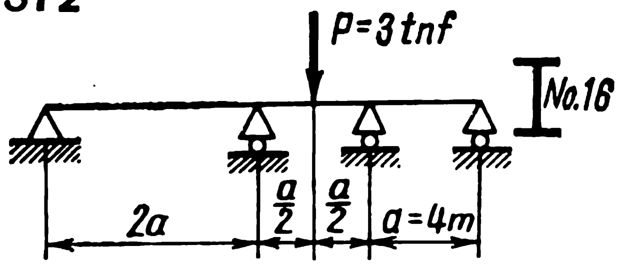


Problems 572 through 575. Determine the safety factors n_y (by load) which were used in designing the beams and systems.

The yield point of the material $\sigma_y = 2400$ kgf/cm².

572



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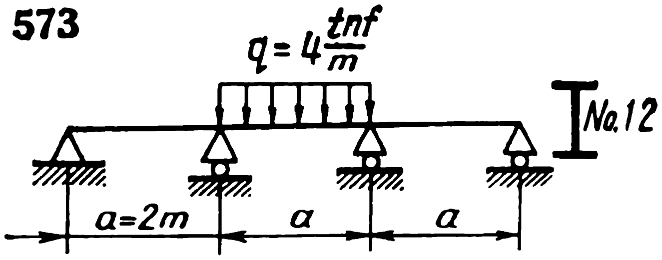

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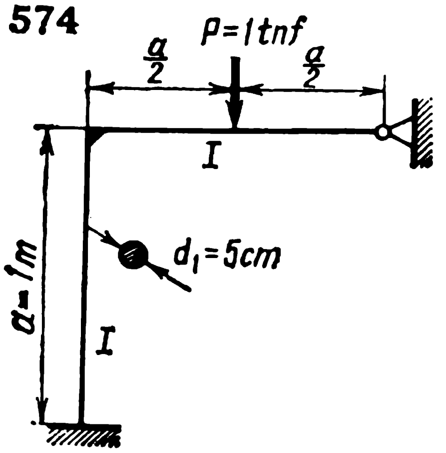

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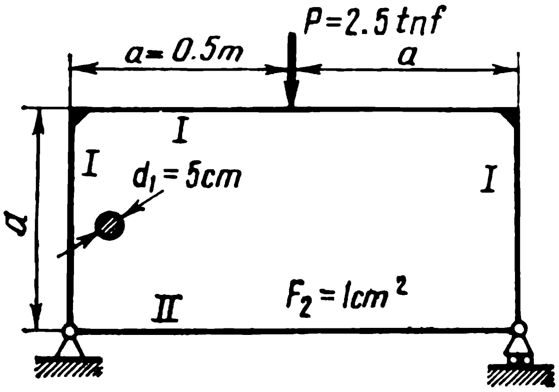


8.9.

Elastic Strain Energy in Bending

The amount of elastic strain energy in a beam subject to plane transverse bending is determined by the formula

$$U=\sum \int \frac{M^2\,dx}{2EI}+\sum k\int \frac{Q^2\,dx}{2GF} \tag{134}$$

in which

$$k=F\int_F \left(\frac{S}{bI}\right)^2 dF \tag{135}$$

is an abstract factor, characterizing the nonuniformity of distribution of shearing stresses in the cross section of the beam, and depending on the shape of the cross section.

In formula (134) integration is carried out over the length of each portion of the beam and summation over all the portions. In formula (135) integration is carried out over the area F of the cross section; S, b and I have the same meaning as in formula (99) for determining the shearing stress; E and G are the moduli of elasticity of the beam material in tension or compression and in shear.

Example 60. Given: a, a_0, h and h_0 (Fig. 113).

Find $k=F\int_F \left(\frac{S}{bI}\right)^2 dF$.

Solution. The area of the given cross section is

$$F = ah - a_0h_0$$

The equatorial moment of inertia of the cross-sectional area about the z-axis is

$$I = \frac{1}{12}(ah^3 - a_0h_0^3)$$

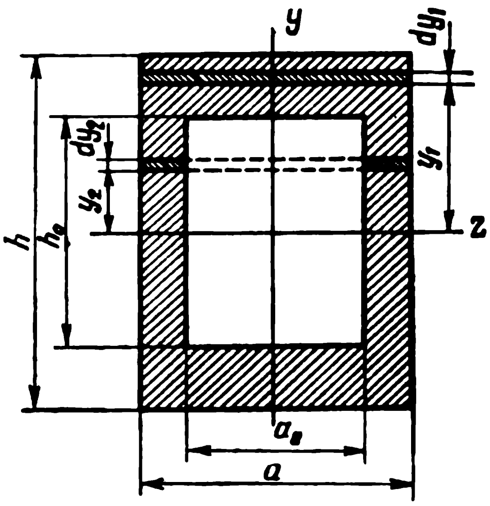


Fig. 113

For the flanges

$$S_1 = \frac{a}{2}\left(\frac{h^2}{4} - y_1^2\right); \quad dF_1 = a\,dy; \quad b = a$$

and for the webs

$$S_2 = \frac{ah^2 - a_0h_0^2}{8} - \frac{a - a_0}{2}y_2^2;$$

$$dF_2 = (a - a_0)\,dy_2;$$

$$b = a - a_0$$

The coefficient of nonuniformity of the shearing stresses is

$$k = \frac{2F}{I^2}\left[\int_{F_1}\left(\frac{S_1}{a}\right)^2 dF_1 + \int_{F_2}\left(\frac{S_2}{a - a_0}\right)^2 dF_2\right]$$

$$= \frac{F}{2I^2}\left\{a\int_{h_0/2}^{h/2}\left(\frac{h^2}{4} - y_1^2\right)^2 dy_1 + (a - a_0)\int_0^{h/2}\left[\frac{ah^2 - a_0h_0^2}{4(a - a_0)}\right.\right.$$

$$\left.\left. - y_2^2\right]^2 dy_2\right\} = \frac{F}{2I^2}\left\{a\left[\frac{h^4}{16}\left(\frac{h}{2} - \frac{h_0}{2}\right) - \frac{h^2}{2}\right.\right.$$

$$\left.\times\left(\frac{h^3}{3\times 8} - \frac{h_0^3}{3\times 8}\right) + \frac{h^5 - h_0^5}{5\times 32}\right] + (a - a_0)$$

$$\times\left[\frac{ah^2 - a_0h_0^2}{16(a - a_0)^2} \times \frac{h_0}{2} - \frac{ah^2 - a_0h_0^2}{2(a - a_0)}\right.$$

$$\left.\left.\times\frac{h_0^3}{3\times 8} + \frac{h_0^5}{5\times 32}\right]\right\}$$

Substituting the values of F and I and after certain transformations we obtain

$$k = \frac{6}{5}\,\frac{1 - \dfrac{a_0h_0}{ah}}{\left(1 - \dfrac{a_0}{a}\right)\left(1 - \dfrac{a_0h_0^3}{ah^3}\right)^2}\left\{1 - \frac{a_0}{a}\left[1 - \frac{1}{8}\left(15\frac{h_0}{h}\right.\right.\right.$$

$$\left.\left.\left. - 30\frac{h_0^3}{h^3} + 7\frac{h_0^5}{h^5} + 8\frac{a_0}{a} \times \frac{h_0^5}{h^5}\right]\right\}$$

Particular cases are:

(1) $a_0 = a$; $k = \frac{6}{5}$

(2) $a = h$ and $a_0 = h_0$; $k = \frac{6}{5} \times \frac{1 + \frac{a_0}{a}}{\left(1 + \frac{a_0^4}{a^4}\right)^2} \left\{1 - \frac{a_0}{a}\right.$

$\left. \times \left[1 - \frac{1}{8} \frac{a_0}{a} \left(15 - 30 \frac{a_0^2}{a^2} + 7 \frac{a_0^4}{a^4} + 8 \frac{a_0^5}{a^5}\right)\right]\right\}$

If, for example, $a_0 = \frac{a}{2}$; then $k = \frac{387}{250} = 1.548$.

(3) I-beam No. 20.

From the data for standard rolled steel shapes, $a = b = 10$ cm;

$$h = 20 \text{ cm}; \; a - a_0 = d = 0.52 \text{ cm}; \; a_0 = 9.48 \text{ cm};$$

$$h_0 = h - 2t = 20 - 2 \times 0.84 = 18.32 \text{ cm}$$

Since $\frac{a_0}{a} \cong 0.948$; $\frac{h_0}{h} = 0.916$; $\frac{a_0 h_0}{ah} \cong 0.868$;

$\frac{h_0^3}{h^3} = 0.769$; $\frac{a_0 h_0^3}{ah^3} \cong 0.729$; $\frac{h_0^5}{h^5} \cong$

$\cong 0.645$ and $\frac{a_0 h_0^5}{ah^5} = 0.611$

then

$$k = \frac{6}{5} \times \frac{1 - 0.868}{(1 - 0.948)(1 - 0.729)^2} \left\{1 - 0.948\left[1 - \frac{1}{8}(15 \times 0.916\right.\right.$$

$$\left.\left. - 30 \times 0.769 + 7 \times 0.645 + 8 \times 0.611)\right]\right\} \cong 2.51$$

For rolled I-beams the factor of nonuniformity of shearing stresses in bending can be approximately found from the formula

$$k = \frac{1 - \frac{a_0 h_0}{ah}}{1 - \frac{a_0}{a}} = \frac{F}{F_w}$$

in which F = cross-sectional area

F_w = area of the web of height h, i.e. $F_w = hd$.

Using this formula, for example, for the No. 20 I-beam we find $F = 26.8$ cm², $F_w = 20 \times 0.52 = 10.4$ cm², $k = \frac{26.8}{10.4} \cong 2.58$

Example 61. Let $P = 60$ kN, $q =$ kN/m, $a = 2$ m, $E = 2 \times 10^5$ MN/m², $G = 8 \times 10^4$ MN/m² and $[\sigma] = 160$ MN/m² (Fig. 114).

Determine the size number of the I-beam and U.

Solution. First we determine the reaction of the supports from the statics equations:

$$A=\frac{2P+2qa}{3}=80\text{ kN};\quad B=P+2qa-A=100\text{ kN}$$

Writing equations for Q and M we obtain:

$Q_{x_1}=A=80$ kN;
$M_x=A_{x_1}=80x_1$ kN-m;
$M_{x_1=0}=0$;
$M_{x_1=a}=160$ kN-m;
$Q_{x_2}=-B+qx_2=-100+30x_2$; $Q_{x_2=0}=-100$ kN;
$Q_{x_2=2a}=-100+30\times 4=20$ kN;
$M_{x_2}=Bx_2-\frac{qx_2^2}{2}=100x_2-\frac{30x_2^2}{2}$; $M_{x_2=0}=0$;

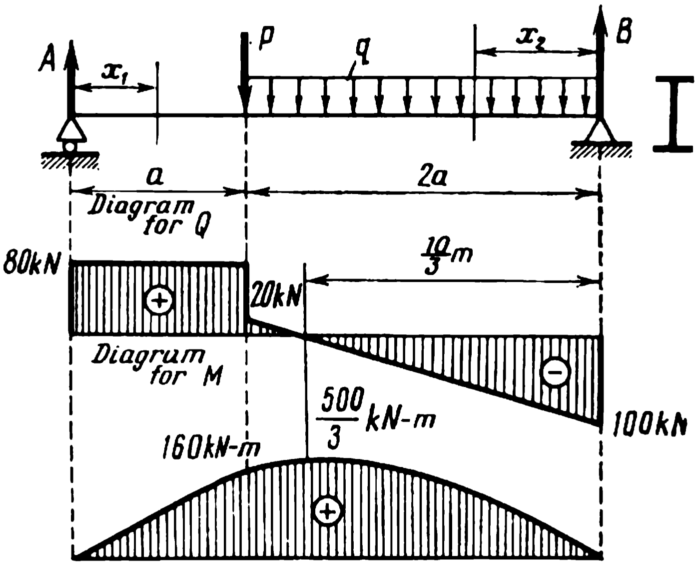


Fig. 114

$$M_{x_2=2a}=100\times 4-\frac{30}{2}\times 16=160\text{ kN-m}$$

Next we determine $M_{\max}$. From the condition

$$Q_{x_2}=-100+30x_2=0,\ x_2=\frac{10}{3}\text{ m}$$

Hence

$$M_{\max\,x_2=\frac{10}{3}}=100\times\frac{10}{3}-\frac{30}{2}\times\frac{10^2}{9}=\frac{500}{3}\text{ kN-m}$$

From the design formula it follows that

$$W=\frac{M_{\max}}{[\sigma]}=\frac{500\times 10^3}{3\times 160\times 10^6}\cong 1042\times 10^{-6}\text{ m}^3\cong 1042\text{ cm}^3$$

From the data for rolled steel shapes we select the No. 45 I-beam for which

$$W=1220\text{ cm}^3;\ I=27{,}450\text{ cm}^4;\ F=83.0\text{ cm}^2;\ h=45\text{ cm};$$

$$d=0.86\text{ cm and } k=\frac{F}{F_w}=\frac{83}{45\times 0.86}\cong 2.14$$

The elastic strain energy of the beam can be found as the sum of energies due to the bending moment (U_M) and transverse (shearing) force (U_Q).

According to formula (134)

$$U_M=\sum\int\frac{M^2\,dx}{2EI}=\frac{1}{2EI}\left(\int_0^a M_{x_1}^2\,dx_1+\int_0^{2a}M_{x_2}^2\,dx_2\right)$$

$$=\frac{1}{2EI}\left[64\int_0^a x_1^2\,dx_1+\frac{1}{4}\int_0^{2a}(20x_2-3x_2^2)^2\,dx_2\right]\times 10^8$$

$$=\frac{10^8}{2\times 2\times 10^{11}\times 27{,}450\times 10^{-8}}\left(\frac{64}{3}\times 8+\frac{800}{3}\times 8-120\times 16\right.$$

$$\left.+\frac{72}{5}\times 32\right)\cong 769.4\ \text{J};$$

$$U_Q=\sum k\int\frac{Q^2\,dx}{2GF}=\frac{k}{2GF}\left(Q_{x_1}^2a+\int_0^{2a}Q_{x_2}^2\,dx_2\right)$$

$$=\frac{k\times 10^8}{2GF}\left[64a+\int_0^{2a}(-10+3x_2)^2\,dx_2\right]$$

$$=\frac{k\times 10^8}{2GF}(64a+200a-120a^2+24a^3)$$

$$=\frac{2.14\times 10^8}{2\times 8\times 10^{10}\times 83.0\times 10^{-4}}$$

$$\times(128+400-480+192)\cong 38.7\ \text{J};$$

$$U=U_M+U_Q=769.4+38.7=808.1\ \text{J}$$

Since for normal beams the strain energy due to the transverse (shearing) force is small in comparison with the strain energy from the bending moment, the former is usually neglected.

In this example

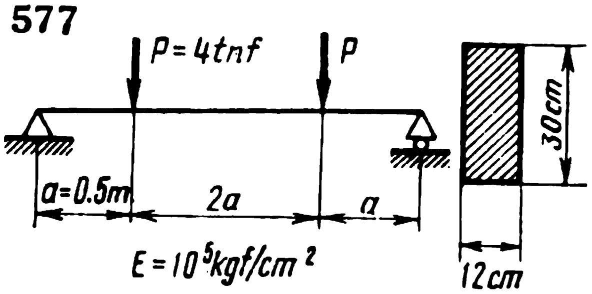
$$\frac{U_Q}{U_M}\times 100=\frac{38.7}{769.4}\times 100\cong 5\%$$

For short beams the strain energy due to the transverse (shearing) force cannot be neglected, since it can reach substantial magnitudes.

Problems 576 through 581. Determine the elastic strain energy U of the beams, taking only the bending moments into account.

The beams for Problem 576 are given in Problems 222 through 231. Assume the loads, lengths l and rigidity of cross sections EI to be known.

577



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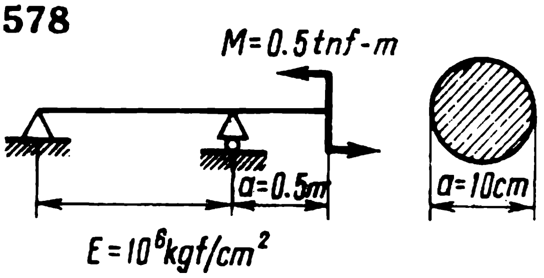

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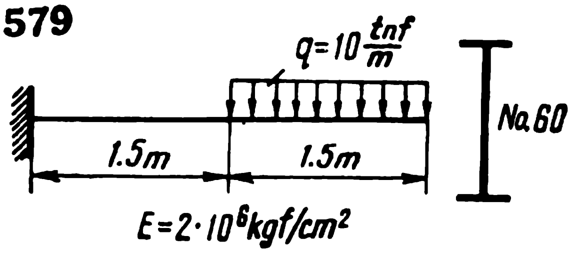

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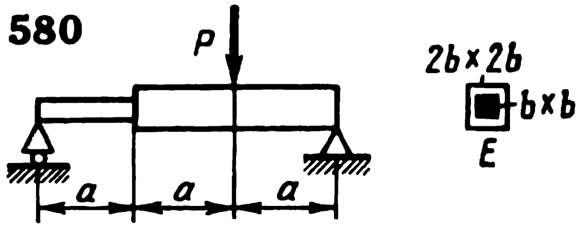

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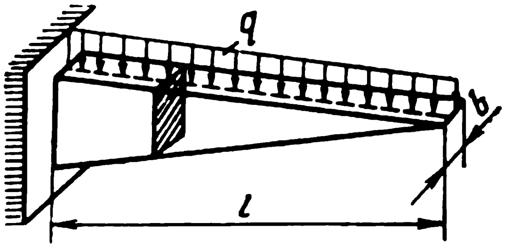


[σ]. E; b; beam of uniform strength

Problem 582. Determine the effect of the supports of a beam with a force applied at the middle on the strain energy due to the bending moment. Consider a beam:

(a) simply supported at the ends;

(b) with one fixed end and the other simply supported;

(c) with fixed ends.

Problem 583. Determine the change in the strain energy due to the bending moment if a rectangular beam on two supports and with force P in the middle is replaced by a beam of constant strength with a constant height of cross section.

Problem 584. Find the shape factors k for the following cross sections: (1) rectangular, (2) round, (3) equilateral triangle, (4) annular, (5) square box shape, and (6) No. 60 I-beam. Prove, using the I-beam section, that factor k for an I-beam can be assumed equal to the ratio of the cross-sectional area to the area of the web with the height of the section.

Problem 585. Determine the strain energy due to the shearing force for the beams of Problems 577, 578 and 579. Evaluate (in per cent) the degree to which the shearing force influences the strain energy in bending.

CHAPTER 9. COMBINED STRENGTH OF STRAIGHT BEAMS OF HIGH RIGIDITY

9.1.

Oblique, or Unsymmetrical, Bending

For beams subject to unsymmetrical, or oblique, bending, which is a combination of transverse bending in two directions, the normal stress σ at an arbitrary point of the cross section with the coordinates y and z (Fig. 115) is determined by the formula

$$\sigma = \frac{M_z y}{I_z} + \frac{M_y z}{I_y} = M\left(\frac{y\cos\alpha}{I_z} + \frac{z\sin\alpha}{I_y}\right) \tag{136}$$

in which I_y and I_z = principal centroidal moments of inertia of the cross section of the beam

M_y and M_z = bending moments about axes y and z, components of the resultant bending moment $M = \sqrt{M_y^2 + M_z^2}$, acting in plane xp which is located at an angle α to the principal plane of inertia xy of the beam.

Here and in the following we shall assume that M_y and M_z are positive, if they denote tensile stresses at points in the first quadrant of the cross section.

The equation for the neutral axis nn can be written in the following form:

$$y = -\frac{I_z}{I_y} \times \frac{M_y}{M_z} z = -\frac{I_z}{I_y} z\tan\alpha = -z\tan\beta \tag{137}$$

in which

$$\tan\beta = \frac{I_z}{I_y} \times \frac{M_y}{M_z} = \frac{I_z}{I_y}\tan\alpha \tag{138}$$

is the tangent of the angle of inclination of the neutral axis nn from the z-axis.

The neutral axis nn always deviates from the z-axis by angle β to the same side to which the trace pp of the plane of action of the forces deviates from the y-axis by angle α.

The maximum and minimum normal stresses are calculated from formula (136) substituting the coordinates (y_1, z_1 and y_2, z_2) of the points of tangency of straight lines, parallel to the neutral line, to the outline of the cross section.

If the maximum or minimum normal stress is developed at a point most remote from both principal centroidal axes of inertia of the section, then

$$\sigma_{max} = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z} \tag{139}$$

and

$$\sigma_{min} = -\left(\frac{|M_y|}{W_y} + \frac{|M_z|}{W_z}\right) \tag{140}$$

in which W_y and W_z are the equatorial section moduli of the section about the axes y and z.

The cross sections of beams subject to unsymmetrical bending are selected on the basis of the normal stresses by the trial-and-error method.

The first trial can be based on plane bending due to the component bending moment which requires the greater size of a beam.

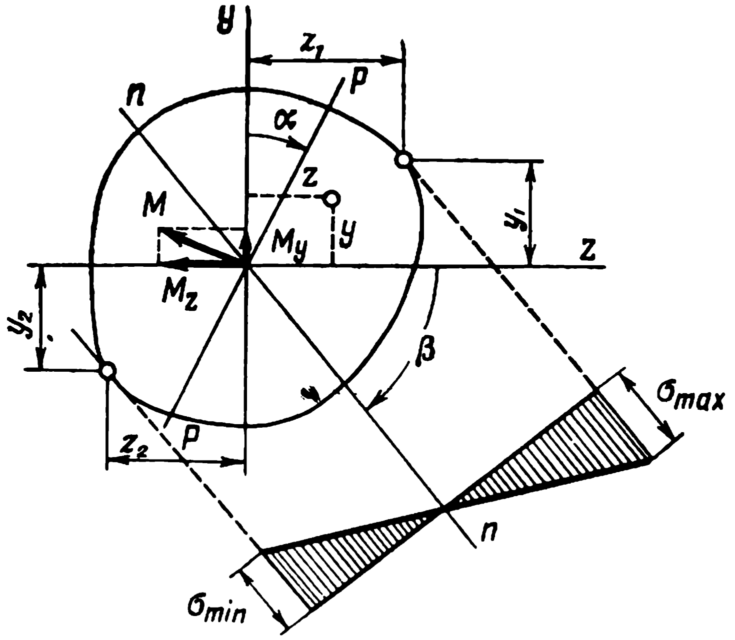


Fig. 115

For sections inscribed within a rectangle the first trial is made according to the formula

$$W_z \geqslant \frac{M_z + cM_y}{[\sigma]} \tag{141}$$

in which

$$c = \frac{W_z}{W_y}$$

For rectangular sections of height h and base b, $c = \frac{h}{b}$; for a rolled I-beam assume that $c = 8$ and for channels that $c = 6$.

The deflection f and angle θ of rotation of a cross section of the beam in unsymmetrical bending are determined as the geometrical sums of deflections and angles of rotation due to the components of the bending moment acting in the principal planes of inertia of the beam, i.e.

$$f = \sqrt{f_y^2 + f_z^2} \quad \text{and} \quad \theta = \sqrt{\theta_y^2 + \theta_z^2}$$

in which f_y and f_z = deflections in the direction of axes y and z
θ_y and θ_z = angles of rotation of the section about axes y and z.

The resultant angle of rotation is about the neutral axis and the resultant deflection, in a plane perpendicular to the neutral axis.

If unsymmetrical bending is due to two different systems of external forces acting in the principal planes of inertia of the beam, then the position of the neutral axis in an arbitrary cross section can be determined by the formula

$$\tan\beta = \frac{I_z}{I_y} \times \frac{M_y}{M_z}$$

and the position of the line of deflection, by the formula

$$\tan\beta' = \frac{f_z}{f_y}$$

since the angle β' between the direction of the resultant deflection and the y-axis is not equal to angle β between the neutral axis and z-axis.

In this case the elastic line of the beam is a space curve.

If the forces acting on the beam are located in several planes passing through the geometrical axis of the beam, then, by projecting the forces on the principal planes of inertia, we shall obtain the preceding case.

Example 62. Given: P, q, l, b, h, E and α (Fig. 116).

Determine the position of the neutral axis; $\sigma_{\substack{max\\min}}$ and f_{max}.

Solution. First we resolve force P and intensity q of the uniformly distributed load into components on the principal centroidal axes of inertia y and z of the cross section.

$$P_y = P\cos\alpha;\quad q_y = q\cos\alpha;\quad P_z = P\sin\alpha;\quad q_z = q\sin\alpha$$

The components of the maximum bending moment at the section directly under the force can be written as

$$M_{z_{max}} = \frac{P_y l}{4} + \frac{q_y l^2}{8} = \frac{l}{4}\left(P + \frac{ql}{2}\right)\cos\alpha;$$

$$M_{y_{max}} = \frac{P_z l}{4} + \frac{q_z l^2}{8} = \frac{l}{4}\left(P + \frac{ql}{2}\right)\sin\alpha$$

The bending moment M_y developed in the principal plane of inertia zx of the beam will stretch the fibres located to the left of axis y and

will compress the fibres to the right of it (Fig. 116*a*). The bending moment M_z developed in the principal plane of inertia yx of the beam will stretch the fibres below axis z and compress the fibres above this axis.

Thus the maximum tensile stresses σ_{max} are developed at point A of the central section of the beam and the maximum compressive

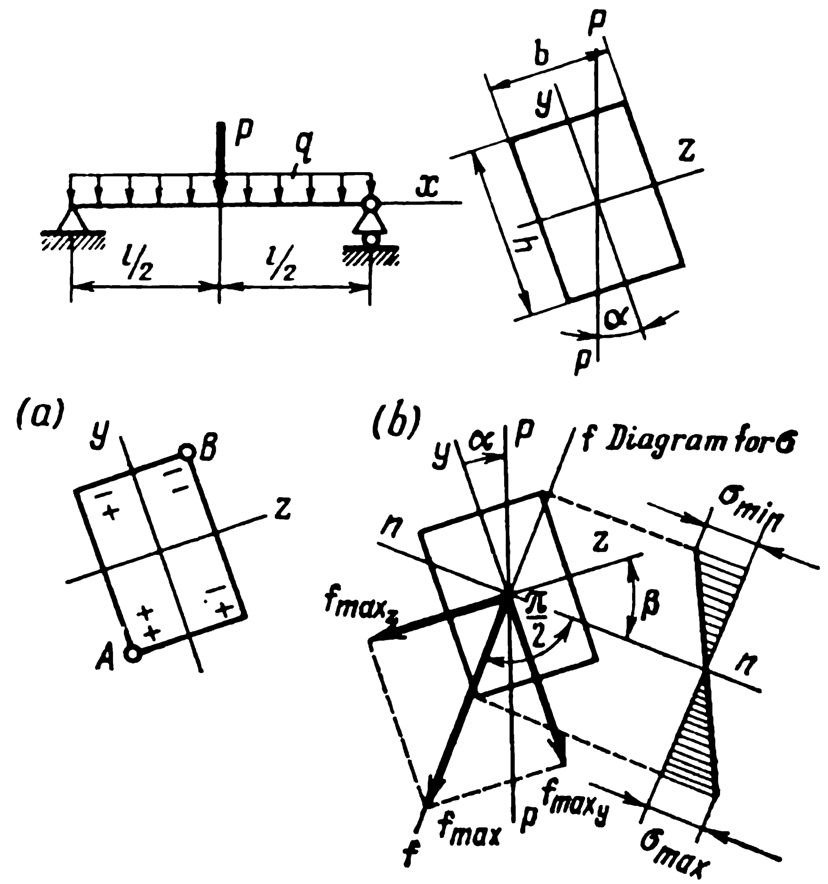


Fig. 116

stresses σ_{min}, at point B. They equal

$$\sigma_{\substack{max \\ min}} = \pm\left(\frac{M_{z\,max}}{W_z} + \frac{M_{y\,max}}{W_y}\right) = \pm\frac{3}{2}\frac{1}{bh}\left(P + \frac{ql}{2}\right)\left(\frac{\cos\alpha}{h} + \frac{\sin\alpha}{b}\right)$$

The position of the neutral axis nn can be determined from the equation

$$\tan\beta = \frac{I_z}{I_y}\tan\alpha = \frac{h^2}{b^2}\tan\alpha$$

It is evident from this equation that the greater the ratio $\frac{h}{b}$, the greater the difference between angle β and angle α.

If, for example, the plane of action of forces px is a diagonal plane of the beam, then $\tan\alpha = \frac{b}{h}$ and $\tan\beta = \frac{h^2 b}{b^2 h} = \frac{h}{b}$, i.e. the neutral plane nx is another diagonal plane of the beam.

In the case considered $h > b$, therefore $\beta > \alpha$ as is evident from Fig. 116*b*. The diagram for σ is also shown here.

From beams (5) and (6) with listed data (see Fig. 99) the maximum deflection due to M_y is

$$f_{z_{\max}}=\frac{5}{384}\frac{q_z l^4}{EI_y}+\frac{P_z l^3}{48EI_y}=\frac{l^3}{4Ehb^3}\left(\frac{5}{8}ql+P\right)\sin\alpha$$

and that due to M_z is

$$f_{y_{\max}}=\frac{5}{384}\frac{q_y l^4}{EI_z}+\frac{P_y l^3}{48EI_z}=\frac{l^3}{4Eh^3b}\left(\frac{5}{8}ql+P\right)\cos\alpha$$

The resultant bending deflection at the middle of the beam is

$$f_{\max}=\sqrt{f^2_{z_{\max}}+f^2_{y_{\max}}}=\frac{\left(\frac{5}{8}ql+P\right)l^3}{4Ehb}\sqrt{\frac{\sin^2\alpha}{b^4}+\frac{\cos^2\alpha}{h^4}}$$

The direction of the deflections is shown in Fig. 116*b*.

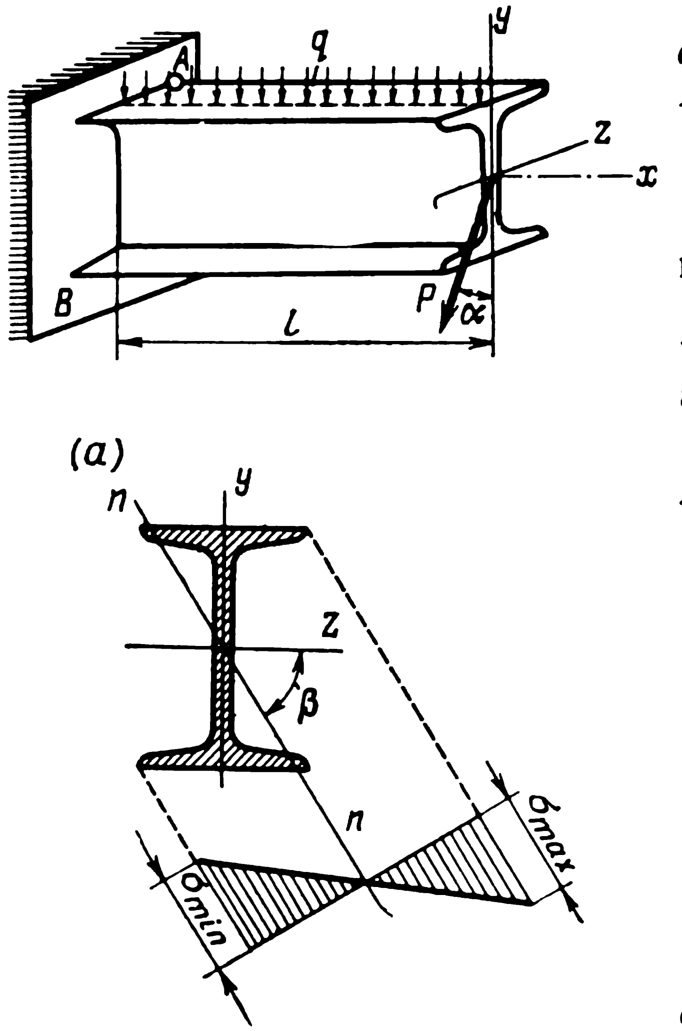


Fig. 117

Example 63. Let $P = 240$ kgf, $q = 400$ kgf/m, $l = 2$ m, $\alpha=30°$, $E = 2\times 10^6$ kgf/cm² and $[\sigma] = 1600$ kgf/cm² (Fig. 117).

Determine the size No. of the I-beam, the position of the neutral axis and $f_{\max}$.

Solution. We begin by finding the maximum bending moments at the fixed cross section

$$M_{y_{\max}}=Pl\sin\alpha=240\times2\times\frac{1}{2}=240\text{ kgf-m};$$

$$M_{z_{\max}}=\frac{ql^2}{2}+Pl\cos\alpha=\frac{400\times4}{2}+240\times2\times0.866\cong1216\text{ kgf-m}$$

As a first trial we assume that $c = 8$.

From formula (141) we have

$$W_z=\frac{M_z+cM_y}{[\sigma]}=\frac{1216+8\times240}{1600}\times100=196\text{ cm}^3$$

We check the nearest smaller I-beam (No. 20) for which

$$W_z=184\text{ cm}^3\quad\text{and}\quad W_y=23.1\text{ cm}^3$$

At points A and B of the fixed cross section we obtain $\sigma_{max} = -\sigma_{min}$, therefore

$$\sigma_{max} = \frac{M_{z\,max}}{W_z} + \frac{M_{y\,max}}{W_y} = \frac{121{,}600}{184} + \frac{24{,}000}{23.1} \cong 1700 \text{ kgf/cm}^2$$

Since

$$\frac{\sigma_{max} - [\sigma]}{[\sigma]} \times 100 = \frac{1700 - 1600}{1600} \times 100 \cong 6.2\% > 5\%$$

we select I-beam No. 20a for which $W_z = 203$ cm^3, $W_y = 28.2$ cm^3 and

$$\sigma_{max} = \frac{121{,}600}{203} + \frac{24{,}000}{28.2} \cong 1450 \text{ kgf/cm}^2$$

The understress is $\frac{1600-1450}{1600} \times 100 \cong 9.4\%$.

For the No. 20a I-beam $I_y = 155$ cm^4 and $I_z = 2030$ cm^4, therefore in the fixed cross section

$$\tan\beta = \frac{I_z}{I_y} \times \frac{M_{y\,max}}{M_{z\,max}} = \frac{2030}{155} \times \frac{240}{1216} \cong 2.58 \text{ and } \beta = 68°50'$$

Shown in Fig. 117*a* are the neutral axis *nn* and the diagram for the normal stress σ in the section in the fixed end of the beam.

The maximum deflection is at the free end of the beam. From beams (2) and (3) with listed data (see Fig. 99) we have

$$f_{y\,max} = \frac{1}{EI_z}\left(\frac{ql^4}{8} + \frac{Pl^3 \cos\alpha}{3}\right) = \frac{1}{2 \times 10^6 \times 2030}\left(\frac{4 \times 16 \times 10^8}{8} + \frac{240 \times 8 \times 10^6 \times 0.866}{3}\right) \cong 0.33 \text{ cm};$$

$$f_{z\,max} = \frac{Pl^3 \sin\alpha}{3EI_y} = \frac{240 \times 8 \times 10^6 \times 0.5}{3 \times 2 \times 10^6 \times 155} = 1.03 \text{ cm}$$

The maximum resultant deflection is

$$f_{max} = \sqrt{f_{y\,max}^2 + f_{z\,max}^2} = \sqrt{0.33^2 + 1.03^2} \cong 1.08 \text{ cm}$$

This deflection is at an angle β' to axis y. Thus

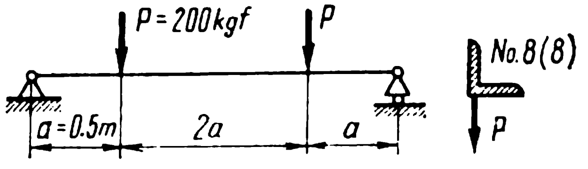
$$\tan\beta' = \frac{f_{z\,max}}{f_{y\,max}} = \frac{1.03}{0.33} = 3.12, \text{ i.e. } \beta' = 72°14'$$

Problems 586 through 595. Determine the maximum absolute value of the normal stresses and the position of the neutral axis in the dangerous cross sections of the beams.

Additionally, in Problem 587, determine the vertical f_v and horizontal f_h deflections of the cross section due to the force, assuming that $E = 2 \times 10^6$ kgf/cm^2; and in Problem 593 determine the magni-

tude and direction of the deflection f due to force P_1, assuming that $E = 10^4$ MN/m².

586



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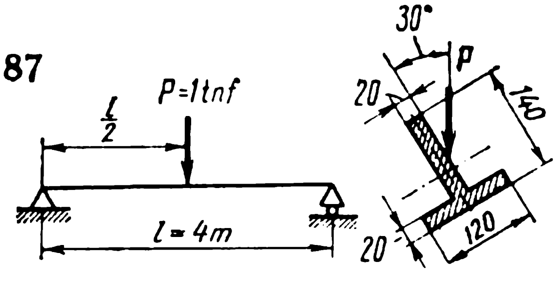

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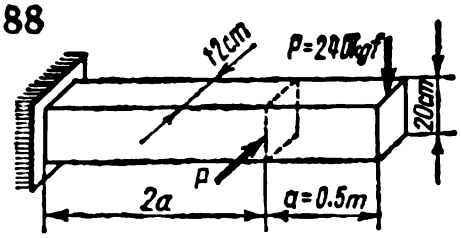

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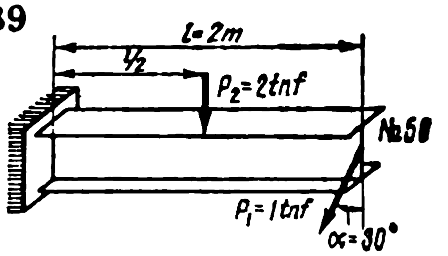

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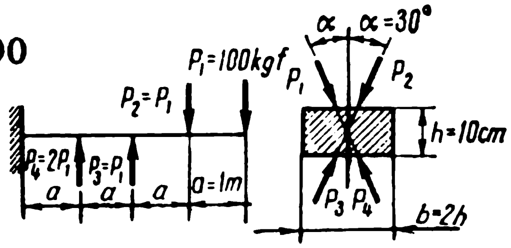

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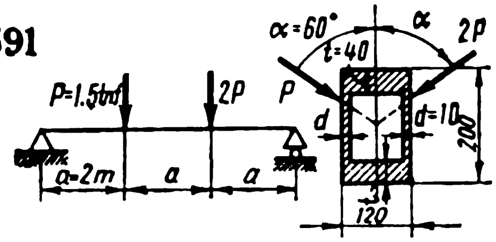

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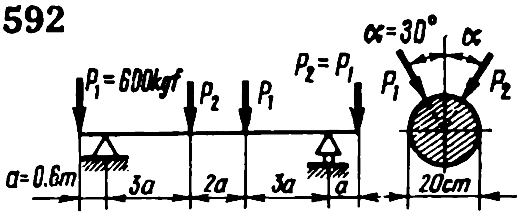

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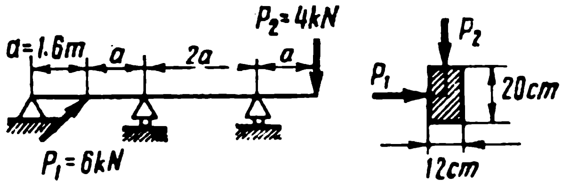

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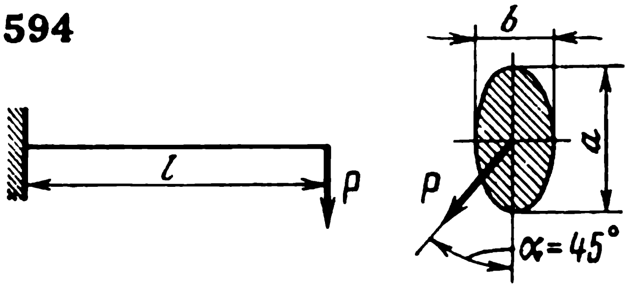

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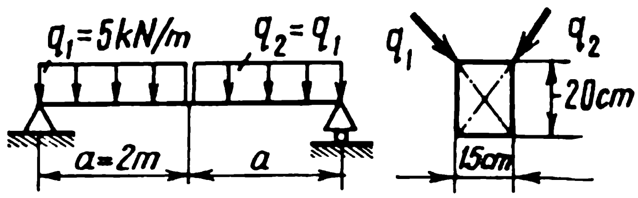
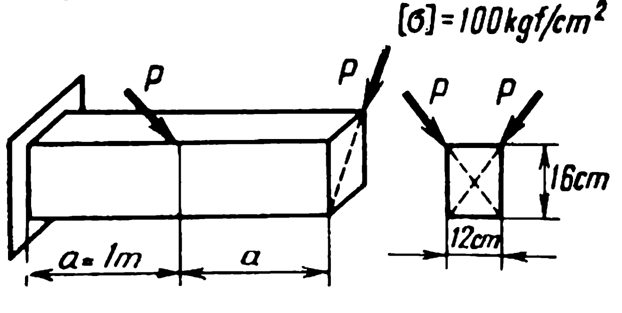


Problems 596 through 601. Determine the permissible forces P and the position of the neutral axis at the dangerous sections of the beams.

Additionally, determine the magnitude and direction of the deflection: in Problem 596, of the free end of the beam ($E = 10^5$ kgf/cm²), in Problem 599, of the middle of the left-hand span of the beam ($E = 10^4$ MN/m²).

596



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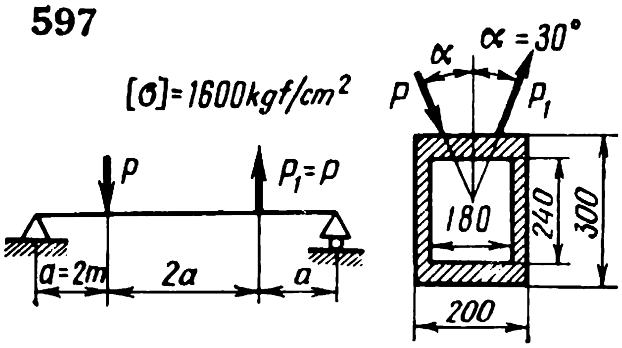

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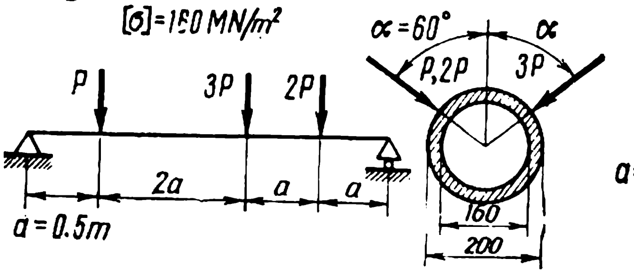

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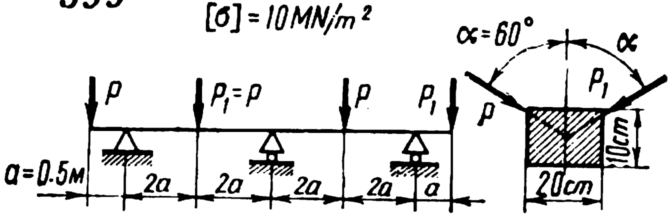

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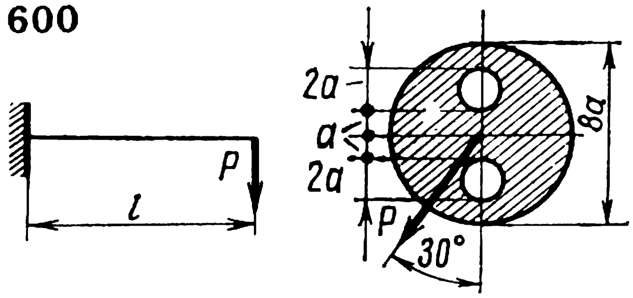

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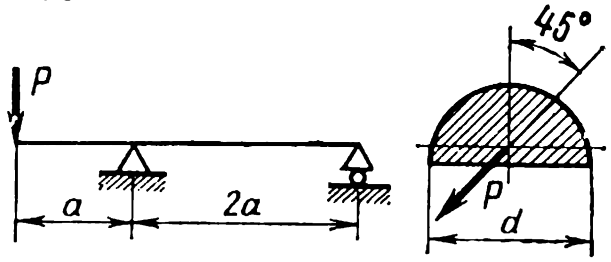
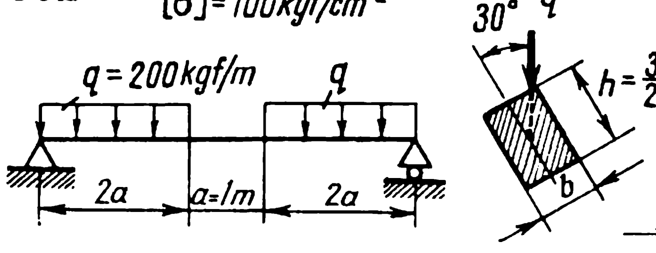


Problems 602 through 605. Select the dimensions of the cross sections and determine the position of the neutral axis in the dangerous sections of the beams.

Additionally, in Problem 604, determine the magnitude and direction of the deflection of the free end of the beam ($E = 10^5$ kgf/cm^2).

602



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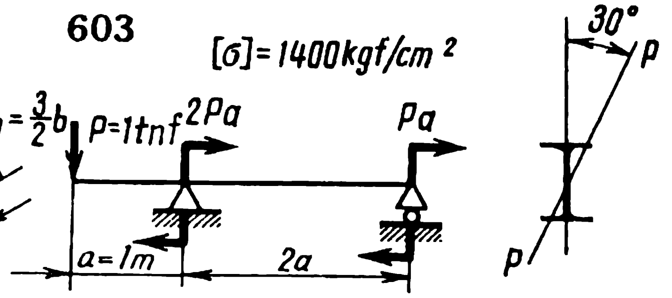

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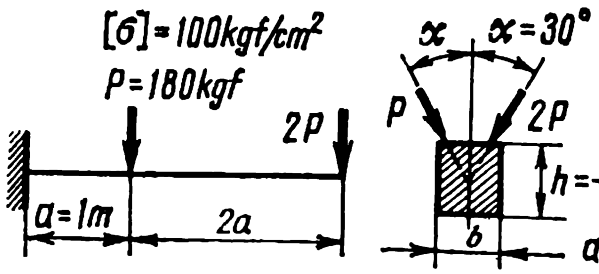

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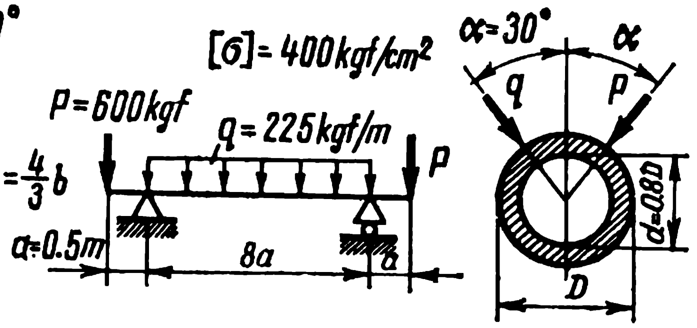
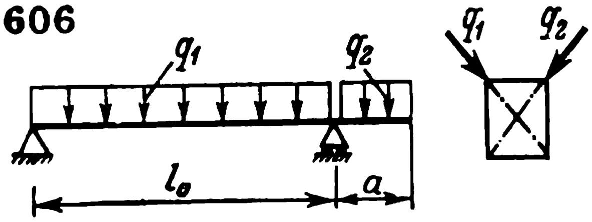


Problem 606. Prove that with $a \leqslant \frac{l_0}{2}$ and $q_2 \leqslant q_1$, the stress σ_{max} is not dependent on a and q_2.

Problem 607. Prove that with $c \leqslant a$ and $q_2 \leqslant q_1$, the stress σ_{max} is not dependent on q_2.

606



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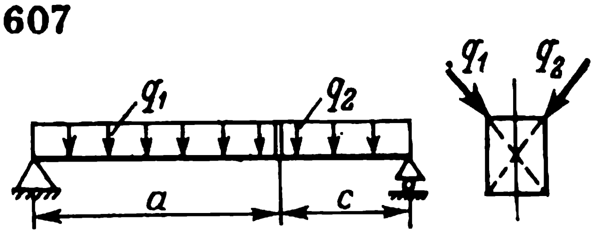
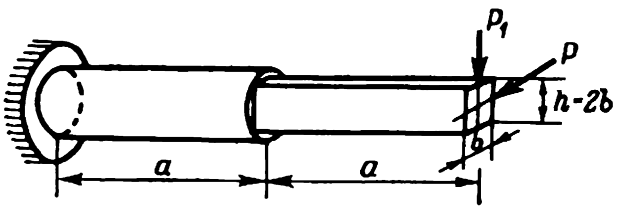


Problem 608. Given: $P_1 = 1.63\ P$, E, a and $h = 2b$. Determine the direction and magnitude of f_{max}.

Problem 609. Given: P, l, a and $\alpha = \arctan 0.5$.
Determine σ_{max}.

608



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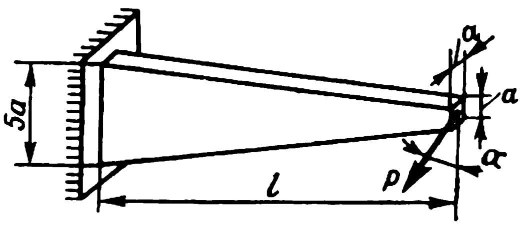
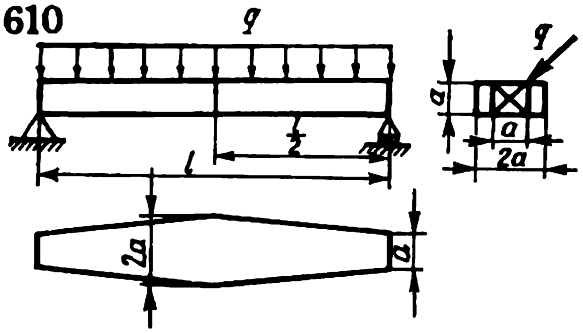


$\alpha = \arctan 0.5$

Problem 610. Given: q, l and a. Determine σ_{max}.

Problem 611. Given: P, a, b and E. Determine f_{max} and the angle of its inclination from the line of action of force P.

610



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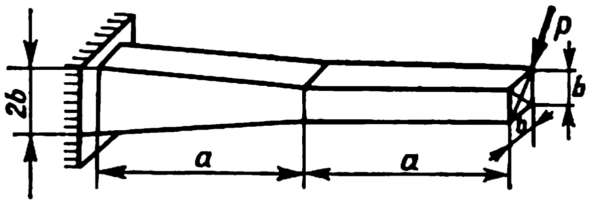


9.2.

Combined Tension or Compression and Bending

In the general case of simultaneous tensile or compressive and bending strains in an arbitrary cross section of a prismatic bar, the internal stresses are reduced to the axial force N_x, acting along the geometric x-axis of the bar, the bending moments M_y and M_z in the principal centroidal planes of inertia xz and xy of the bar, and the

transverse (shearing) forces Q_y and Q_z, acting along the y- and z-axes (Fig. 118).

The normal stresses in a cross section of the bar are found from the following equation of a plane which does not pass through the origin of coordinates:

$$\sigma=\frac{N_x}{F}+\frac{M_y}{I_y}z+\frac{M_z}{I_z}y=\frac{N_x}{F}\left(1+\frac{M_y z}{N_x i_y^2}+\frac{M_z y}{N_x i_z^2}\right) \qquad (142)$$

in which F = cross-sectional area

I_y and I_z = principal centroidal moments of inertia of the cross section

i_y and i_z = principal radii of inertia of the cross section

y and z = coordinates of a point in the plane of the cross section.

The equation of the neutral axis is of the form

$$1+\frac{M_y z}{N_x i_y^2}+\frac{M_z y}{N_x i_z^2}=0 \qquad (143)$$

The normal stresses reach their maximum and minimum values at the points of tangency of straight lines, parallel to the neutral axis, to the outline of the cross section.

The resultant shearing stresses can be determined approximately by geometrical summation of the shearing stresses at the given point of cross section due to Q_y and Q_z.

As a rule, these stresses are not great and are of minor importance in practical design.

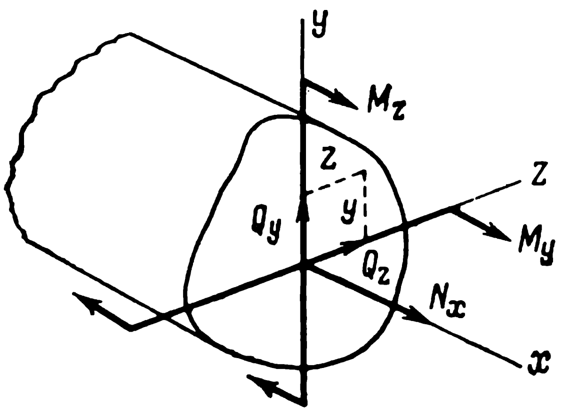


Fig. 118

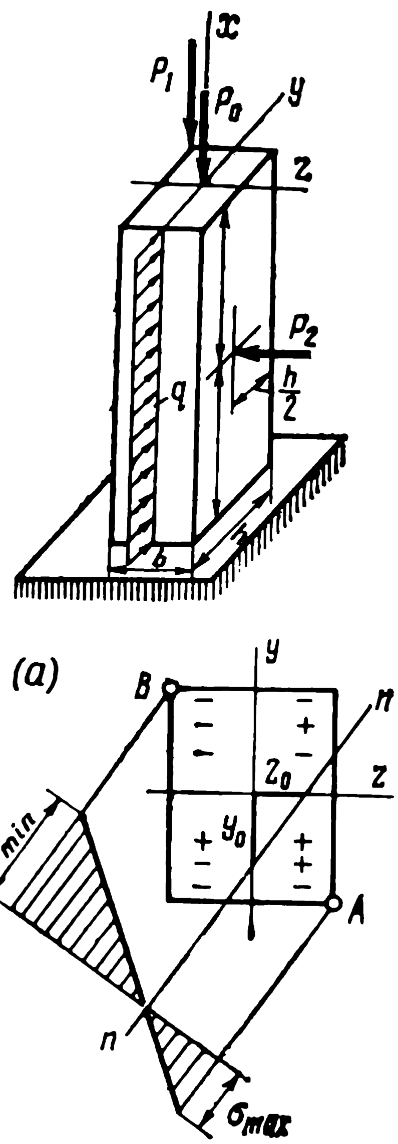


Fig. 119

The cross-sectional area of the bar is selected on the basis of the normal stresses by the trial-and-error method. The first trial can be made on the basis of plane bending alone due to the component bend-

ing moment which requires the larger size. The data of the first trial should be checked, taking into account the second component bending moment and the axial force. In the selected cross section the overstress must not exceed 5%.

Example 64. Let $q = 200$ kgf/m, $P_0 = 24$ tnf, $P_1 = 16$ tnf, $P_2 = 400$ kgf, $b = 12$ cm, $h = 16$ cm and $l = 2$ m (Fig. 119).

Determine σ_{max}, σ_{min} and the position of the neutral axis.

Solution. For the dangerous fixed section of the beam we can write

$$N_x = -P_0 - P_1 = -24\times10^3 - 16\times10^3 = -40\times10^3 \text{ kgf};$$

$$M_y = P_1\frac{b}{2} + P_2\frac{l}{2} = 16\times10^3\times6 + 4\times10^2\times10^2 = 136\times10^3 \text{ kgf-cm};$$

$$M_z = -P_1\frac{h}{2} - \frac{ql^2}{2} = -16\times10^3\times8 - \frac{2\times4\times10^4}{2} = -168\times10^3 \text{ kgf-cm}$$

The signs of the stresses at the points of the dangerous section of the beam, due to N_x, M_y and M_z, are indicated in Fig. 119a. Therefore

$$\sigma_{\substack{max\\min}} = \frac{N_x}{F} \pm \frac{M_y}{W_y} \mp \frac{M_z}{W_z} = -\frac{40\times10^3}{12\times16} \pm \frac{136\times10^3}{16\times12^2}\times6$$

$$\pm \frac{168\times10^3}{12\times16^2}\times6 \cong \begin{cases} 475 \\ -891 \end{cases} \text{kgf/cm}^2$$

In formula (143) it is assumed that $N_x > 0$ and due to M_y and M_z in the first quadrant of the section $\sigma > 0$. In the problem being considered we have

$$N_x < 0; \quad M_y > 0; \quad M_z < 0; \quad \text{and } i_y^2 = \frac{b^2}{12} = 12 \text{ cm}^2$$

$$\text{and } i_z^2 = \frac{h^2}{12} \cong 21.3 \text{ cm}^2$$

Therefore the segments intercepted by the neutral axis nn on the z- and y-axes are

$$z_0 = -\frac{N_x i_y^2}{M_y} = \frac{40\times10^3}{136\times10^3}\times12 \cong 3.53 \text{ cm};$$

$$y_0 = -\frac{N_x i_z^2}{M_z} = -\frac{40\times10^3}{168\times10^3}\times21.3 \cong -5.07 \text{ cm}$$

In Fig. 119a the neutral axis nn is passed through the ends of these segments and the diagram for the normal stresses is constructed.

Example 65. Let $q = 6$ kN/m, $L = 6$ m, $\alpha = 30°$ and $[\sigma] = 140$ MN/m^2 (Fig. 120).

Determine the required size number of the I-beam.

Solution. The projections of load q on axes x and y (Fig. 120a) are $q_x = q \sin\alpha$; $q_y = q\cos\alpha$.

The component q_x is uniformly distributed over the length l to the left, thereby compressing the beam. The component q_y produces plane transverse bending of the beam.

The maximum bending moment in the middle section of the beam is

$$M_{\max}=\frac{q_y l^2}{8}=\frac{qL^2}{8\cos\alpha}$$

As a first trial the cross section is selected on the basis of this moment. Then

$$W_z=\frac{M_{\max}}{[\sigma]}=\frac{qL^2}{8[\sigma]\cos\alpha}=\frac{6000\times 36}{8\times 14\times 10^7\times 0.866}=2.23\times 10^{-4}\ \text{m}^3=223\ \text{cm}^3$$

The nearest larger section modulus is listed for the No. 22 I-beam in the data for standard rolled steel shapes. For this beam $W_z=$ $=232$ cm^3 and $F=$ $=30.6$ cm^2.

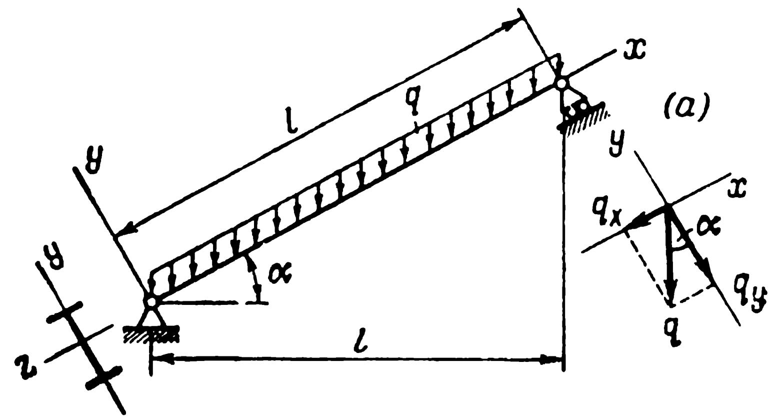


Fig. 120

At the cross section with $M_{\max}$, the axial compressive force is

$$N_{x=\frac{l}{2}}=-\frac{q_x l}{2}=-\frac{qL}{2}\tan\alpha$$

Next we check the selected section, taking the axial force into account:

$$|\sigma|_{\max}=\frac{|N_x|}{F}+\frac{M_{\max}}{W_z}=\frac{6000\times 6}{2\times 30.6\times 10^{-4}\sqrt{3}}+\frac{6000\times 36}{8\times 232\times 10^{-6}\times 0.866}$$

$$=137.8\times 10^6\ \text{N/m}^2=137.8\ \text{MN/m}^2$$

The understress is

$$\frac{[\sigma]-|\sigma|_{\max}}{[\sigma]}\times 100=\frac{2.02}{140}\times 100\cong 1.6\%$$

Actually, the section with the maximum stress is somewhat to the left of the middle section of the beam, but is so close to it that the effect of the difference can be neglected in calculations. Indeed, for an arbitrary cross section

$$|\sigma|_{\max}=\frac{1}{W}\left(\frac{qlx}{2}\cos\alpha-\frac{qx^2}{2}\cos\alpha\right)+\frac{q(l-x)}{F}\sin\alpha$$

Since

$$\frac{d|\sigma|_{\max}}{dx}=\frac{ql}{2W}\cos\alpha-\frac{qx}{W}\cos\alpha-\frac{q}{F}\sin\alpha=0$$

the distance from the dangerous cross section to the left support is

$$x = \frac{l}{2} - \frac{W}{F}\tan\alpha = \frac{6\times 2}{2\sqrt{3}} - \frac{232\times 10^{-6}}{30.6\times 10^{-4}\sqrt{3}} = 3.420 \text{ m}$$

i.e. 4.4 cm from the middle of the beam.

At this section (for $x = 3.420$ cm) for the selected beam

$$|\sigma|_{max} = \frac{1}{232\times 10^{-6}}\left(\frac{6000\times 6}{2}\times 3.420 - \frac{6000}{2}\times 3.420^2\frac{\sqrt{3}}{2}\right)$$

$$+\frac{6000\times 3.508}{30.6\times 10^{-4}\times 2} \cong 137.85\times 10^6 \text{ N/m}^2 = 137.85 \text{ MN/m}^2$$

which is only 0.05% more than at the middle section.

Example 66. Let $P_1 = 4$ tnf (located in plane xy), $P_2 = 1$ tnf, $P_3 = 0.8$ tnf, $M = 2$tnf-m, $l = 2$ m, $\alpha = 15°$, $\beta = 30°$ and $[\sigma] = 1600$ kgf/cm^2 (Fig. 121).

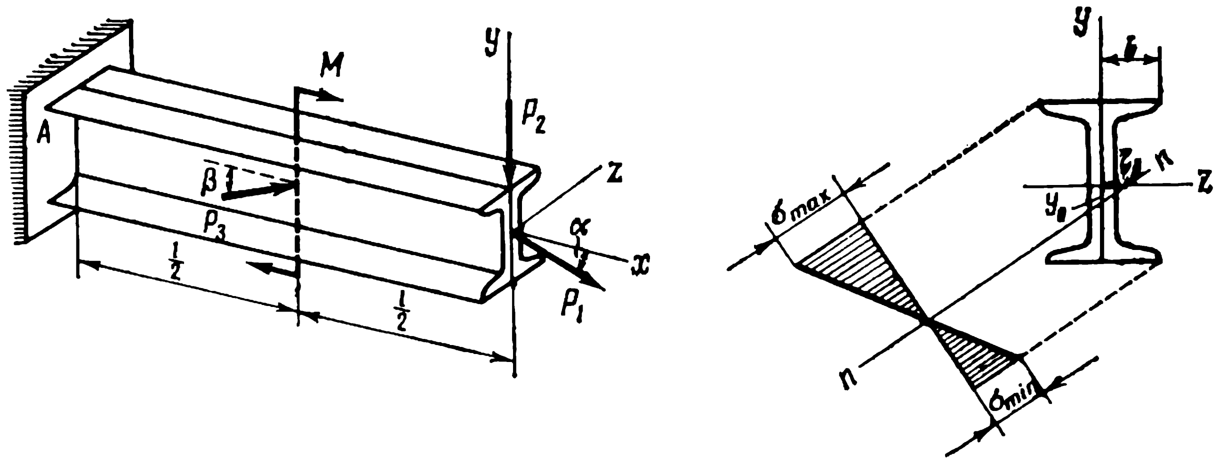


Fig. 121

Determine the size No. of the channel, y_0 and z_0.

Solution. In the dangerous (fixed) section

$$N_x = P_1\cos\alpha + P_3\cos\beta = 4\times 0.966 + 0.8\times 0.866 \cong 4.557 \text{ tnf};$$

$$M_y = -P_3\frac{l}{2}\sin\beta = -0.8\times 0.5\times 100 = -40 \text{ tnf-cm};$$

$$M_z = P_1 l\sin\alpha + P_2 l + M = 4\times 200\times 0.259 + 1\times 200 + 200 = 607.2 \text{ tnf-cm}$$

We make the first trial for determining the dimensions of the section on the basis of plane bending due to moment M_z. Thus

$$W_z = \frac{M_z}{[\sigma]} = \frac{607.2\times 10^3}{16\times 10^2} \cong 380 \text{ cm}^3$$

The nearest larger section modulus from tne data for standard rolled steel shapes is for the No. 22 channel for which $W_z' = 192$ cm^3. For the two channels $W_z = 384$ cm^3.

Taking into account the fact that, in addition to M_z, there are also a large bending moment M_y and axial force N_x in the beam, we select the No. 24 channel for this trial. For this channel $W_z = 242$ cm³. Then

$$W_z = 2 \times 242 = 484 \text{ cm}^3; \quad F = 2F' = 2 \times 30.6 = 61.2 \text{ cm}^2;$$

$$I_y = 2I'_y = 2 \times 387.2 = 774.4 \text{ cm}^4; \quad W_y = \frac{I_y}{b} = \frac{774.4}{9.0} = 86.0 \text{ cm}^3$$

Checking the strength of the No. 24 channels at the most stressed point A of the fixed section, we obtain

$$\sigma_{\max} = \frac{N_x}{F} - \frac{M_y}{W_y} + \frac{M_z}{W_z} = \frac{4557}{61.2} + \frac{40 \times 10^3}{86.0} + \frac{607.2 \times 10^3}{484}$$
$$\cong 1794 \text{ kgf/cm}^2$$

Since the overstress is equal to

$$\frac{\sigma_{\max} - [\sigma]}{[\sigma]} \times 100 = \frac{194}{16} \cong 12.1\%$$

it is clear that the No. 24 channel is insufficiently strong.

Next we check the No. 24a channel, for which

$$W'_z = 265 \text{ cm}^3; \quad W_z = 2 \times 265 = 530 \text{ cm}^3; \quad F = 2 \times 32.9 = 65.8 \text{ cm}^2;$$

$$I_y = 2I'_y = 2 \times 488.5 = 977 \text{ cm}^4 \text{ and } W_y = \frac{977}{9.6} \cong 102.8 \text{ cm}^3$$

In checking the section made up of these channels we obtain

$$\sigma_{\max} = \frac{4557}{65.8} + \frac{40 \times 10^3}{102.8} + \frac{607.2 \times 10^3}{530} \cong 160 \quad \text{kgf/cm}^2$$

The overstress is $\frac{4}{16} = 0.25\%$ which is within the permissible value. Thus the No. 24a channels can be used.

Now we determine the position of the neutral axis in the dangerous section.

For channel No. 24a: $i_z = 9.84$ cm; $i_z^2 \cong 96.8$ cm²;

$$i_y^2 = \frac{I_y}{F} = \frac{977}{65.8} \cong 14.8 \text{ cm}^2$$

According to formula (143) the segments cut off by the neutral axis on axes y and z are

$$y_0 = -\frac{4.557}{607.2} \times 96.8 \cong -0.73 \text{ cm}; \quad z_0 = -\frac{4.557}{-40} \times 14.8 \cong 1.69 \text{ cm}$$

In Fig. 121 (to the right) the neutral axis nn is passed through the ends of these segments and a diagram of normal stresses is constructed.

Eccentric load. In a general case of eccentric loading a prismatic bar is subject to simultaneous tensile or compressive deformation and pure oblique (unsymmetrical) bending.

The internal forces in each cross section of the bar are reduced to the axial force $N_x = P$ and to two bending moments $M_y = Pz_p$ and $M_z = Py_p$ developed in the principal centroidal planes of inertia xz and xy of the bar. Here P are the effective tensile (or compressive) forces applied, not in the centre of gravity of the end sections of the bar, but at points with the coordinates y_p and z_p (Fig. 122).

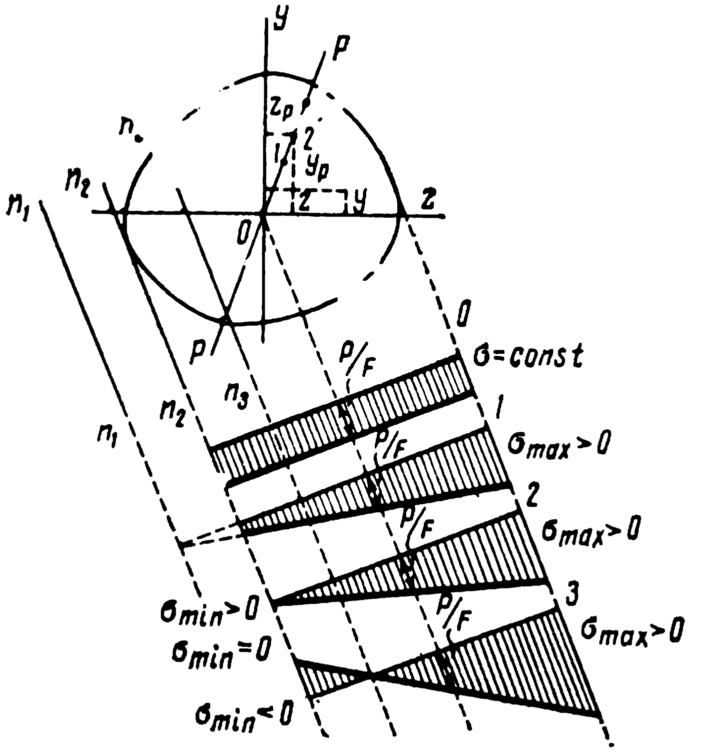


Fig. 122

The normal stresses in a cross section of the bar are found from the following equation for a plane which does not pass through the origin of coordinates:

$$\sigma = \frac{N_x}{F} + \frac{M_y z}{I_y} + \frac{M_z y}{I_z} = \frac{P}{F}\left(1 + \frac{zz_p}{i_y^2} + \frac{yy_p}{i_z^2}\right) \quad (144)$$

in which F = cross-sectional area

I_y and I_z = principal centroidal moments of inertia of the area

i_y and i_z = principal radii of inertia

y and z = coordinates of an arbitrary point in the plane of the section.

The equation for the neutral axis nn can be written as

$$1 + \frac{zz_p}{i_y^2} + \frac{yy_p}{i_z^2} = 0 \quad \text{or} \quad \frac{z}{-\frac{i_y^2}{z_p}} + \frac{y}{-\frac{i_z^2}{y_p}} = 1 \quad (145)$$

in which

$$-\frac{i_y^2}{z_p} \quad \text{and} \quad -\frac{i_z^2}{y_p} \quad (146)$$

are the segments intercepted by the neutral axis on the principal centroidal axes of inertia z and y of the cross section of the bar.

For points of a line, parallel to the neutral axis and passing through the centre of gravity of the section, the normal stresses are $\sigma = \frac{P}{F}$.

The maximum and minimum normal stresses are developed at points of tangency of straight lines, parallel to the neutral axis, to the outline of the cross section.

For symmetrical sections with points most remote from both principal centroidal axes of inertia, the maximum stresses occur at these points.

Therefore, for such sections

$$|\sigma|_{max} = P\left(\frac{1}{F} + \frac{y_p}{W_z} + \frac{z_p}{W_y}\right) \tag{147}$$

If the point of force application moves along the straight line pp, passing through the centre of gravity O of the section, then the neutral axis nn will, without rotation, either approach the centre of gravity or move away from it, depending on whether the point of force application moves away from or approaches the centre of gravity.

Figure 122 illustrates the position of the neutral axes (n_1n_1, n_2n_2 and n_3n_3; n_0n_0 is at infinity) and the respective diagrams for σ for cases when the tensile force P is applied at points 0, 1, 2 and 3 of the straight line pp.

If the point of force application moves along the straight line pp (Fig. 123) which does not pass through the centre of gravity of the section, then the neutral axis turns about a stationary point K whose coordinates are

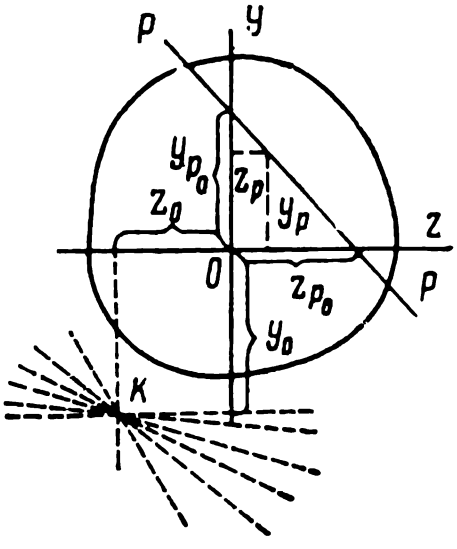


Fig. 123

$$y_0 = -\frac{i_z^2}{y_{p_0}} \quad \text{and} \quad z_0 = -\frac{i_y^2}{z_{p_0}}$$

The reverse is also true: if the neutral axis turns about a stationary point whose coordinates are y_0 and z_0, then the point of force application moves along the straight line pp, which does not pass through the centre of gravity of the section, and the equation of this line is of the form

$$\frac{y_p}{-\frac{i_z^2}{y_0}} + \frac{z_p}{-\frac{i_y^2}{z_0}} = 1$$

If the point of force application lies on one of the principal centroidal axes of inertia of the section, then the bar is subject to simultaneous axial tensile or compressive strain and pure plane bending. All the preceding formulas hold true but, in them, z_p should be equated to zero (if the point of application is on axis y) or y_p equated to zero (if the point of application is on axis z).

To ensure sufficient strength for bars made of brittle materials of low tensile strength, care should be taken to prevent the eccentric compressive force from producing tensile stresses in the cross section.

The portion of the plane of the cross section, which surrounds the centre of gravity and is bounded by a closed contour inside which the applied force sets up stresses of a single sign at all points of the cross section, is called the *core of the cross section.*

The contour of the cross-sectional core is the locus of such points of application of the eccentric force for which the neutral axes may touch the outline of the cross section but do not intersect it.

Example 67. Let $P = 6.4$ tnf, $b = 4$ cm, $h = 8$ cm, $y_p = 2$ cm and $z_p = 1$ cm (Fig. 124).

Determine σ_{max}, σ_{min}, y_0 and z_0.

Solution. The forces developed in the cross section are $N_x = -P = -6.4$ tnf; $M_y = -Pz_p = -6.4 \times 1 = -6.4$ tnf-cm and $M_z = -Py_p = -6.4 \times 2 = -12.8$ tnf-cm.

Since in the first quadrant of the cross section the stresses due to all the forces are compressive, σ_{min} will be in the upper right-hand

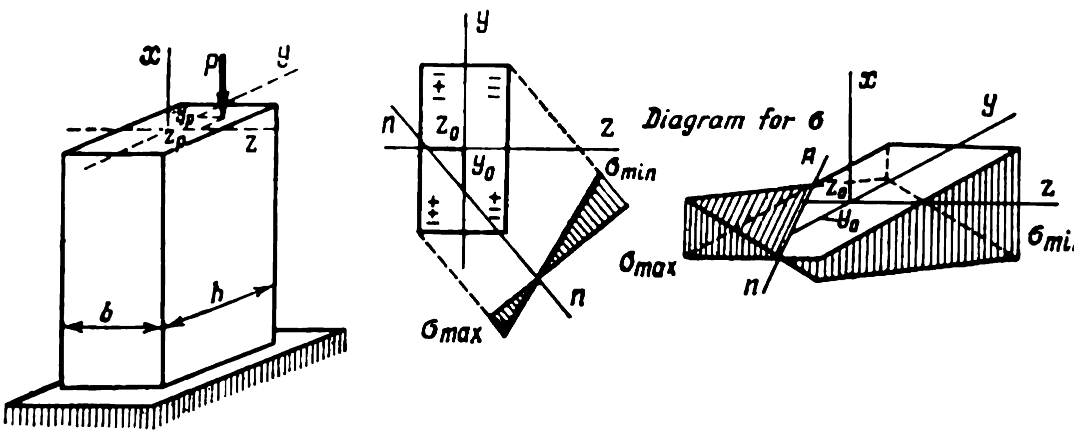


Fig. 124

corner of the section and σ_{max}, in the lower left-hand corner (Fig. 124*a*). Then

$$\sigma_{\substack{max\\min}} = \frac{N}{F} \mp \frac{M_y}{W_y} \mp \frac{M_z}{W_z}$$

Since

$$F = bh = 4 \times 8 = 32 \text{ cm}^2;\quad W_y = \frac{hb^2}{6} = \frac{8 \times 16}{6} = \frac{64}{3} \text{ cm}^3 \text{ and}$$

$$W_z = \frac{bh^2}{6} = \frac{4 \times 64}{3} = \frac{128}{3} \text{ cm}^3$$

we find that

$$\sigma_{\substack{max\\min}} = -\frac{6.4 \times 10^3}{32} \pm \frac{6.4 \times 10^3 \times 3}{64} \pm \frac{12.8 \times 10^3 \times 3}{128}$$

or

$$\sigma_{max} = 400 \text{ kgf/cm}^2 \quad \text{and} \quad \sigma_{min} = -800 \text{ kgf/cm}^2$$

Finally, using formulas (146), we find the segments intercepted by the neutral axis nn on the principal centroidal axes of inertia y and z of the section:

$$y_0 = -\frac{i_z^2}{y_p} = -\frac{h^2}{12y_p} = -\frac{64}{12\times 2} = -\frac{8}{3} \cong -2.67 \text{ cm};$$

$$z_0 = -\frac{i_y^2}{z_p} = -\frac{b^2}{12z_p} = -\frac{16}{12\times 1} = -\frac{4}{3} \cong 1.33 \text{ cm}$$

The position of the neutral axis and the diagram for σ are shown in Fig. 124, *a*, *b*.

Example 68. Construct the core of an arbitrary cross section (Fig. 125) symmetrical about the z-axis and inscribed in the rectangle

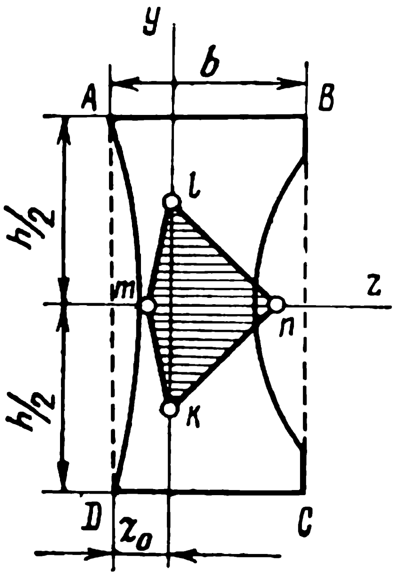


Fig. 125

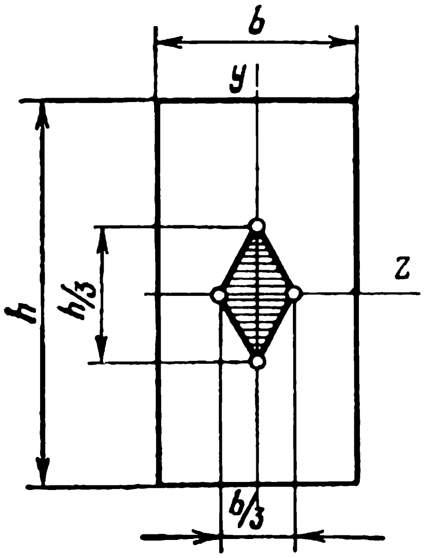


Fig. 126

$ABCD$ with sides $b < h$; if the principal centroidal radii of inertia of the section $i_y < i_z$ and the position of the centre of gravity of the section is determined by the distance z_0.

Solution. If the neutral axes touch the smaller sides, AB and CD, of the rectangle the corresponding points of application of the eccentric force (points k and l on the outline of the core of the cross section) should lie on the y-axis and are determined by the ordinates $y_c = \mp \frac{2i_z^2}{h}$.

If the neutral axes touch the larger sides BC and DA of the rectangle, then the corresponding points of application of the eccentric force (points m and n of the outline of the core of the cross section) should lie on the z-axis and are determined by the abscissas

$$z_c' = -\frac{i_y^2}{b - z_0} \quad \text{and} \quad z_c'' = \frac{i_y^2}{z_0}$$

Since the horizontal positions of the neutral axis are converted into vertical ones (and vice versa) due to the rotation about fixed points coinciding with the vertices of the rectangle, then, during the rotation of the neutral axis, the point of application of the eccentric force should move along straight lines.

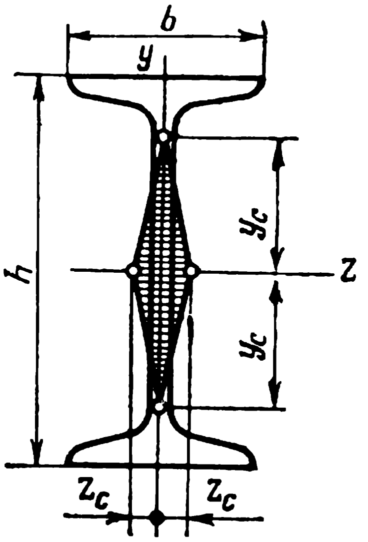


Fig. 127

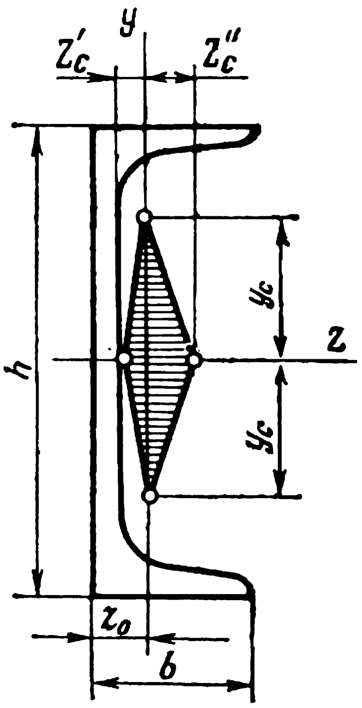


Fig. 128

Connecting points k, l, m and n by straight lines we obtain the core of the cross section (in Fig. 125 the core of the cross section is hatched with horizontal lines).

Special cases: (a) A rectangular section with sides b and h (Fig. 126). For the rectangular section $i_z^2 = \frac{h^2}{12}$, $i_y^2 = \frac{b^2}{12}$ and $z_0 = \frac{b}{2}$.

Therefore $y_c = \mp \frac{h}{6}$ and $z_c = \mp \frac{b}{6}$.

(b) An I-beam section (No. 20a I-beam, Fig. 127). From the data for standard rolled steel shapes: for the No. 20a I-beam, $i_z = 8.37$ cm, $i_y = 2.32$ cm, $h = 20$ cm, $b = 11$ cm and $z_0 = \frac{b}{2}$.

Therefore

$$y_c = \mp \frac{2 \times 8.37^2}{20} \cong \mp 7.01 \text{ cm} \quad \text{and} \quad z_c = \mp \frac{2 \times 2.32^2}{11} \cong \mp 0.979 \text{ cm}$$

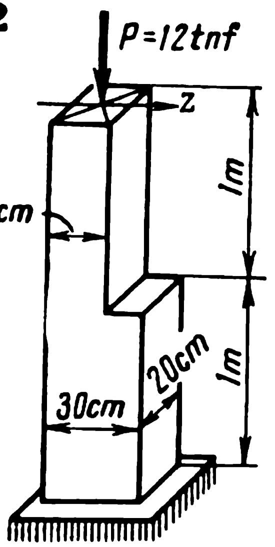
(c) A channel section (No. 22a channel, Fig. 128). From the data for standard rolled steel shapes: for the No. 22a channel, $i_z = 8.99$ cm, $i_y = 2.55$ cm, $h = 22$ cm, $b = 8.7$ cm and $z_0 = 2.46$ cm. Therefore

$$y_c = \mp \frac{2 \times 8.99^2}{22} \cong \mp 7.35 \text{ cm}; \quad z'_c = -\frac{2.55^2}{8.7 - 2.46} \cong -1.04 \text{ cm};$$

$$z''_c = \frac{2.55^2}{2.46} \cong 2.64 \text{ cm}$$

Problems 612 through 620. Determine the maximum σ_{max} and the minimum σ_{min} normal stresses and the position of the neutral axis in the dangerous sections of the straight bars.

612



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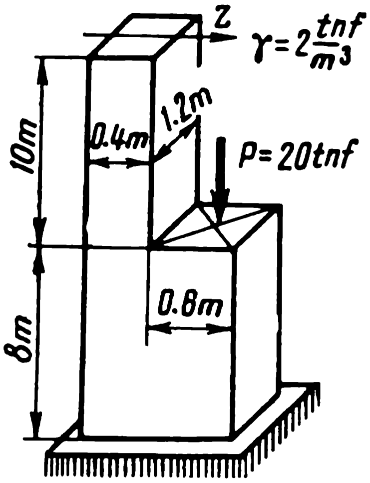

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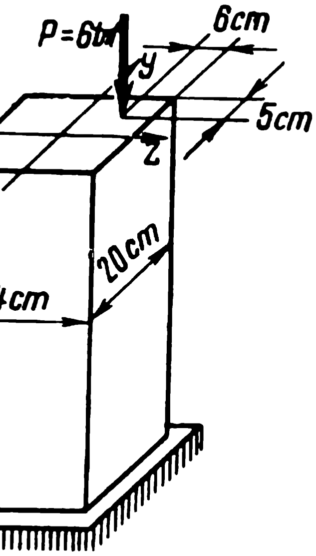

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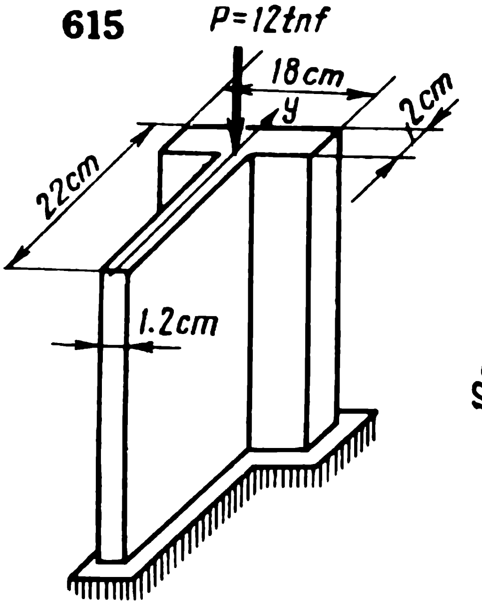

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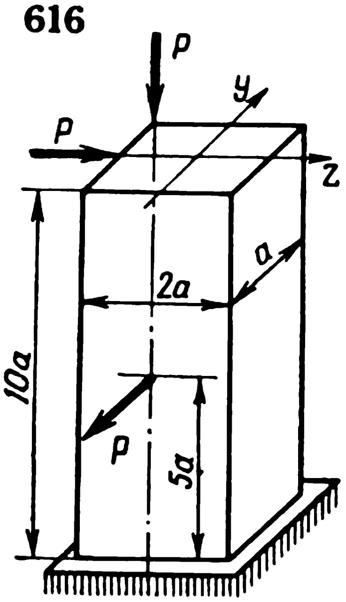

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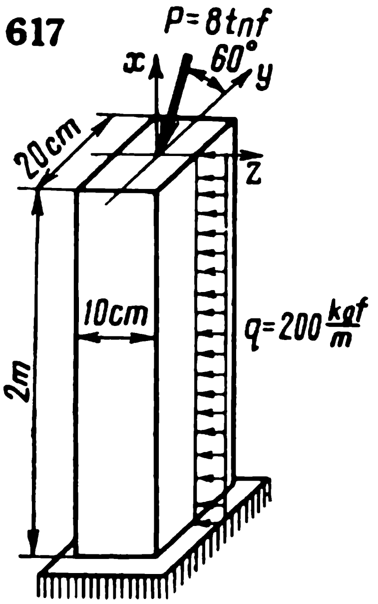

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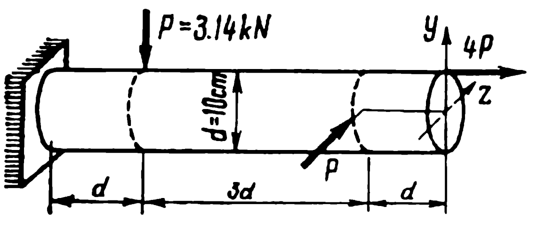

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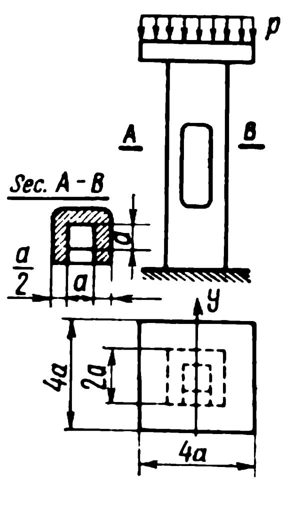

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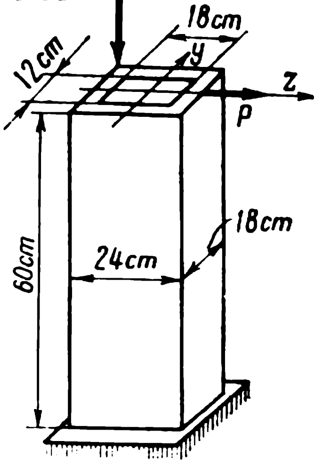
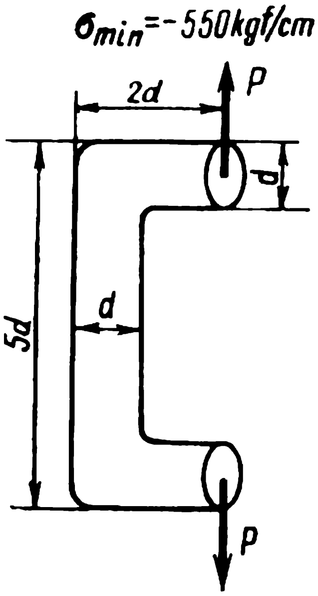


Problems 621 through 623. Determine the maximum normal stress σ_{max} and the position of the neutral axis in the dangerous sections of the bars.

621



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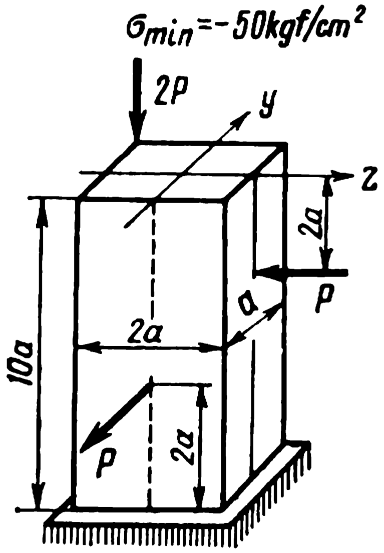

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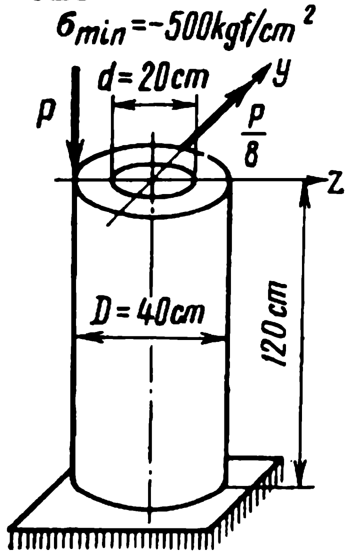
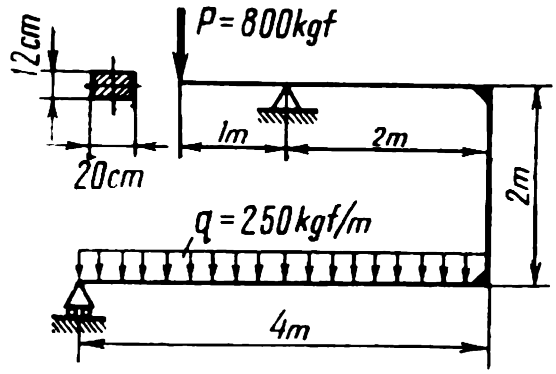


Problems 624 through 629. Determine the maximum σ_{max} and the minimum σ_{min} normal stresses in the dangerous cross sections of the bars and in the members of the system subject to combined stresses.

624



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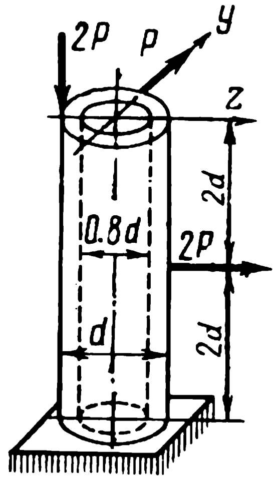

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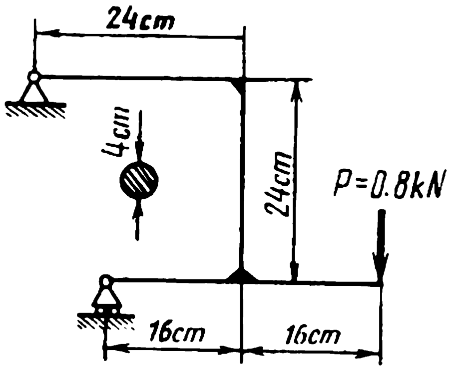


Problems 630 through 635. Determine the required dimensions of the cross sections of the bars, beams and members of the systems.

Problem 636. Determine the thickness t_1 for which the maximum normal stresses will be the same in the strips and in the cover plate.

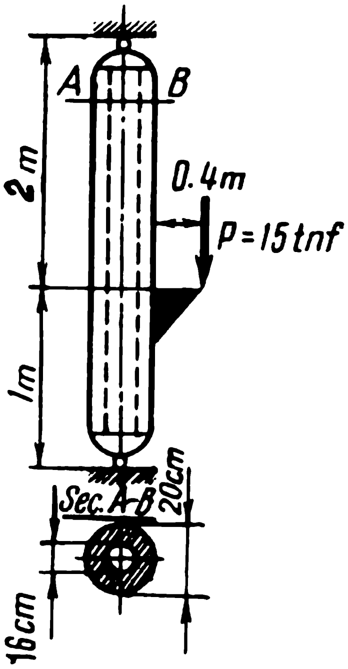
Problem 637. The specific weight of the material of the upright is $\gamma = 2.5$ tnf/m^3 and the lateral pressure on its diametral section is $p = 90$ kgf/m^2.

Determine the diameter d of the cross section proceeding from the condition that it is not subject to tensile stresses.

Problems 638 through 647. Determine the maximum safe load P.

Problem 648. Determine the force P_1 which will eliminate tensile stresses in the lower section of the upright.

627



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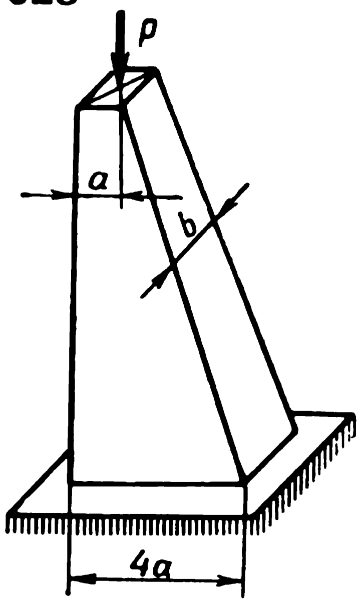

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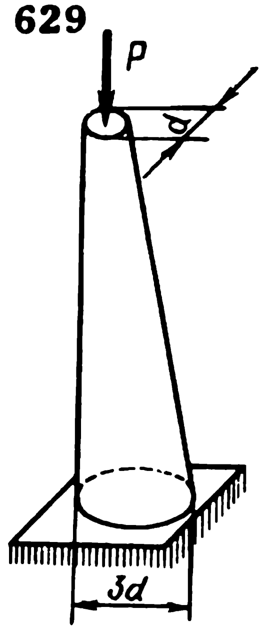

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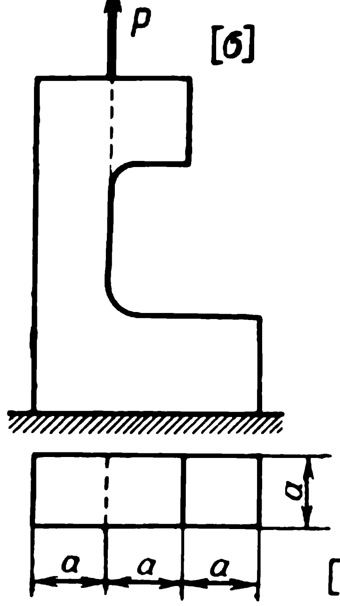

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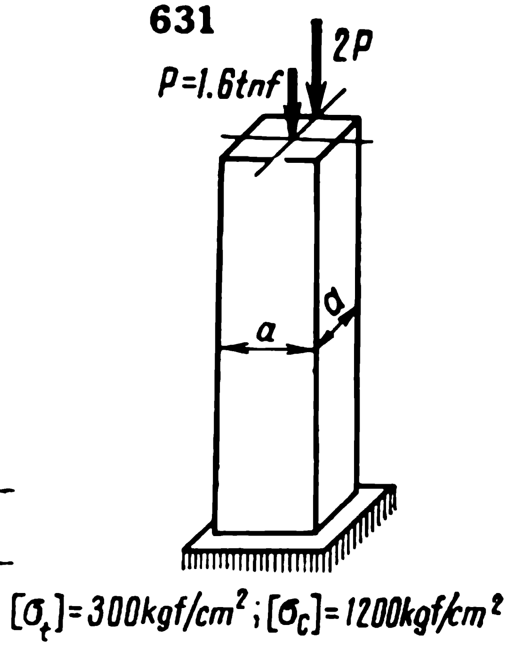

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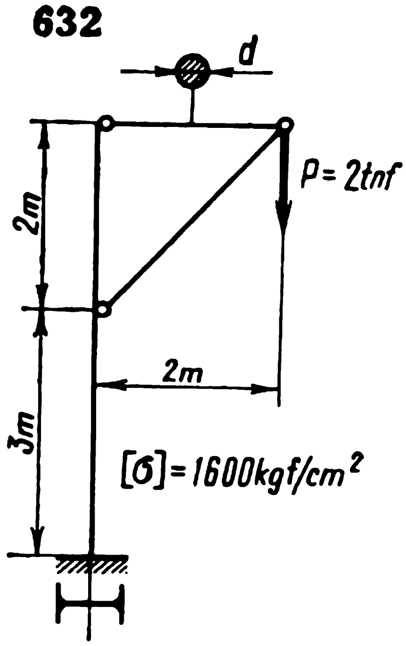

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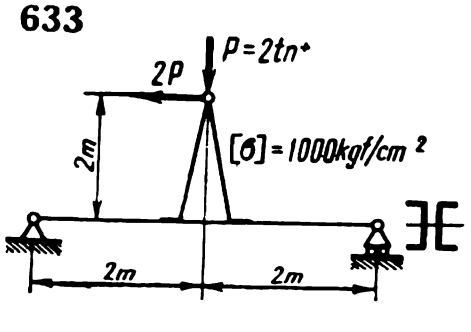

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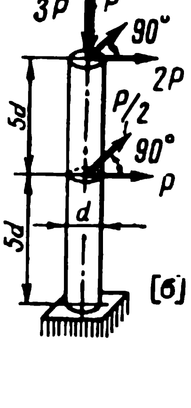

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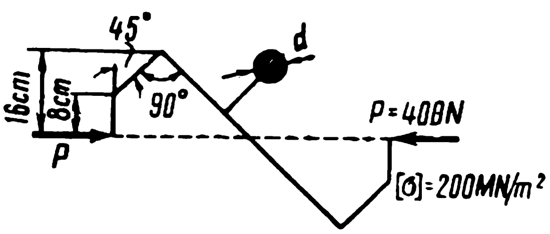

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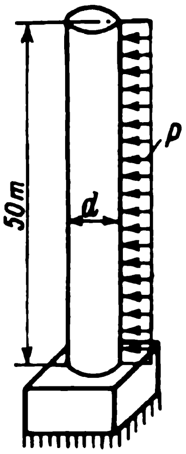

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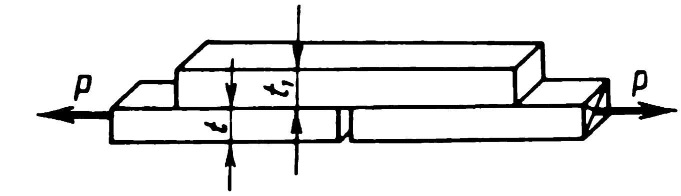

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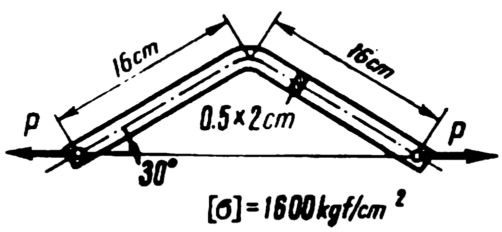

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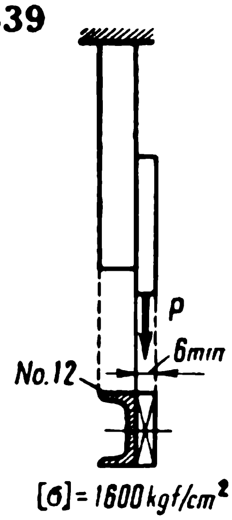

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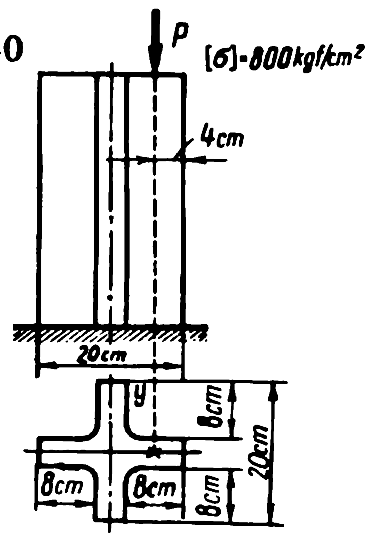

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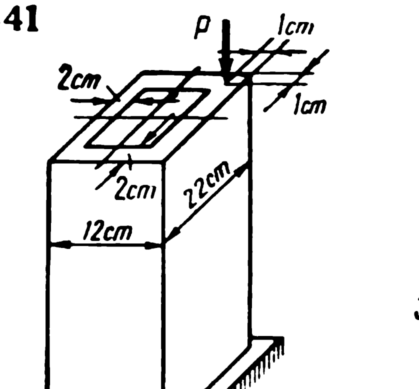

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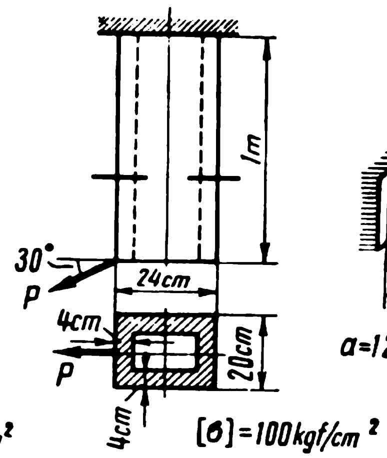

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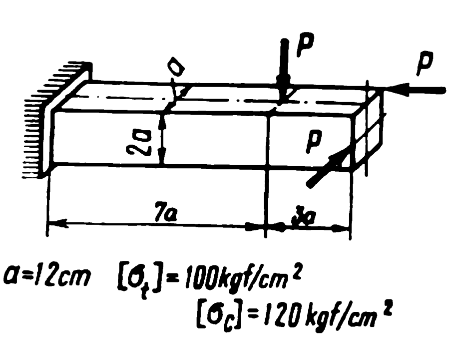

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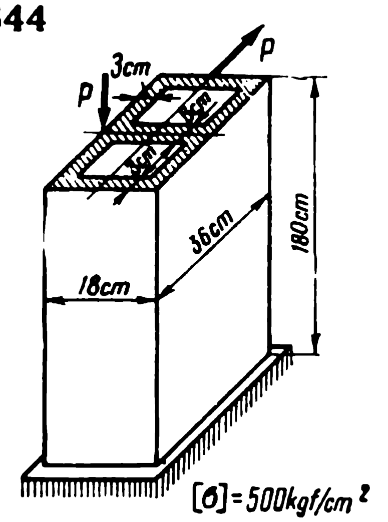

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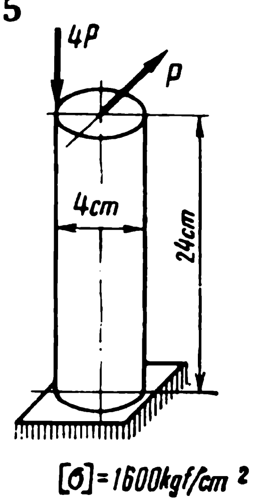

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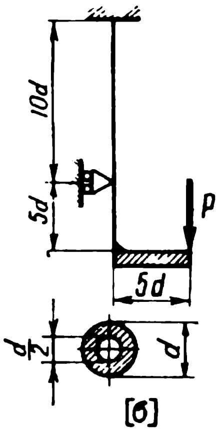

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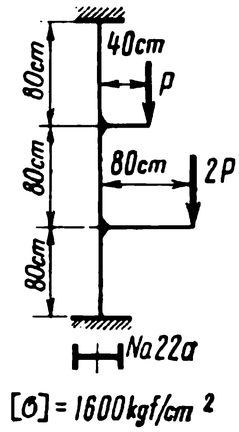
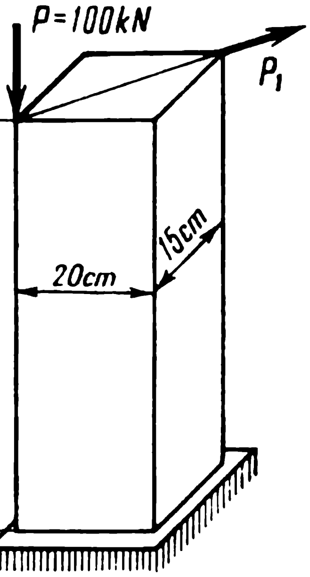


Problem 649. Determine force P_1 and stress σ_{min}, proceeding from the condition that σ_{max} in the upright is zero.

648



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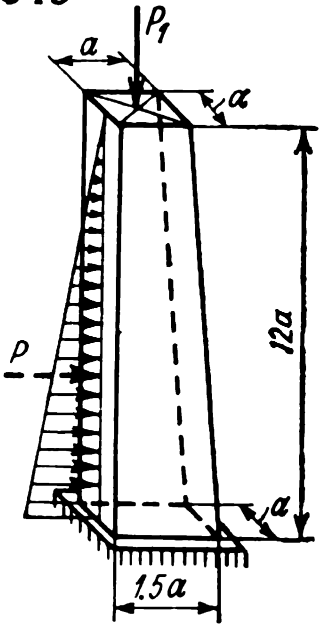
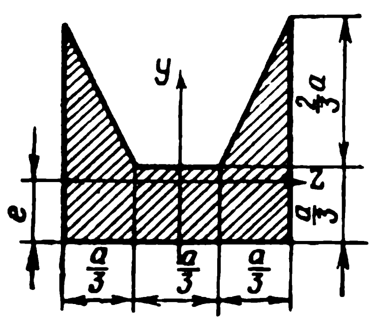


Problems 650 through 655. Construct the cores of the cross sections.

650



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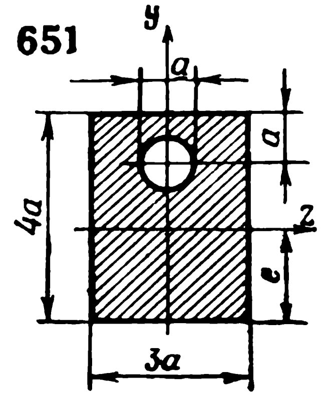

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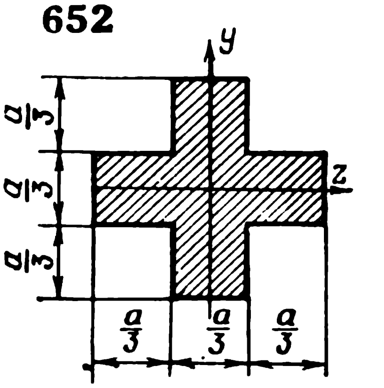

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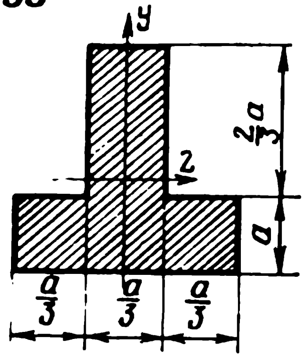

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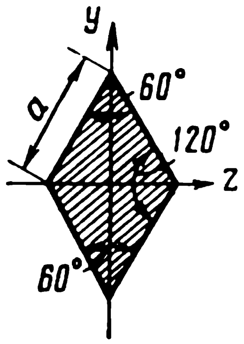

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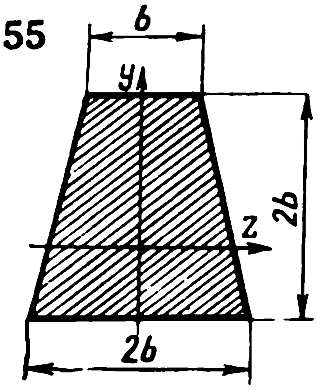


9.3.

Combined Tension or Compression and Torsion

If external forces produce axial stresses N_x and torques M_t in the cross sections of a bar, then it is subject to simultaneous tensile or compressive and torsional strains.

Here x is the geometrical axis of the bar, y and z are the principal centroidal axes of inertia of its cross section.

At the dangerous point (y, z) of the cross section being considered, the normal stresses are due to the axial force N_x

$$\sigma = \frac{N_x}{F}$$

in which F is the cross-sectional area; and the shearing stresses are set up by the torque

$$\tau = \frac{M_t}{W_t}$$

in which W_t is the section modulus of the section in torsion.

The principal normal stresses at this point are found by formula (107). Thus

$$\sigma_{\substack{1\\3}} = \frac{1}{2}\left(\sigma \pm \sqrt{\sigma^2 + 4\tau^3}\right)$$

The various strength theories are applied in design and analysis.

For plastic materials, use is made of the third or fourth strength theory, from which

$$\sigma_{eq_{III}} = \sigma_1 - \sigma_3 = \sqrt{\sigma^2 + 4\tau} \leqslant [\sigma] \tag{148}$$

and

$$\sigma_{eq_{IV}} = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3} = \sqrt{\sigma^2 + 3\tau^2} \leqslant [\sigma] \tag{149}$$

For materials which differ in their strength in tension and compression, when $\frac{[\sigma_t]}{[\sigma_c]} = \nu$, use is made of the fifth theory of strength.

According to this theory

$$\sigma_{eq_V}=\sigma_1-\nu\sigma_3=\frac{1-\nu}{2}\sigma+\frac{1+\nu}{2}\sqrt{\sigma^2+4\tau^2} \tag{150}$$

For a round bar of diameter d

$$F=\frac{\pi d^2}{4} \quad \text{and} \quad W_t=W_p=\frac{\pi d^3}{16}$$

Example 69. Let $P = 2$ tnf, $M = 4$ tnf-cm, $d = 4$ cm, $[\sigma_t] =$ $= 350$ kgf/cm^2 and $[\sigma_c] = 1400$ kgf/cm^2 (Fig. 129).

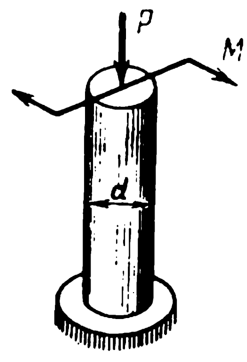


Fig. 129

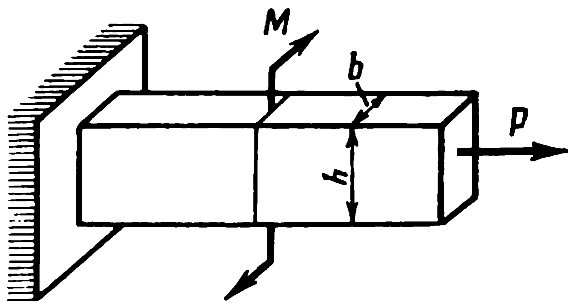


Fig. 130

Determine whether the bar is sufficiently strong.

Solution. The axial force is

$$N_x=-P=-2\times10^3 \text{ kgf}$$

and the torque is

$$M_t=M=4\times10^3 \text{ kgf-cm}$$

The normal stresses at all the points of the section are

$$\sigma=\frac{N_x}{F}=-\frac{2\times10^3\times4}{\pi\times4^2}\cong-159 \text{ kgf/cm}^2$$

The maximum shearing stresses at the points along the outline of the section are

$$\tau_{max}=\frac{M_t}{W_p}=\frac{4\times10^3\times16}{\pi\times4^3}\cong318 \text{ kgf/cm}^2$$

Since

$$\nu=\frac{[\sigma_t]}{[\sigma_c]}=\frac{350}{1400}=0.25$$

the equivalent stress obtained by formula (150) is

$$\sigma_{eq_V}=-\frac{1-0.25}{2}159+\frac{1+0.25}{2}\sqrt{159^2+4\times318^2}\cong$$
$$\cong 350 \text{ kgf/cm}^2=[\sigma_t]$$

Consequently, the strength of the bar is sufficient.

Example 70. Let $P = 160$ kN, $M = 4$ kN-m, $h = 8$ cm, $b = 4$ cm and $\sigma_y = 360$ MN/m^2 (Fig. 130).

Determine the factor of safety n_y.

Solution. Since $N_x = P = 16 \times 10^4$ N, $M_t = M = 4 \times 10^3$ N-m and $\frac{h}{b} = \frac{8}{4} = 2$, then at all points of the cross sections $\sigma = \frac{N_x}{F} = \frac{16 \times 10^4}{0.04 \times 0.08} = 50 \times 10^6$ N/m^2 $= 50$ MN/m^2 and at the points most highly stressed in torsion, in the middle of the larger sides of the rectangle of sections in the left half of the bar

$$\tau_{\max} = \frac{M_t}{W_t} = \frac{M_t}{\beta b^3} = \frac{4 \times 10^3 \times 10^6}{0.493 \times 64} \cong 127 \times 10^6 \text{ N/m}^2 = 127 \text{ MN/m}^2$$

At the dangerous points of the bar the equivalent stress according to the third strength theory and formula (148) is

$$\sigma_{eq_{III}} = \sqrt{50^2 + 4 \times 127^2} \cong 259 \text{ MN/m}^2$$

The factor of safety then is

$$n_y = \frac{\sigma_y}{\sigma_{eq_{III}}} = \frac{360}{259} \cong 1.39$$

9.4.

Combined Torsion and Bending

In case of bending strain combined with torsion, the internal forces at the cross section of a bar are reduced to five components: torque $M_x = M_t$ about the geometrical axis x of the bar (Fig. 131), bending moments M_y and M_z about the principal centroidal axes of inertia y and z of the section and to transverse (shearing) forces Q_y and Q_z acting along these axes.

If a bar is of round cross section of diameter d, the shearing stresses, due to Q_y and Q_z, are of minor importance and are commonly neglected in calculations.

The shearing stresses due to the torque reach their maximum and equal value

$$\tau_{\max} = \frac{M_t}{W_p} = \frac{16M_t}{\pi d^3}$$

at all points along the outline of the section (Fig. 132) and the maximum normal stresses due to the bending moments M_y and M_z occur at two points (A and B) along the outline of the section which lie at the ends of the diameter perpendicular to the vector of the resultant bending moment $M = \sqrt{M_y^2 + M_z^2}$; these stresses equal

$$\sigma = \pm \frac{M}{W} = \pm \frac{32M}{\pi d^3}$$

At the dangerous points A and B, the principal normal stresses σ_1 and σ_3 are obtained by formula (107) and the strength conditions are written in the form of the inequalities (148), (149) and (150).

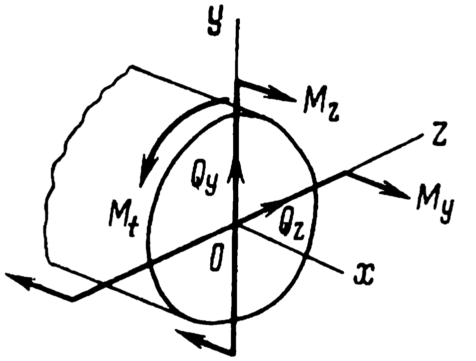


Fig. 131

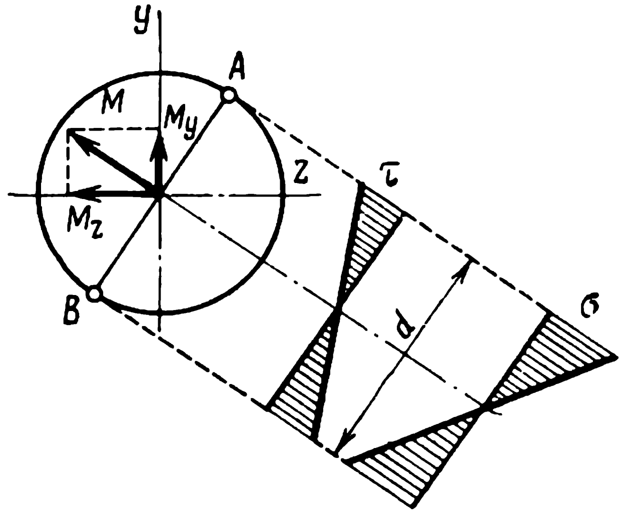


Fig. 132

Taking into account the values of σ and τ and also that $W_p = 2W$, the design formulas are reduced to the form

$$W \geqslant \frac{M_{eq}}{[\sigma]} \tag{151}$$

in which M_{eq} is the equivalent (design) bending moment. According to the third strength theory

$$M_{eq_{III}} = \sqrt{M^2 + M_t^2} \tag{152}$$

the fourth strength theory

$$M_{eq_{IV}} = \sqrt{M^2 + 0.75 M_t^2} \tag{153}$$

the fifth strength theory

$$M_{eq_V} = \frac{1-\nu}{2} M + \frac{1+\nu}{2} \sqrt{M^2 + M_t^2} \tag{154}$$

in which $\nu = \frac{[\sigma_t]}{[\sigma_c]}$.

If a bar of non-circular cross section is subject to torsion combined with bending, the dangerous points will also be located on the outline of the section. However, since points with the maximum shearing stresses due to torsion may not coincide with points at which the maximum normal stresses are due to bending, the dangerous points may be those with the maximum shearing stresses, points with the maximum normal stresses and certain intermediate points on the outline of the section.

These dangerous points (y, z) are ones at which the equivalent stresses obtained according to the selected strength theory reach their maximum value.

In formulating the equivalent stresses, σ and τ should be computed from the formulas

$$\sigma = \frac{M_y}{I_y} z + \frac{M_z}{I_z} y \quad \text{and} \quad \tau = \gamma \frac{M_t}{W_t}$$

in which γ is an abstract coefficient depending on the shape and size of the section and on coordinates (y, z) of the point on the outline of the section being considered.

For more accurate calculations the shearing stresses due to bending (τ_{Q_y} and τ_{Q_z}) can also be taken into account. Then the total shearing stress should be determined by geometrical summation.

Example 71. Let diameters of pulleys $D' = 20$ cm; $D'' = 60$ cm; weights of the pulleys $P_0' = 200$ kgf; $P_0'' = 400$ kgf, the tension of the belts $P_1' = 800$ kgf; $P_2' = 400$ kgf; $P_1'' = 1000$ kgf; $P_2'' = 600$ kgf; shaft length $l = 2$ m; the angle of inclination of the belt on the second pulley to the z-axis $\alpha = 45°$ and the permissible stress of the shaft material $[\sigma] = 1000$ kgf/cm² (Fig. 133).

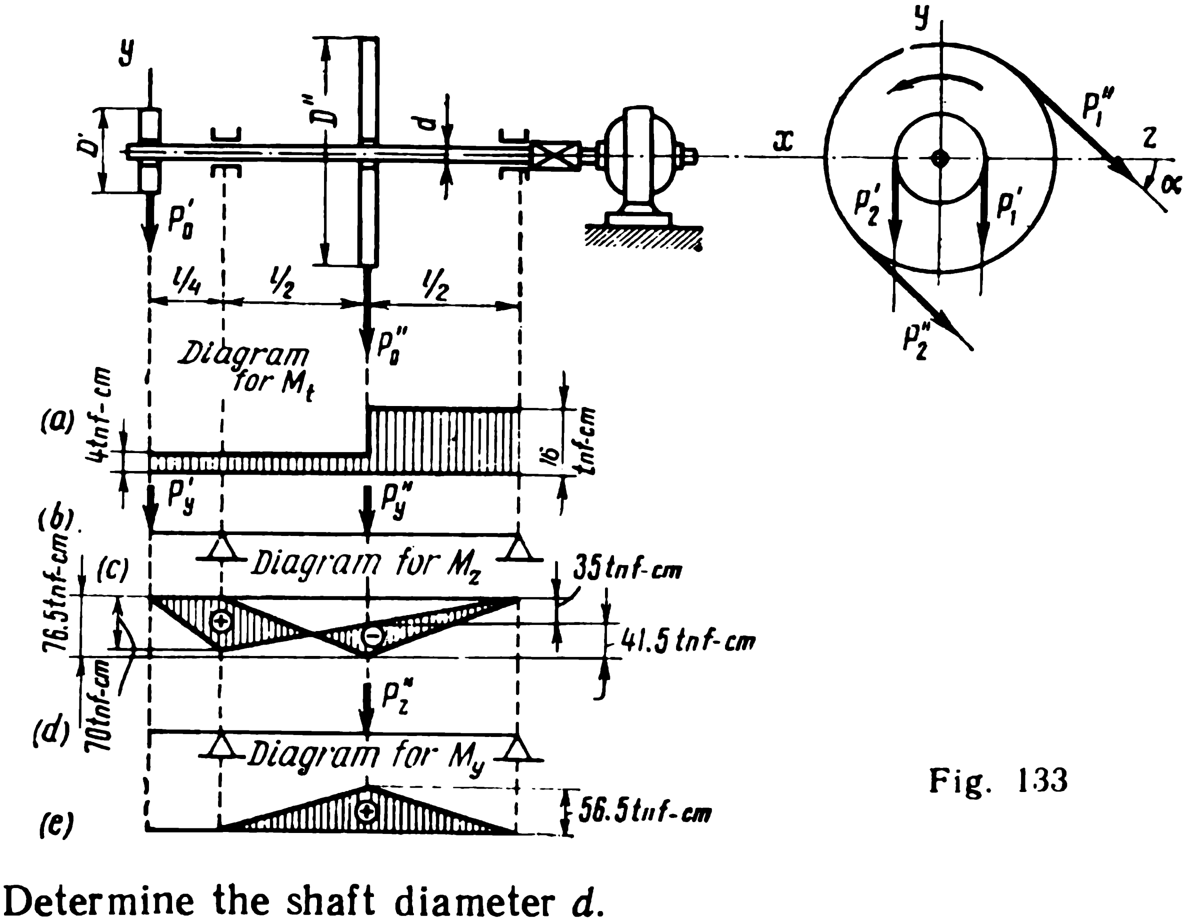


Fig. 133

Determine the shaft diameter d.

Solution. By referring the tension of the belts of each pulley to the centre of the shaft we can find the moments about the x-axis in the sections of the first and second pulleys. Thus

$$M_x' = (P_1' - P_2') \frac{D'}{2} = 400 \times 10 = 4 \times 10^3 \text{ kgf-cm};$$

$$M_x'' = (P_1'' - P_2'') \frac{D''}{2} = 400 \times 30 = 12 \times 10^3 \text{ kgf-cm}$$

Projecting the forces acting in the section of each pulley on the y- and z-axes we find the resultant transverse (shearing) forces acting in the direction of these axes:

$$P'_y = (P'_1 + P'_2) + P'_0 = 1200 + 200 = 1400 \text{ kgf}; \quad P'_z = 0;$$

$$P''_y = (P''_1 + P''_2)\cos 45^\circ + P''_0 \cong 1600 \times 0.707 + 400 = 1530 \text{ kgf};$$

$$P''_z = (P''_1 + P''_2)\sin 45^\circ \cong 1600 \times 0.707 = 1130 \text{ kgf}$$

Moments M'_x and M''_x produce torsion. The diagram for torque M_t is plotted in Fig. 133*a*.

Forces P'_y and P''_y tend to bend the shaft like a cantilever beam in plane xy (Fig. 133*b*). Using the method of superposition of diagrams we construct the diagram for the bending moment M_z about the z-axis (Fig. 133*c*).

Force P''_z tends to bend the shaft in plane xz as shown in Fig. 133*d*. The diagram of the bending moment M_y about the y-axis is given in Fig. 133*e*.

The dangerous section of the shaft is the one at which the second pulley is located.

Next we apply the third strength theory for design calculations. First we find the equivalent bending moment by formula (152)

$$M_{eq_{III}} = \sqrt{M^2 + M_t^2} = \sqrt{M_y^2 + M_z^2 + M_t^2}$$
$$= \sqrt{56.5^2 + 41.5^2 + 16^2} \cong 71.9 \text{ tnf-cm}$$

According to the design formula (151)

$$W = \frac{\pi d^3}{32} \cong 0.1d^3 \geqslant \frac{M_{eq_{III}}}{[\sigma]} = \frac{71{,}900}{1000}$$

Then the shaft diameter is

$$d \geqslant \sqrt[3]{71.9} \cong 4.2 \text{ cm}$$

Example 72. Given: P, a, $h = 2b$ and $[\sigma]$ (Fig. 134). Find b and h.

Solution. First we construct the diagrams for the bending moment M and torque M_t (Fig. 134*a* and *b*).

At point A of the dangerous section in the fixed portion (Fig. 134*c*)

$$\sigma = \frac{M}{W} = \frac{36Pa}{bh^2} = 9\frac{Pa}{b^3} \quad \text{and} \quad \tau = \gamma\frac{M_t}{W_t} = 0.795\frac{2Pa}{0.493b^3} \cong 3.23\frac{Pa}{b^3}$$

According to the third strength theory

$$\sigma_{eq_{III}} = \sqrt{\sigma^2 + 4\tau^2} = \frac{Pa}{b^3}\sqrt{81 + 4 \times 10.4} \cong 11.07\frac{Pa}{b^3} \leqslant [\sigma]$$

from which

$$b \geqslant \sqrt[3]{11.07\frac{Pa}{[\sigma]}} \cong 2.23\sqrt[3]{\frac{Pa}{[\sigma]}}$$

At point B of the dangerous section $\sigma = 0$ due to the bending moment and due to the torque

$$\tau_{max} = \frac{M_t}{W_t} = \frac{2Pa}{0.493b^3}$$

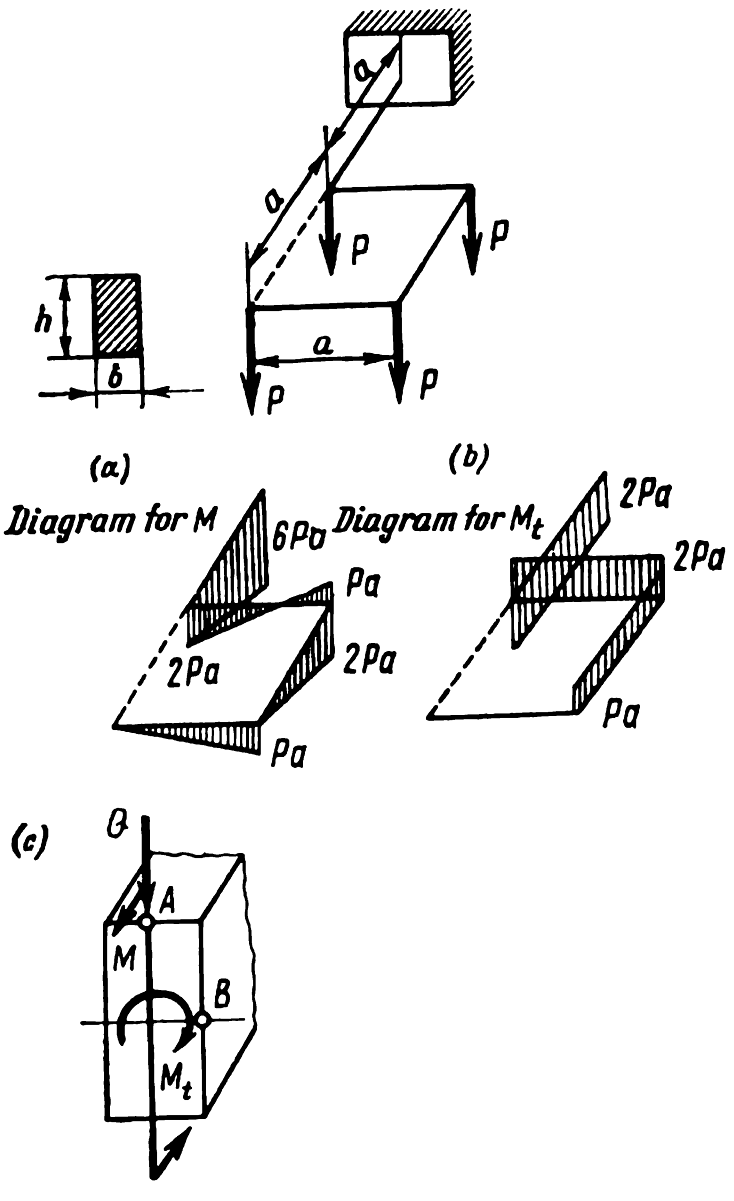


Fig. 134

According to the third strength theory

$$\tau_{max} = \frac{2Pa}{0.493b^3} \leqslant \frac{[\sigma]}{2}$$

from which

$$b \geqslant \sqrt[3]{8.11\frac{Pa}{[\sigma]}}$$

Point A turns out to be the more dangerous and therefore we take

$$b \cong 2.23\sqrt[3]{\frac{Pa}{[\sigma]}}$$

and $h \cong 4.46\sqrt[3]{\frac{Pa}{[\sigma]}}$

Then we check the selected dimensions of b and h for point B, taking the shearing stresses due to bending into consideration.

Since at the dangerous section the transverse (shearing) force $Q = 4P$, the following shearing stresses develop at point B:

$$\tau_Q = \frac{3}{2}\cdot\frac{Q}{bh} = 3\frac{P}{b^2}$$

The resultant shearing stress at point B is

$$\tau = \tau_Q + \tau_{max} = \frac{3P}{b^2} + \frac{2Pa}{0.493b^3} = \frac{Pa}{b^3}\left(4.06 + 3\frac{b}{a}\right)$$

The following condition must be observed in accordance with the third strength theory:

$$\tau = \frac{Pa}{b^3}\left(4.06 + 3\frac{b}{a}\right) \leqslant \frac{[\sigma]}{2}$$

therefore

$$\frac{Pa}{b^3} \leqslant \frac{[\sigma]}{2\left(4.06 + 3\frac{b}{a}\right)}$$

Calculations for point A indicate that $\frac{Pa}{b^3} \leqslant \frac{[\sigma]}{11.07}$.

Thus greater dimensions for the section are obtained from calculations for point B when $\frac{[\sigma]}{2\left(4.06+3\frac{b}{a}\right)} \leqslant \frac{[\sigma]}{11.07}$ or $4.06+3\frac{b}{a} > 5.54$ or $\frac{b}{a} > \frac{1.48}{3} \cong 0.493$.

This means that point B is more dangerous than point A only at such small lengths of portions of the bar for which $a < \frac{b}{0.493} \cong 2b$.

Example 73. Four prismatic bars of length l with their geometric axes lying in a single plane are fixed at one end and joined together at the other end by an absolutely rigid plate (Fig. 135a).

Determine the angle of rotation φ of the plate due to moment M_0 applied in its plane, if the distance between the axes of the bars is a,

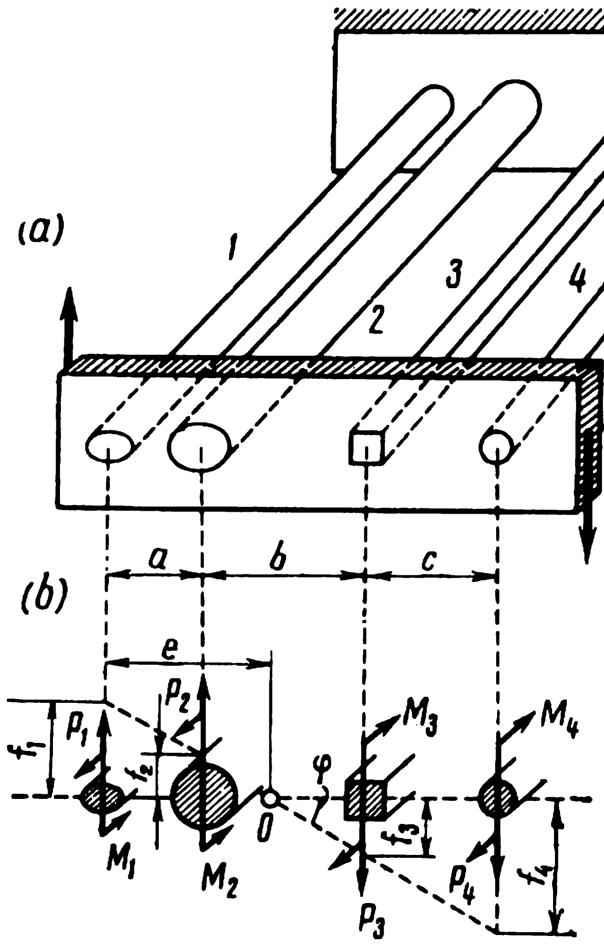


Fig. 135

b and c, torsional rigidity of the bars is C_1, C_2, C_3 and C_4 and their flexural rigidity, B_1, B_2, B_3 and B_4.

Assume that the connecting plate remains vertical and the horizontal straight line passing through the centres of gravity of the end sections of the bars corresponds to the direction of one of the principal centroidal axes of inertia of the bars.

Solution. Since the connecting plate is perfectly rigid, it will turn in its plane about some point O (Fig. 135*b*) through an angle φ. This angle is the angle of twist for each of the bars, i.e.

$$\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi \tag{a}$$

Since

$$\varphi_1 = \frac{M_{t_1}}{C_1}; \quad \varphi_2 = \frac{M_{t_2}}{C_2}; \quad \varphi_3 = \frac{M_{t_3}}{C_3}; \quad \varphi_4 = \frac{M_{t_4}}{C_4} \tag{b}$$

in which M_{t_i} is the torque for each bar $\left(\text{of rigidity } C_i = \frac{G_i I_{t_i}}{l}\right)$, then equations (a) can be written as

$$\frac{M_{t_1}}{C_1} = \frac{M_{t_2}}{C_2} = \frac{M_{t_3}}{C_3} = \frac{M_{t_4}}{C_4} = \varphi \tag{c}$$

Using a property of equal ratios, we obtain

$$M_{t_1} + M_{t_2} + M_{t_3} + M_{t_4} = (C_1 + C_2 + C_3 + C_4)\,\varphi \tag{d}$$

The deflections of the ends of the bars should lie in the plane of the connecting plate and should equal (Fig. 135*b*):

$$f_1 = e\varphi = \frac{1}{B_1}\left(P_1 - \frac{3}{2l} M_1\right); \quad f_2 = (e-a)\,\varphi = \frac{1}{B_2}\left(P_2 - \frac{3}{2l} M_2\right);$$

$$f_3 = (a+b-e)\,\varphi = \frac{1}{B_3}\left(P_3 - \frac{3}{2l} M_3\right) \text{ and } f_4 = (a+b+c-e)\,\varphi =$$

$$= \frac{1}{B_4}\left(P_4 - \frac{3}{2l} M_4\right) \tag{e}$$

in which P_1, P_2, P_3, P_4 and M_1, M_2, M_3, M_4 are the forces and moments applied to the end sections of the respective bars, and their rigidities are

$$B_i = \frac{3E_i I_i}{l^3}$$

Assuming that the connecting plate remains vertical, the angles of rotation of the end sections of the bars due to bending must equal zero, i.e.

$$\left.\begin{aligned} \theta_1 &= \frac{1}{B_1}\left(\frac{3}{2l} P_1 - \frac{3}{l^2} M_1\right) = 0; \quad \theta_2 = \frac{1}{B_2}\left(\frac{3}{2l} P_2 - \frac{3}{l^2} M_2\right) = 0; \\ \theta_3 &= \frac{1}{B_3}\left(\frac{3}{2l} P_3 - \frac{3}{l^2} M_3\right) = 0; \quad \theta_4 = \frac{1}{B_4}\left(\frac{3}{2l} P_4 - \frac{3}{l^2} M_4\right) = 0 \end{aligned}\right\} \tag{f}$$

whence

$$M_1 = \frac{P_1 l}{2}; \quad M_2 = \frac{P_2 l}{2}; \quad M_3 = \frac{P_3 l}{2}; \quad M_4 = \frac{P_4 l}{2} \tag{g}$$

Substituting these values into equations (e) and solving the latter for P_1 we obtain

$$\left.\begin{aligned} P_1 &= 4B_1 e\varphi; \quad P_2 = 4B_2(e-a)\varphi; \\ P_3 &= 4B_3(a+b-e)\varphi; \quad P_4 = 4B_4(a+b+c-e)\varphi \end{aligned}\right\} \tag{h}$$

From the conditions of statics

$$P_1 + P_2 = P_3 + P_4 \tag{i}$$

or

$$\begin{aligned} B_1 e\varphi + B_2(e-a)\varphi &= B_3(a+b-e)\varphi \\ &+ B_4(a+b+c-e)\varphi \end{aligned} \tag{j}$$

therefore

$$e = \frac{B_2 a + B_3(a+b) + B_4(a+b+c)}{B_1 + B_2 + B_3 + B_4} \tag{k}$$

Next we find the sum of the moments about point O:

$$\begin{aligned} M_{t_1} + M_{t_2} + M_{t_3} + M_{t_4} + P_1 e + P_2(e-a) + P_3(a+b-e) \\ + P_4(a+b+c-e) = M_0 \end{aligned} \tag{l}$$

Taking equations (d) and (k) into account we can rewrite equation (l) in the following form

$$\begin{aligned} \varphi(C_1 + C_2 + C_3 + C_4) + 4\varphi B_1 e^2 + 4\varphi B_2(e-a)^2 + 4\varphi B_3(a+b-e)^2 \\ + 4\varphi B_4(a+b+c-e)^2 = M_0 \end{aligned} \tag{m}$$

from which

$$\varphi = \frac{M_0}{C_1 + C_2 + C_3 + C_4 + 4[B_1 e^2 + B_2(e-a)^2 + B_3(a+b-e)^2 + B_4(a+b+c-e)^2]} \tag{n}$$

Knowing e and φ, we can find M_{t_i} from equations (b), P_i, from equations (h) and M_i from equations (g).

The maximum bending moments for each bar are developed in the end sections. They equal

$$\max M_i = P_i \frac{l}{2}$$

The required strength of each of the bars is established by calculations involving combined torsion with bending.

Special case. If all the bars are round and

$$d_1 = d_2 = d_3 = d_4 = d; \quad a = b = c = d; \quad l = 10d;$$

$$E_1 = E_2 = E_3 = E_4 = E; \quad G_1 = G_2 = G_3 = G_4 = G = \frac{2}{5}E$$

then

$$I = \frac{1}{2} I_p; \quad C_1 = C_2 = C_3 = C_4 = C = \frac{GI_p}{l};$$

$$B_1 = B_2 = B_3 = B_4 = B = \frac{3EI}{l^3} = \frac{15}{4}\frac{GI_p}{l^3}$$

From formulas (k), (n), (c) and (h) we find, respectively,

$$e=\frac{3}{2}d;$$

$$\varphi=\frac{M_0}{4\frac{GI_p}{l}+15\frac{GI_p}{l^3}d^2\left(\frac{9}{4}+\frac{1}{4}+\frac{1}{4}+\frac{9}{4}\right)}=\frac{M_0l}{GI_p}\cdot\frac{1}{4+75\frac{d^2}{l^2}}=\frac{4}{19}\frac{M_0l}{GI_p};$$

$$M_{t_1}=M_{t_2}=M_{t_3}=M_{t_4}=M_t=\frac{4}{19}M_0;$$

$$P_1=P_4=\frac{4}{19}\times\frac{M_0l}{GI_p}4\times\frac{15}{4}\times\frac{GI_p}{l^3}\times\frac{3}{2}d=\frac{90}{19}M_0\frac{d}{l^2}=\frac{9}{19}\cdot\frac{M_0}{l};$$

$$P_2=P_3=\frac{3}{19}\frac{M_0}{l}$$

The maximum bending moments in the bars are

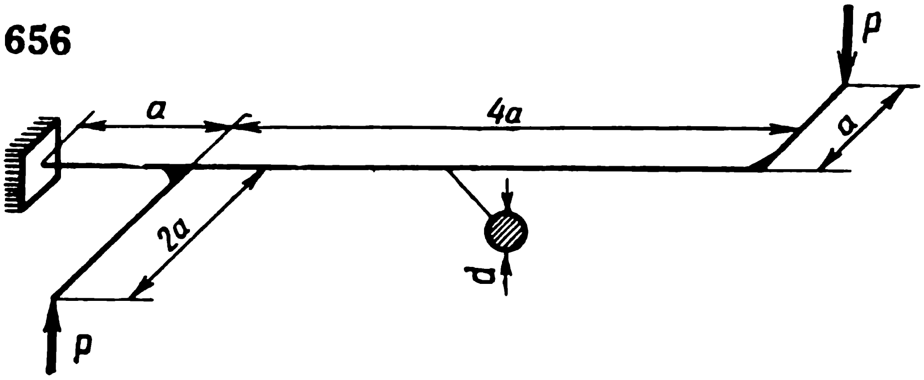
$$\max M_1=\max M_4=\frac{9}{38}M_0;\quad \max M_2=\max M_3=\frac{3}{38}M_0$$

Design equations should be set up for the lst and 4th bars Using he third strength theory we obtain

$$\frac{M_0}{38W}\sqrt{81+64}\cong 0.317\frac{M_0}{W}\leqslant[\sigma]$$

Problems 656 through 664. Construct the diagrams for the torque (M_t) and bending moment M, and determine the equivalent stress on the basis of the third strength theory.

656



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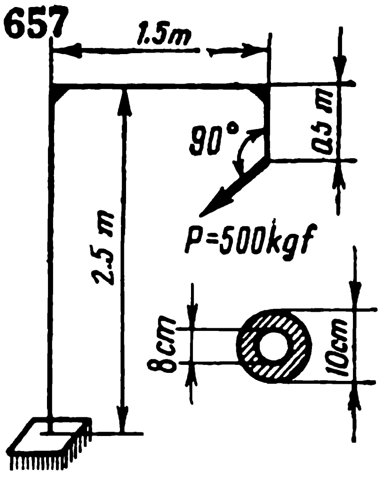

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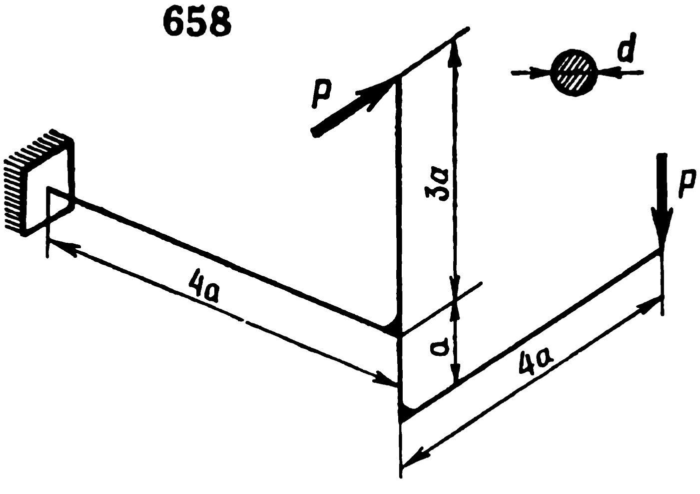

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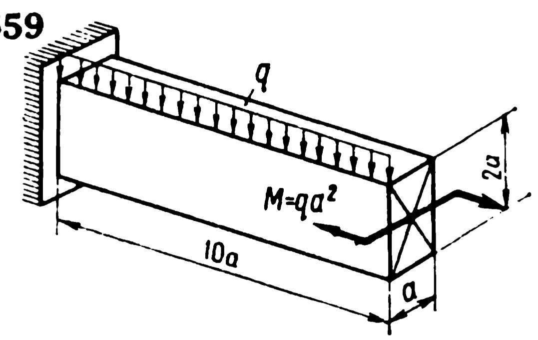

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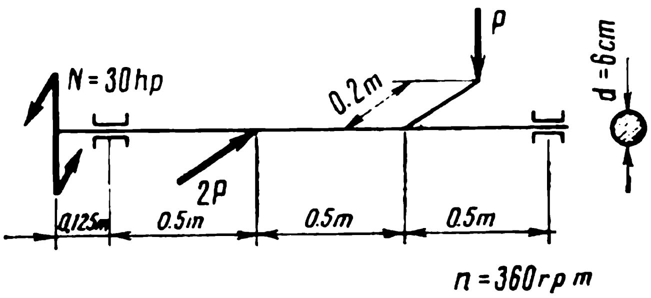

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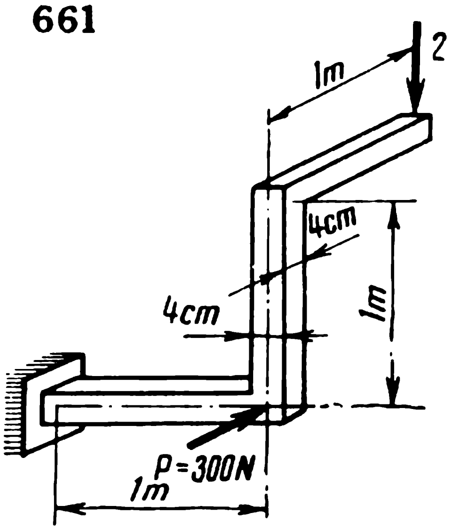

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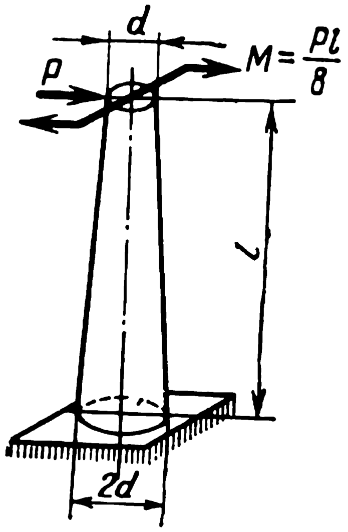

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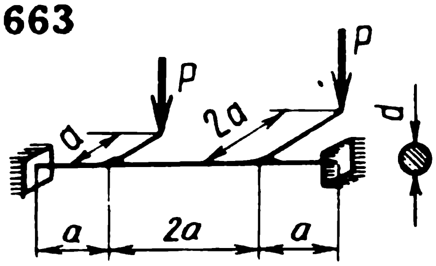

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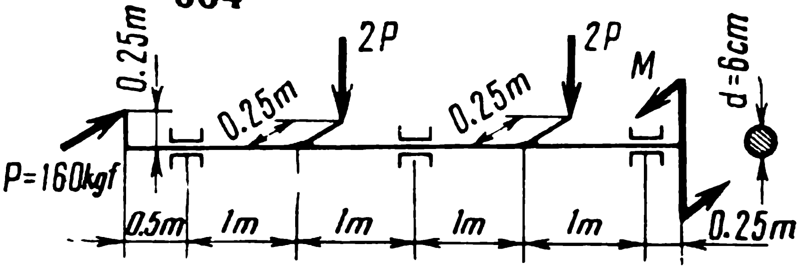
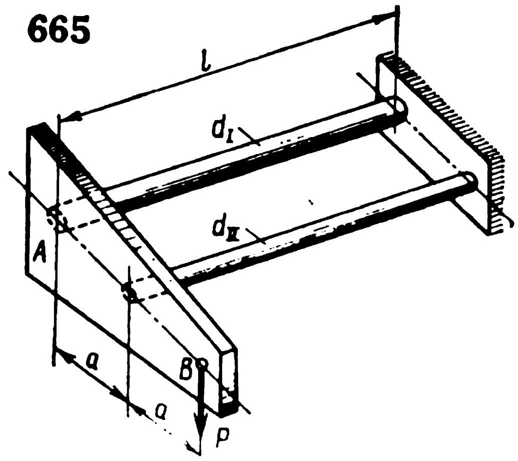


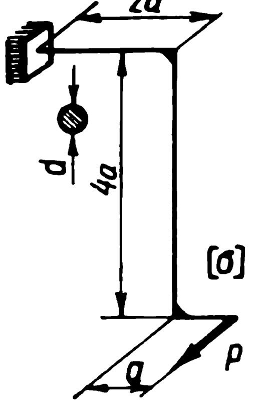
Problem 665. Let $P = 60$ kgf; $d_1 = 2$ cm; $d_2 = 1$ cm; $a = 8$ cm; $l = 20$ cm and $E = \frac{5}{2} G = 2 \times 10^6$ kgf/cm^2. Plate AB is perfectly rigid and remains in a vertical plane. Determine the maximum and minimum principal normal stresses in the bars ($\sigma^{\max}_{\min\, I,\, II}$) and the vertical displacement (δ_p) of the point of application of force P.

665



Problems 666 through 680. Determine the cross-sectional dimensions of the bars in accordance with the third strength theory.

666



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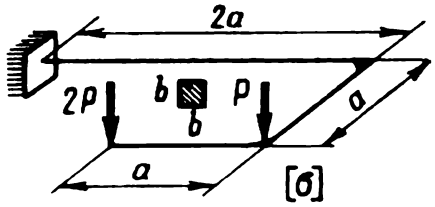

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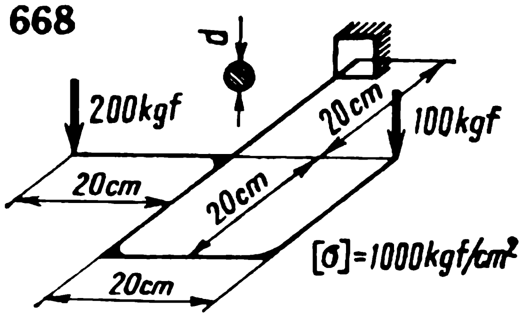

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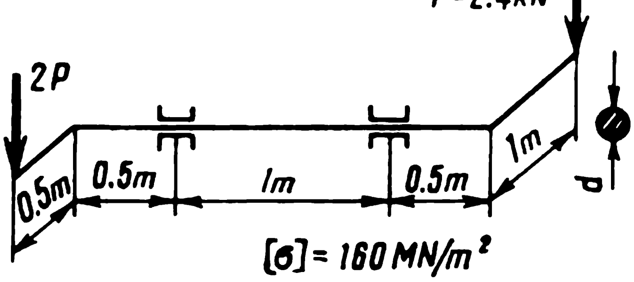

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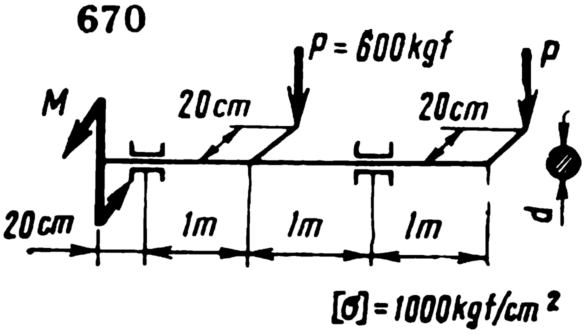

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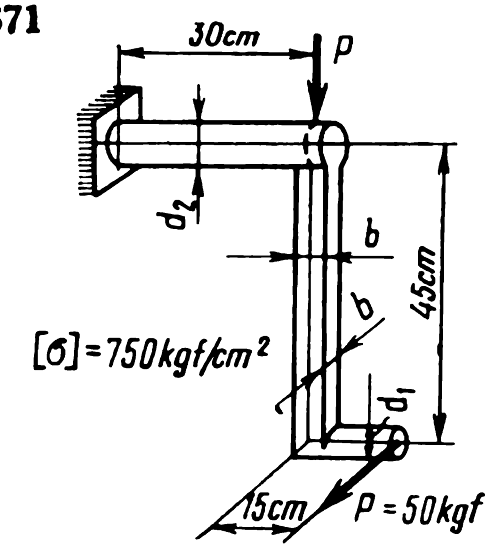

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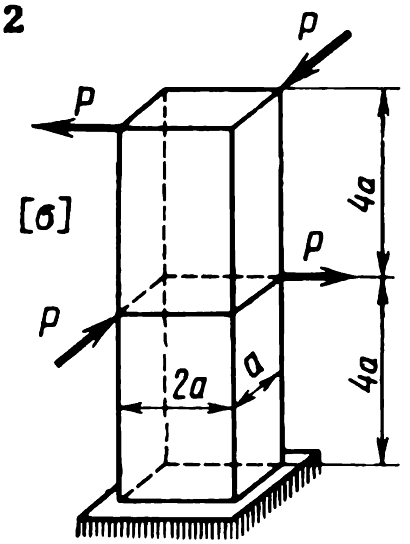

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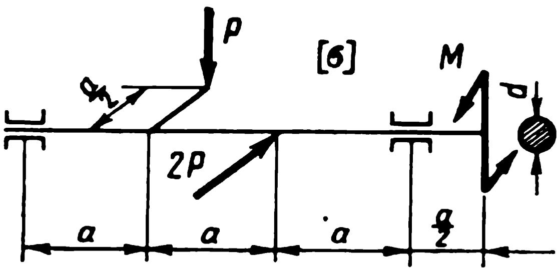

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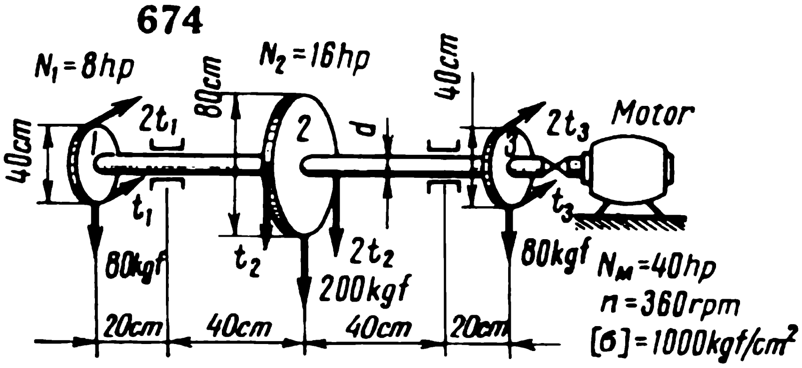

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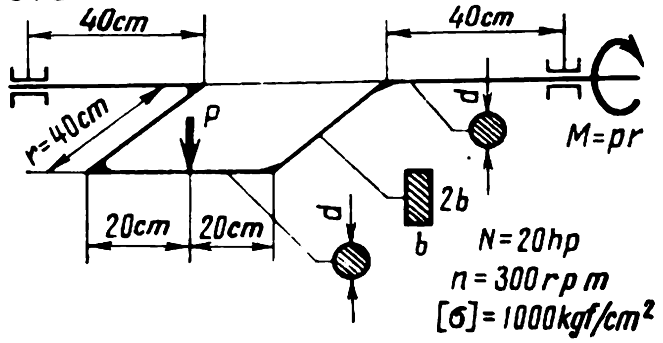

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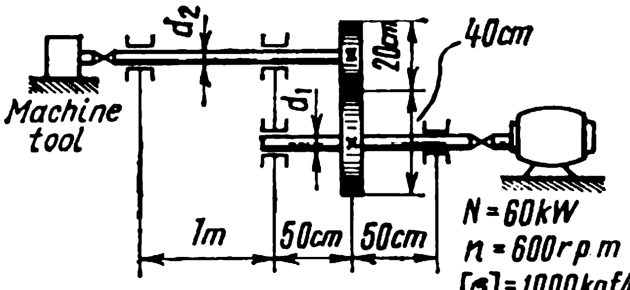

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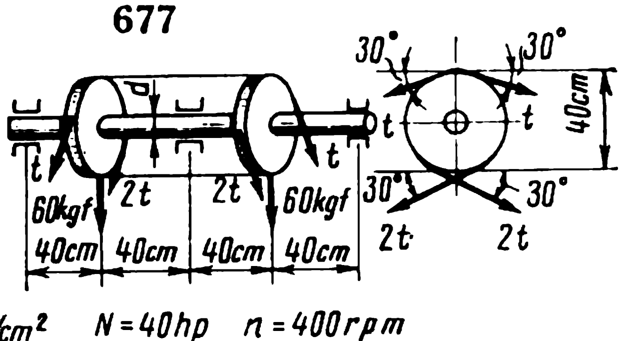

Radial pressure $= \frac{1}{3}$ circumferential force

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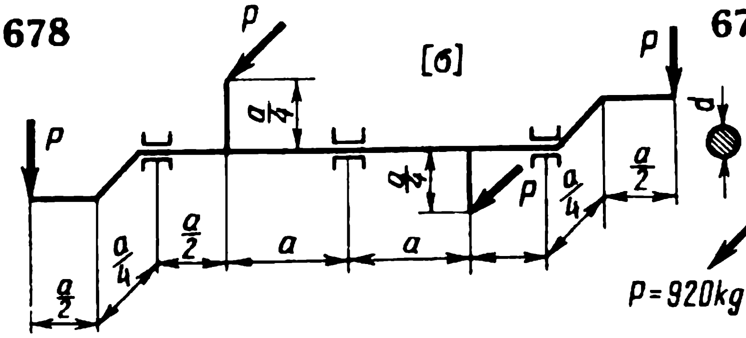

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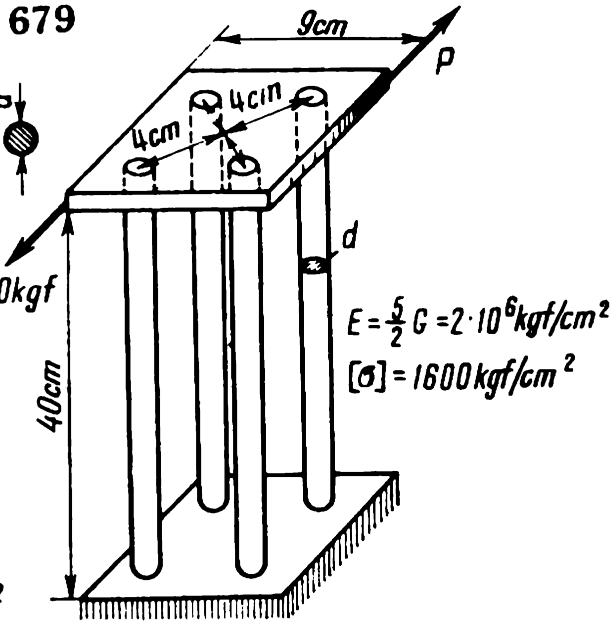

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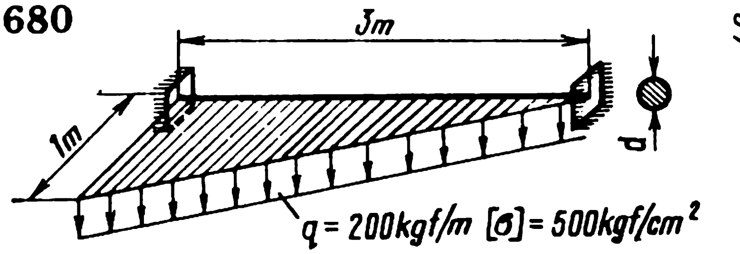

Problems 681 through 688. Construct the torque and bending moment diagrams and determine the safe load in accordance with the third strength theory.

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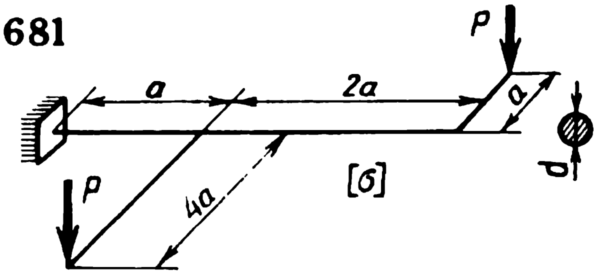

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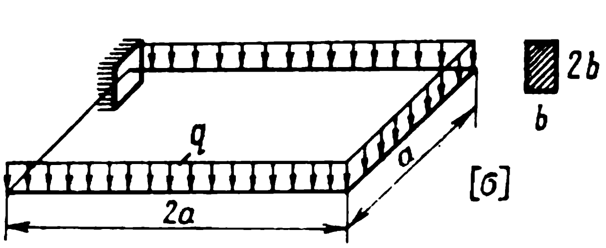

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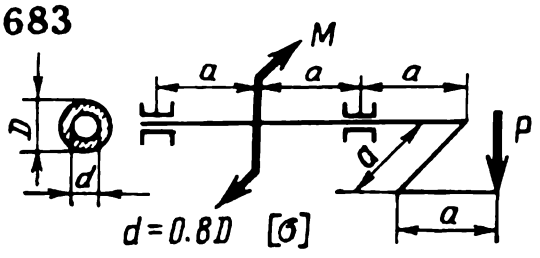

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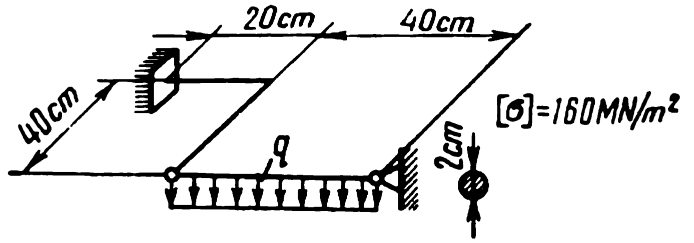

In Problem 688 the connecting plate AB is assumed to be perfectly rigid and to remain in a vertical plane.

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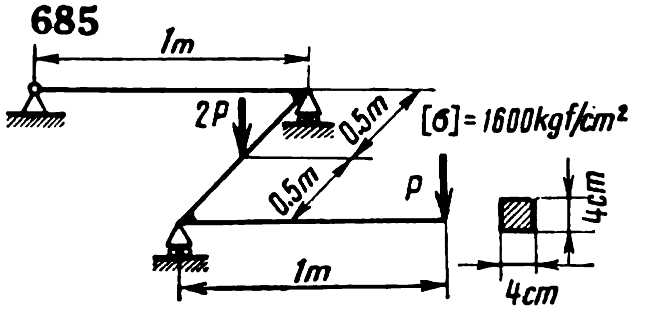

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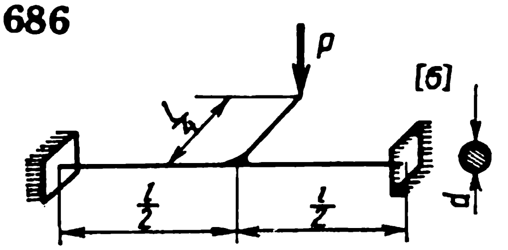

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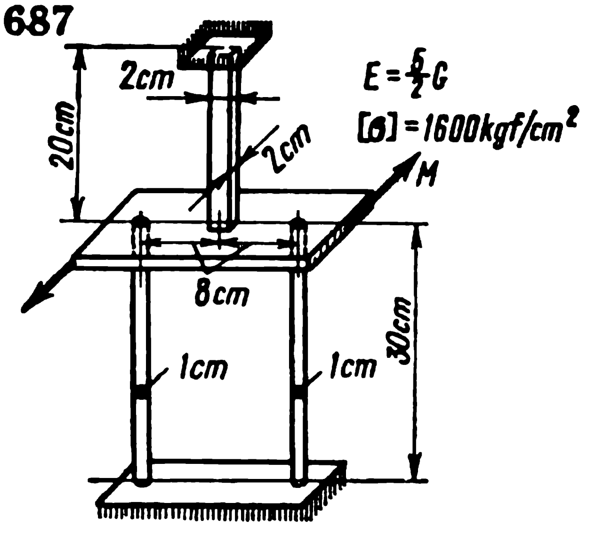

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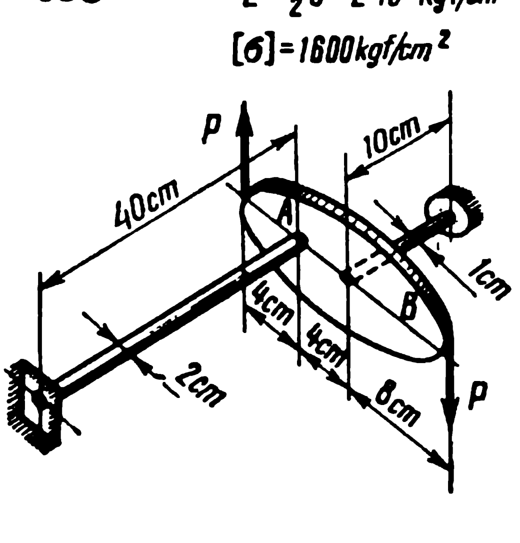


9.5.

General Case of Combined Stress

In the general case of three-dimensional action of forces on a prismatic bar, the internal forces in a cross section are reduced to six components: axial force N_x, torque M_t, the transverse (shearing) forces Q_y and Q_z, and the bending moments M_y and M_z (Fig. 136).

If the x-axis is the geometric axis of the bar and the y- and z-axes are the principal centroidal axes of inertia of the cross section whose centre of gravity coincides with the shear (flexural) centre, then Q_y and M_z determine the transverse bending in plane xy, whereas Q_z and M_y, the transverse bending in plane xz. Thus the bar undergoes

simultaneous tensile or compressive strain, torsional strain and two plane bending strains.

At an arbitrary point (y, z) of the cross section of the bar the normal stress is found by formula (142), and the resultant shearing stress, by the geometrical summation of shearing stresses due to torsion and bending.

The neutral axis equation is of the form given in expression (143).

The position of the dangerous point cannot always be readily determined. Therefore it is necessary to compare the degree of danger of several points along the outline of the section. The most dangerous will be the point of the outline for which the equivalent stress, found by using the selected strength theory is of maximum value.

Comparing this design stress with the permissible one we can either determine the required cross section of the bar or check the strength of a bar with given dimensions of its cross section.

Example 74. Let $P_0 = 40$ tnf, $P_1 = 8$ tnf, $P_2 = 4$ tnf, $P_3 = 2$ tnf, $l = 1$ m, $h = 24$ cm, $b = 8$ cm and $[\sigma] = 1400$ kgf/cm^2 (Fig. 137).

Check the strength of the bar.

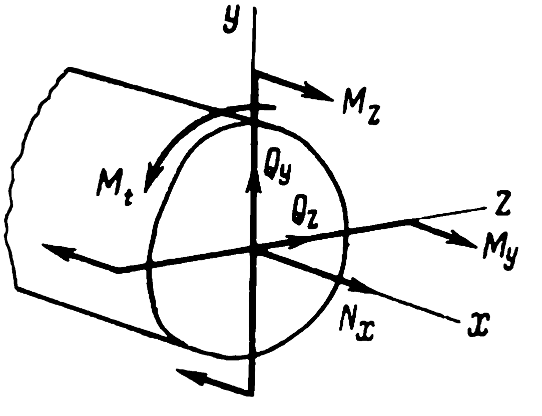


Fig. 136

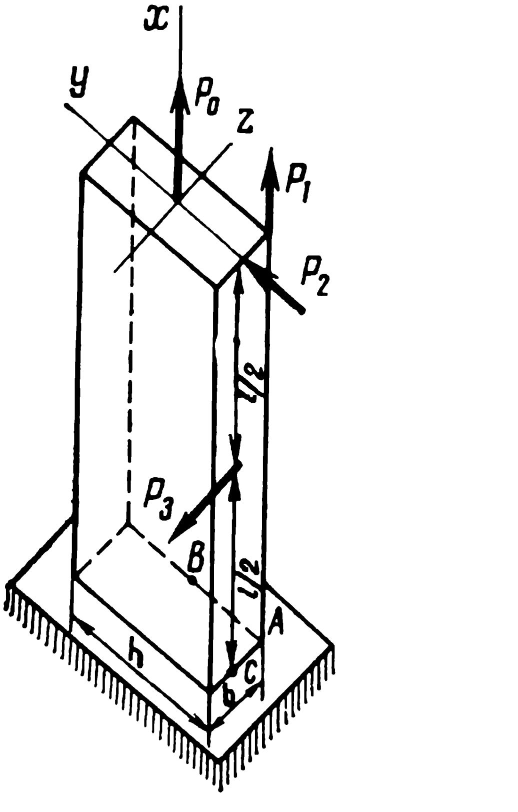


Fig. 137

Solution. In the lower dangerous section the bar is subject to the following stresses:

$$N_x = P_0 + P_1 = 40 + 8 = 48 \text{ tnf};$$

$$M_z = P_1 \frac{h}{2} + P_2 l = 8 \times 12 + 4 \times 100 = 496 \text{ tnf-cm};$$

$$M_y = P_1 \frac{b}{2} + P_3 \frac{l}{2} = 8 \times 4 + 2 \times 50 = 132 \text{ tnf-cm};$$

$$M_t = P_3 \frac{h}{2} = 2 \times 12 = 24 \text{ tnf-cm};$$

$$Q_y = P_2 = 4 \text{ tnf}; \ Q_z = P_3 = 2 \text{ tnf}$$

Figure 138 illustrates the diagrams and values of normal and shearing stresses corresponding to the found internal forces.

The normal stresses developed at corner A of the lower cross section of the bar (Fig. 137) reach the maximum value of

$$\sigma_{eq_A} = \sigma_{\max} = \sigma' + \sigma'' + \sigma''' = 250 + 646 + 516 = 1412 \text{ kgf/cm}^2$$

The normal and shearing stresses at point B in the middle of the right-hand longer side are

$$\sigma = \sigma' + \sigma''' = 250 + 516 = 766 \text{ kgf/cm}^2;$$

$$\tau = \tau_{t\max} - \tau_y = 59 - 31 = 28 \text{ kgf/cm}^2$$

The equivalent stress according to the third strength theory is

$$\sigma_{eq_B} = \sqrt{\sigma^2 + 4\tau^2} = \sqrt{766^2 + 4 \times 28^2} \cong 768 \text{ kgf/cm}^2$$

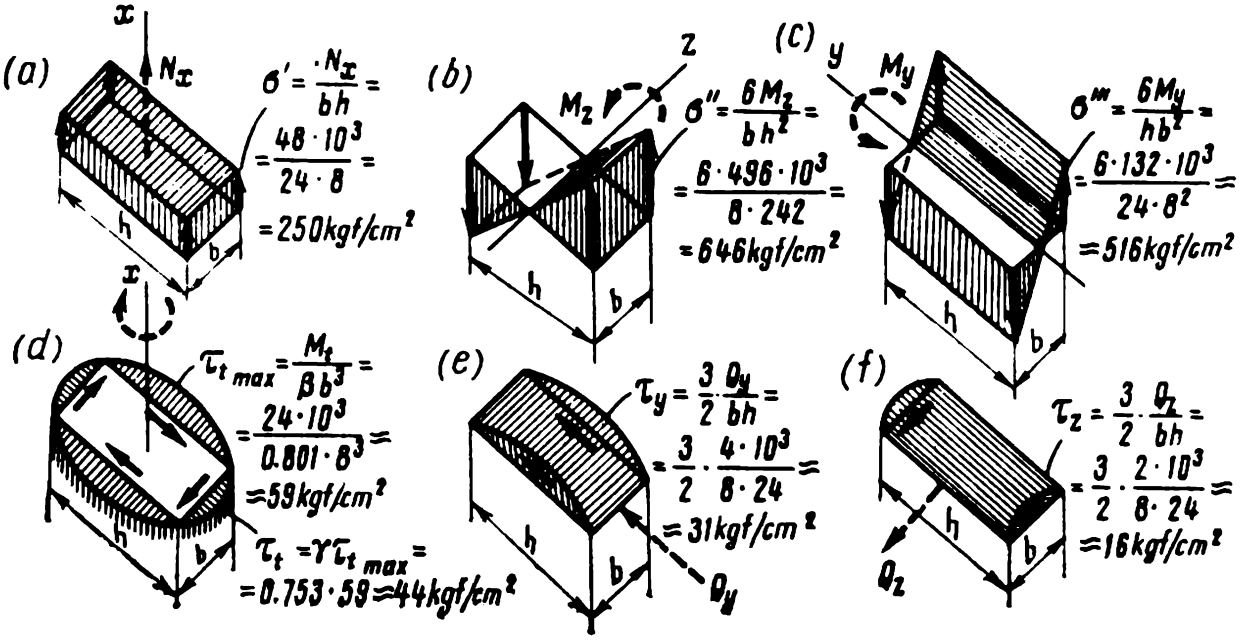


Fig. 138

The following normal and shearing stresses are developed at point C in the middle of the lower smaller side:

$$\sigma = \sigma' + \sigma \;\; = 250 + 646 = 896 \text{ kgf/cm}^2;$$

$$\tau = \tau_t + \tau_z = 44 + 16 = 60 \text{ kgf/cm}^2$$

The equivalent stress according to the third strength theory is

$$\sigma_{eq_C} = \sqrt{896^2 + 4 \times 60^2} \cong 904 \text{ kgf/cm}^2$$

Comparing values σ_{eq} at the dangerous points A, B and C, it is evident that the most dangerous point is A.

Since $\sigma_{eq_A} = 1412$ kgf/cm² which exceeds $[\sigma]$ by less than 1%, the bar is sufficiently strong.

Example 75. Let $P_1 = 200$ kgf; $P_2 = 100$ kgf; $P_3 = 240$ kgf; $l_1 = 30$ cm; $l_2 = 40$ cm; $l_3 = 60$ cm; $l_4 = 80$ cm and $[\sigma] = 1000$ kgf/cm² (Fig. 139).

Determine a, h, b, d and d_0.

Solution. 1. First, using the six conditions of statics we determine the components of the reaction at the fixed end (Fig. 140). Thus

$$\sum X = A_x - P_1 = 0; \quad A_x = P_1 = 200 \text{ kgf};$$

$$\sum Y = A_y - P_2 = 0; \quad A_y = P_2 = 100 \text{ kgf};$$

$$\sum Z = -A_z + P_3 = 0; \quad A_z = P_3 = 240 \text{ kgf};$$

$$\sum M_x = M_{A_x} - P_3 l_3 = 0; \quad M_{A_x} = P_3 l_3 = 240 \times 0.6 = 144 \text{ kgf-m};$$

$$\sum M_y = M_{A_y} + P_1 l_1 - P_3 l_4 = 0;$$

$$M_{A_y} = P_3 l_4 - P_1 l_1 = 240 \times 0.8 - 200 \times 0.3 = 132 \text{ kgf-m};$$

$$\sum M_z = M_{A_z} - P_1 l_3 + P_2 (l_4 - l_2) = 0;$$

$$M_{A_z} = P_1 l_3 - P_2 (l_4 - l_2) = 200 \times 0.6 - 100 (0.8 - 0.4) = 80 \text{ kgf-m}$$

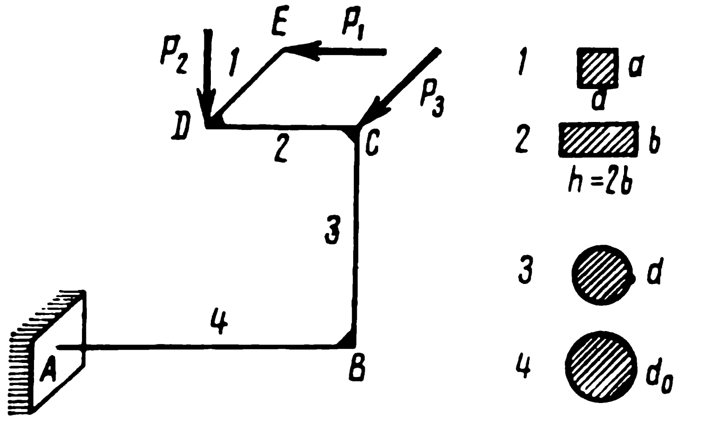


Fig. 139

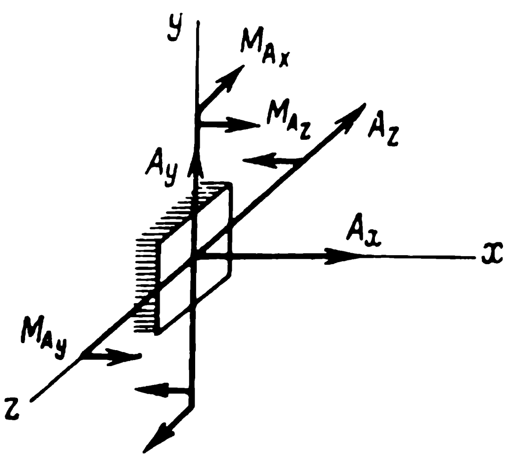


Fig. 140

2. Next we determine the axial forces N, torques M_t and bending moments M_x, M_y, M_z.

Part 1. We take $z = 0$ at point E and $z = l_1$ at point D.

Then for the bending in plane xz we obtain

$$M_y = P_1 z; \quad M_{y_{z=0}} = 0; \quad M_{y_{z=l_1}} = P_1 l_1 = 200 \times 0.3 = 60 \text{ kgf-m}$$

Part 2. We take $x = 0$ at point D and $x = l_2$ at point C.

Then for the tension and bending in planes xz and xy we obtain $N = P_1 = 200$ kgf; $M_y = P_1 l_1 = 200 \times 0.3 = 60$ kgf-m; $M_z = P_2 x$;

$$M_{z_{x=0}} = 0; \quad M_{z_{x=l_2}} = P_2 l_2 = 100 \times 0.4 = 40 \text{ kgf-m}$$

Part 3. We take $y = 0$ at point C and $y = l_3$ at point B.

Then for the compression, torsion and bending in planes yz and xy we obtain

$$N = -P_2 = -100 \text{ kgf}; \quad M_t = P_1 l_1 = 200 \times 0.3 = 60 \text{ kgf-m};$$

$$M_x = P_3 y$$

$$M_{x_{y=0}} = 0; \quad M_{x_{y=l_3}} = P_3 l_3 = 240 \times 0.6 = 144 \text{ kgf-m}; \quad M_z = P_2 l_2 + P_1 y;$$

$$M_{z_{y=0}} = P_2 l_2 = 100 \times 0.4 = 40 \text{ kgf-m};$$

$$M_{z_{y=l_3}} = P_2 l_2 + P_1 l_3 = 100 \times 0.4 + 200 \times 0.6 = 160 \text{ kgf-m}$$

Part 4. We take $x = 0$ at point A and $x = l_4$ at point B.

Then for the compression, torsion and bending in planes xz and xy we obtain

$$N = -A_x = -200 \text{ kgf}; \quad M_t = M_{Ax} = 144 \text{ kgf-m};$$

$$M_y = M_{Ay} - A_z x;$$

$$M_{y_{x=0}} = M_{Ay} = 132 \text{ kgf-m}; \quad M_{y_{x=l_4}} = M_{Ay} - A_z l_4 =$$

$$= 132 - 240 \times 0.8 = -60 \text{ kgf-m};$$

$$M_z = M_{Az} + A_y x; \quad M_{z_{x=0}} = M_{Az} = 80 \text{ kgf-m};$$

$$M_{z_{x=l_4}} = M_{Az} + A_y l_4 = 80 + 100 \times 0.8 = 160 \text{ kgf-m}$$

3. Next we construct the diagrams for N, M_t, M_x, M_y and M_z at the parts.

In accordance with the magnitudes of N, M_t, M_x, M_y and M_z determined for each part of the system the diagrams for these quantities are shown in Fig. 141*a*, *b*, *c*, *d* and *e*.

4. Now we can select the cross sections of the parts:

Part 1. The dangerous section is at point D (Figs. 141*f* and 139):

$$M_y = 60 \text{ kgf-m}$$

We select the section on the basis of plane transverse bending:

$$W = \frac{a^3}{6} \geqslant \frac{M_y}{[\sigma]}; \quad a \geqslant \sqrt[3]{6 \frac{M_y}{[\sigma]}} = \sqrt[3]{6 \frac{6 \times 10^3}{10^3}} \cong 3.3 \text{ cm}$$

Part 2. The dangerous section is at point C (Figs. 141*g* and 139). $N = 200$ kgf; $M_y = 60$ kgf-m and $M_z = 40$ kgf-m.

We select the section on the basis of oblique bending:

$$W_y = \frac{bh^2}{6} = \frac{h^3}{12}; \quad W_z = \frac{hb^2}{6} = \frac{h^3}{24}; \quad c = \frac{W_y}{W_z} = 2; \; W_y \geqslant \frac{M_y + cM_z}{[\sigma]};$$

$$\frac{h^3}{12} \geqslant \frac{6 \times 10^3 + 2 \times 4 \times 10^3}{10^3} = 14; \quad h \geqslant \sqrt[3]{12 \times 14} \cong 5.52 \text{ cm}$$

We take $h = 5.6$ cm and $b = \frac{h}{2} = 2.8$ cm, and check the part, taking the axial force into account. Thus

$$F = bh = 2.8 \times 5.6 \cong 15.7 \text{ cm}^2;$$

$$W_y = \frac{h^3}{12} \cong 14.6 \text{ cm}^3; \qquad W_z = \frac{h^3}{24} \cong 7.3 \text{ cm}^3;$$

$$\sigma_{\max} = \frac{N}{F} + \frac{M_y}{W_y} + \frac{M_z}{W_z} = \frac{200}{15.7} + \frac{6 \times 10^3}{14.6} + \frac{4 \times 10^3}{7.3} \cong 972 \text{ kgf/cm}^2$$

The understress is $\frac{[\sigma] - \sigma_{\max}}{[\sigma]} \times 100 = \frac{28 \times 100}{10^3} = 2.8\%$

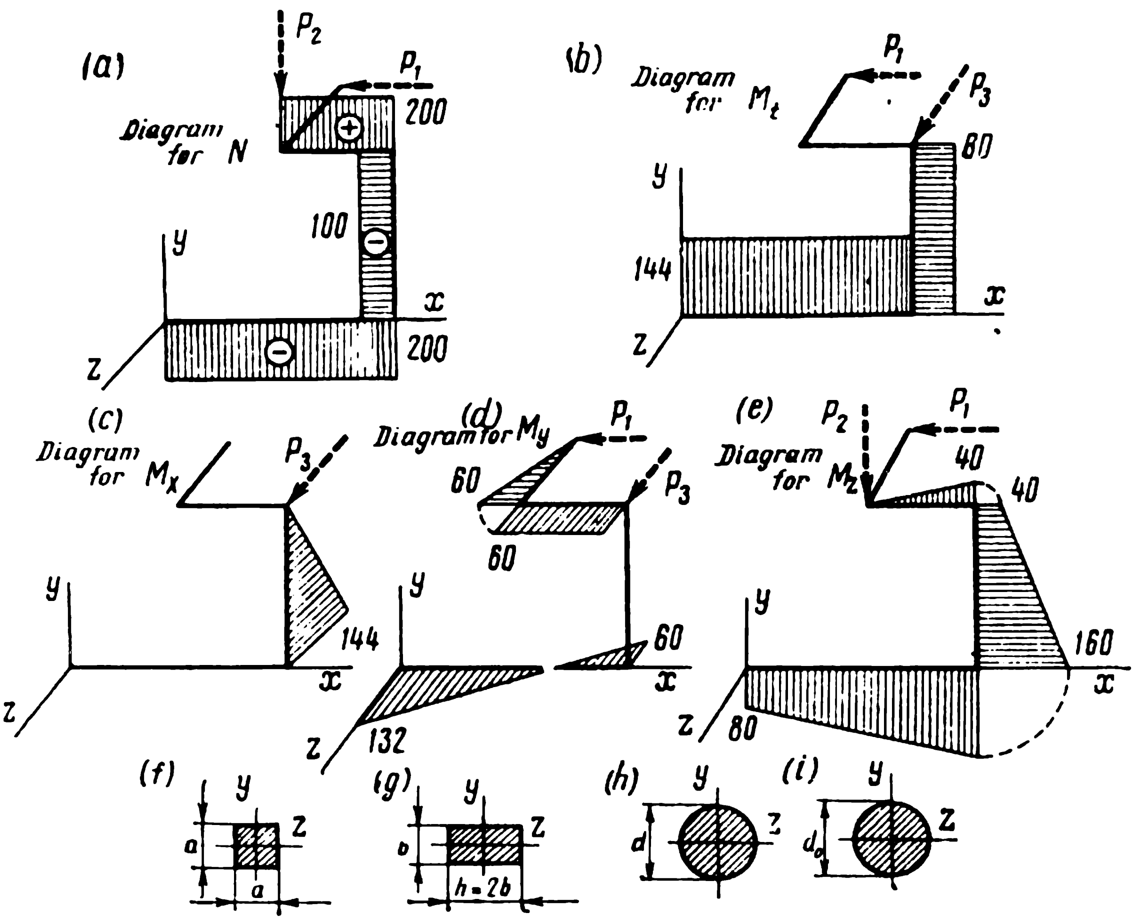


Fig. 141

Part 3. The dangerous section is at point B (Figs. 141*h* and 139). $N = 100$ kgf; $M_t = 60$ kgf-m; $M_x = 144$ kgf-m; $M_z = 160$ kgf-m. We select the required diameter of the cross section from considerations of torsion and bending. The equivalent bending moment according to the third strength theory is

$$M_{eq_{III}} = \sqrt{M_t^2 + M_x^2 + M_z^2} = \sqrt{60^2 + 144^2 + 160^2} \cong 223 \text{ kgf-m};$$

$$W \cong 0.1d^3 \geqslant \frac{M_{eq_{III}}}{[\sigma]} = \frac{223 \times 10^2}{10^3}; \quad d \geqslant \sqrt[3]{223} \cong 6.1 \text{ cm}$$

Then we check the section, taking the axial force into account:

$$W=\frac{\pi d^3}{32}=\frac{\pi\times 6.1^3}{32}\cong 22.3\ \text{cm}^3;\quad W_p=2W=44.6\ \text{cm}^3;$$

$$F=\frac{\pi d^2}{4}=\frac{\pi\times 6.1^2}{4}\cong 29.2\ \text{cm}^2$$

The bending moment is

$$M=\sqrt{M_x^2+M_z^2}=\sqrt{144^2+160^2}\cong 215\ \text{kgf-m}$$

The normal stress is

$$\sigma=\frac{N}{F}+\frac{M}{W}=\frac{100}{29.2}+\frac{215\times 10^2}{22.3}\cong 967\ \text{kgf/cm}^2$$

The shearing stress is

$$\tau=\frac{M_t}{W_p}=\frac{60\times 10^2}{44.6}\cong 135\ \text{kgf/cm}^2$$

The equivalent stress, calculated on the basis of the third strength theory is

$$\sigma_{eq_{III}}=\sqrt{\sigma^2+4\tau^2}=\sqrt{967^2+4\times 135^2}\cong 1004\ \text{kgf/cm}^2$$

The overstress is 0.4% which is quite permissible.

Part 4. The dangerous section is at point B (Figs. 141*i* and 139). $N = 200$ kgf; $M_t = 144$ kgf-m; $M_y = 60$ kgf-m; $M_z = 160$ kgf-m. We calculate the diameter of the cross section from considerations of torsion and bending.

The equivalent moment according to the third strength theory is

$$M_{eq_{III}}=\sqrt{M_t^2+M_y^2+M_z^2}=\sqrt{144^2+60^2+160^2}\cong 223\ \text{kgf-m}$$

On the basis of the calculations for part *3* we take $d_0 = 6.1$ cm and check the section, taking the axial force into account:

$$W=22.3\ \text{cm}^3;\quad W'_p=44.6\ \text{cm}^3;\quad F=29.2\ \text{cm}^2$$

The bending moment is

$$M=\sqrt{M_y^2+M_z^2}=\sqrt{60^2+160^2}\cong 171\ \text{kgf-m}$$

The normal stress is

$$\sigma=\frac{N}{F}+\frac{M}{W}=\frac{200}{29.2}+\frac{171\times 10^2}{22.3}\cong 774\ \text{kgf/cm}^2$$

The shearing stress is

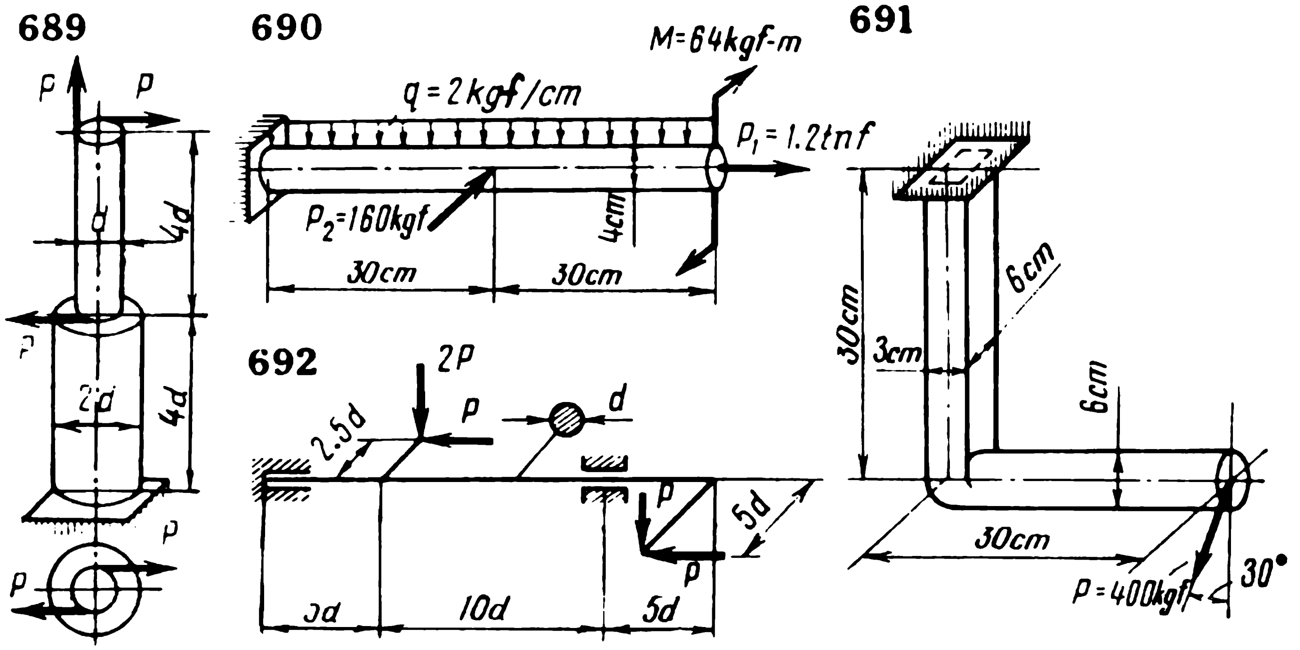
$$\tau=\frac{M_t}{W_p}=\frac{144\times 10^2}{44.6}\cong 323\ \text{kgf/cm}^2$$

The equivalent stress according to the third strength theory is

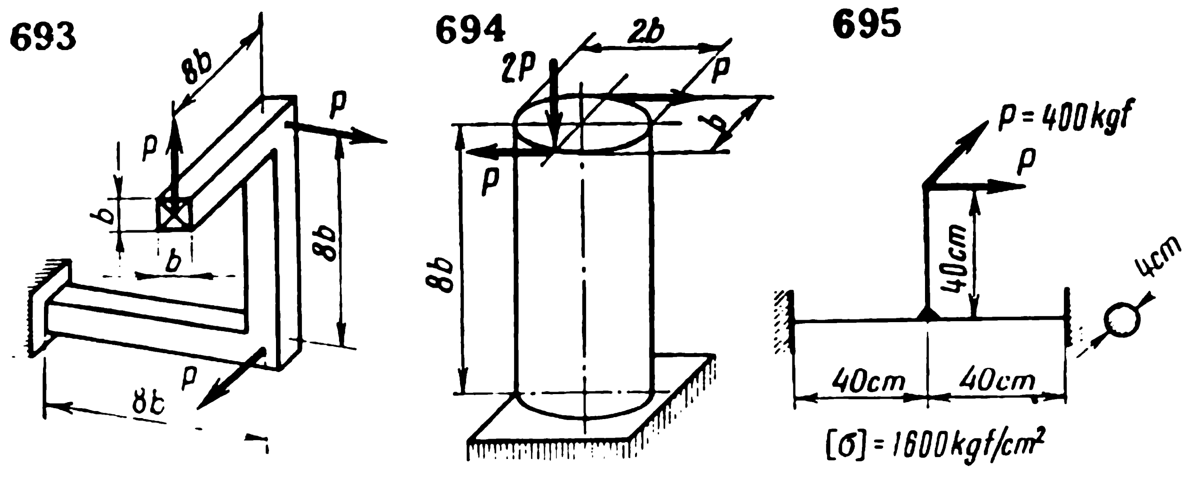
$$\sigma_{eq_{III}} = \sqrt{774^2 + 4 \times 323^2} \cong 1008 \text{ kgf/cm}^2$$

Since the overstress is less than 1%, we can take $d_0 = 6.1$ cm.

Problems 689 through 692. Determine the equivalent stresses according to the third strength theory.

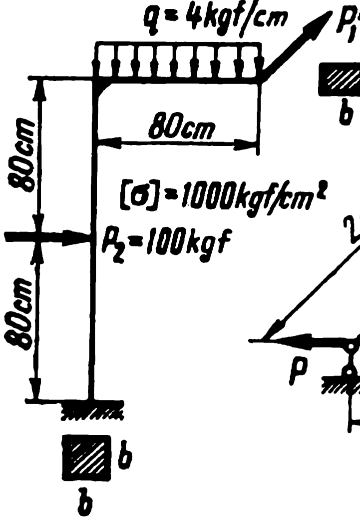


Problems 693 through 695. Check the strength of the bars on the basis of the third strength theory.



Problems 696, 697 and 698. Determine the required dimensions of the cross sections of the bars, using the third strength theory.

696



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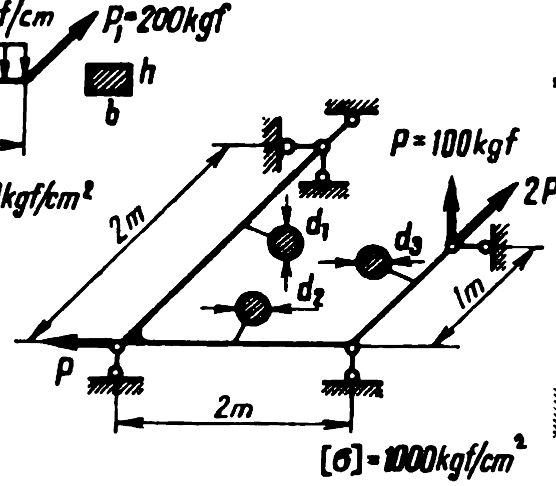

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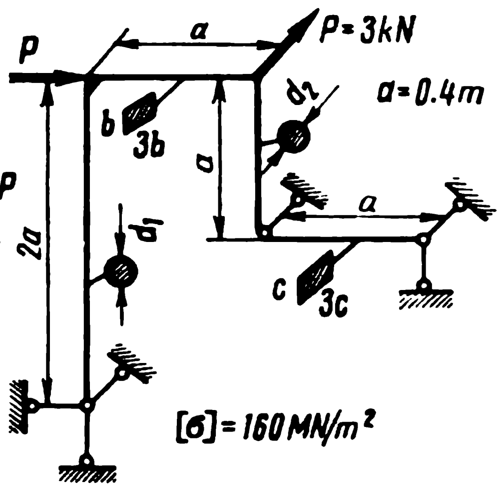


9.6.

Helical Extension or Compression Springs

A *helical spring* is a prismatic rod wound on a round cylinder of constant radius (Fig. 142).

Let us consider a spring made of a round rod of diameter d. We shall denote the mean diameter of the turn (coil) by D and the number of coils by n. The pitch of the spring depends on the angle α of inclination of the plane (xz) of a coil to the horizontal plane.

The x-axis is tangent to the central line of the coil and the y-axis is perpendicular to plane xz.

If the ends of the rod are bent into a line with the centres of the coils and a tensile force P is applied to these ends, then the internal forces in each cross section of the rod comprise a constant tensile

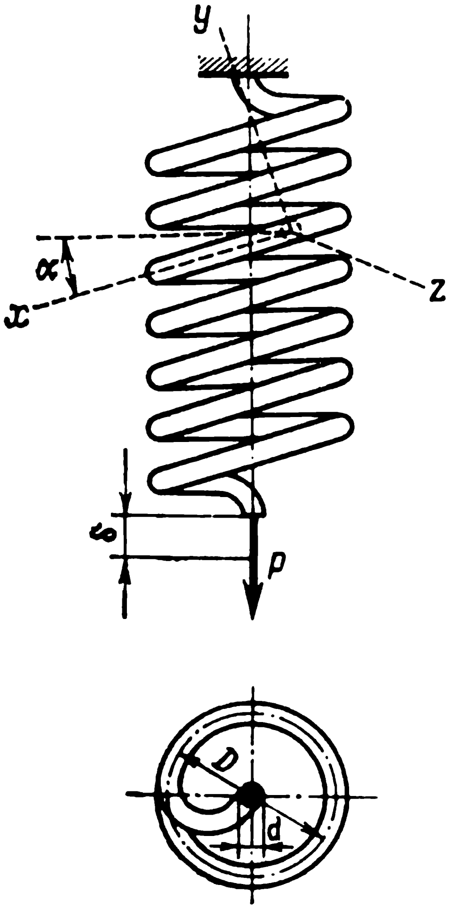


Fig. 142

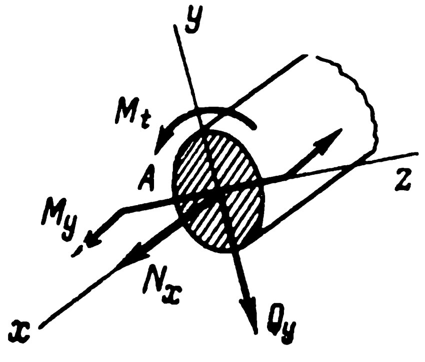


Fig. 143

force $N_x = P \sin\alpha$, a transverse shearing force $Q_y = P\cos\alpha$, torque $M_t = P\frac{D}{2}\cos\alpha$ and bending moment $M_y = P\frac{D}{2}\sin\alpha$ (Fig. 143).

The dangerous point is point A on the internal surface of the coil. At this point

$$\sigma = \frac{N_x}{F} + \frac{M_y}{W} = \frac{16PD}{\pi d^3}\left(1 + \frac{d}{4D}\right)\sin\alpha \tag{155}$$

and

$$\tau = \frac{Q_y}{F} + \frac{M_t}{W_p} = \frac{8PD}{\pi d^3}\left(1 + \frac{d}{2D}\right)\cos\alpha \tag{156}$$

In accordance with the third strength theory the design conditions can be written as follows

$$\sqrt{\sigma^2 + 4\tau^2} = \frac{16PD}{\pi d^3}\sqrt{\left(1 + \frac{d}{4D}\right)^2 \sin^2\alpha + \left(1 + \frac{d}{2D}\right)^2 \cos^2\alpha} \leqslant [\sigma] \tag{157}$$

If the spring pitch is small ($\alpha < 14°$) and $\frac{D}{d} > 10$, only torsional stress is taken into account, using the formula

$$\frac{8PD}{\pi d^3} \leqslant [\tau] \tag{158}$$

With a large curvature of the coil $\left(\frac{D}{d} < 10\right)$ the effect of the curvature and of the force factors can be conveniently accounted for by using the coefficient

$$k = \frac{\frac{D}{d} - 0.25}{\frac{D}{d} - 1} + \frac{0.615}{\frac{D}{d}} \tag{159}$$

which is included into the strength condition. Thus

$$k\frac{8PD}{\pi d^3} \leqslant [\tau] \tag{160}$$

The general expression for determining the axial displacement δ of the free end of the spring is of the form

$$\delta = \frac{8PD^3 n}{d^4 \cos\alpha}\left[2\left(1 + \frac{d^2}{4D^2}\right)\frac{\sin^2\alpha}{E} + \left(1 + \frac{d^2}{2D^2}\right)\frac{\cos^2\alpha}{G}\right] \tag{161}$$

in which E and G are Young's and the shear moduli of elasticity of the spring material.

For small-pitch springs the displacement δ can be sufficiently accurately determined on the basis of only the torsional deformation, using the formula

$$\delta = \frac{8PD^3 n}{Gd^4} \tag{162}$$

Example 76. Let $D = 80$ mm; $d = 20$ mm; $\alpha = 15°$; $n = 10$ co
$E = 2 \times 10^6$ kgf/cm^2; $G = 8 \times 10^5$ kgf/cm^2 and $[\sigma] = 6000$ kgf/c
(Fig. 144).
Find P and δ.
Solution. Since $\alpha > 14°$, we shall make use of formula (157) t
which the permissible force is

$$P \leqslant \frac{\pi \times 2^3 \times 6000}{16 \times 8 \sqrt{\left(1+\frac{2}{4\times 8}\right)^2 0.067+\left(1+\frac{2}{2\times 8}\right) 0.933}} \cong 1050 \text{ k}$$

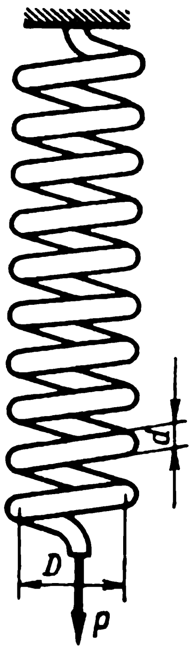


Fig. 144

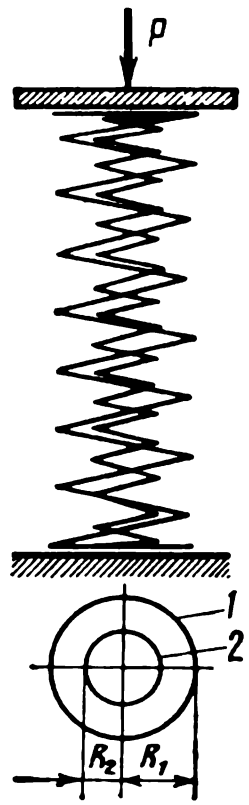


Fig. 145

The axial displacement δ of the free end of the spring can be found by formula (161). Substituting the given numerical values, we obtain

$$\delta = \frac{8 \times 1050 \times 8^3 \times 10}{2^4 \times 0.966} \left[2\left(1+\frac{2^2}{4\times 8^2}\right)\frac{0.067}{2\times 10^6} + \left(1+\frac{2^2}{2\times 8^2}\right)\frac{0.933}{8\times 10^5}\right] \cong 3.54 \text{ cm}$$

If the approximate formula (162) is resorted to we obtain

$$\delta = \frac{8 \times 1050 \times 8^3 \times 10}{8 \times 10^5 \times 2^4} = 3.36 \text{ cm}$$

This is approximately 5% less than the actual displacement.

Example 77. Two helical springs of small pitch are concentrically inserted one into the other (Fig. 145). They are subject to an axial

compressive force $P = 430$ kgf. For the outer spring 1: $D_1 = 160$ mm; $d_1 = 16$ mm and $n_1 = 8$ coils. For the inner spring 2: $D_2 = 80$ mm; $d_2 = 12$ mm and $n_2 = 12$ coils. The shear modulus of the spring material is $G = 8 \times 10^5$ kgf/cm^2.

Determine the deflection (shortening) δ and maximum shearing stresses τ_{max_1} and τ_{max_2}.

Solution. We shall denote the forces acting on the springs by P_1 and P_2. Then, from the equation of equilibrium $P_1 + P_2 = P$.

The condition of combined displacements is expressed in the equality of the deflection of the first (δ_1) and second (δ_2) springs, i.e. $\delta_1 = \delta_2$.

Taking into account the fact that both springs are of small pitch, we use formula (162) for calculating the deflection, rewriting it as follows:

$$\frac{8P_1D_1^3n_1}{Gd_1^4} = \frac{8P_2D_2^3n_2}{Gd_2^4}$$

from which

$$P_2 = P_1\frac{D_1^3n_1d_2^4}{D_2^3n_2d_1^4} = P_1\frac{16^3\times 8\times 1.2^4}{8^3\times 12\times 1.6^4} = \frac{27}{16}P_1$$

Hence

$$P_1 = \frac{16}{43}P = 160 \text{ kgf} \quad \text{and} \quad P_2 = \frac{27}{43}P = 270 \text{ kgf}$$

The deflection of the springs equals

$$\delta = \frac{8P_1D_1^3n_1}{Gd_1^4} = \frac{8P_2D_2^3n_2}{Gd_2^4} = \frac{8\times 160\times 16^3\times 8}{8\times 10^5\times 1.6^4} = 8 \text{ cm}$$

Since the outer spring has a small pitch and $\frac{D_1}{d_1} = 10$, using formula (158) we find

$$\tau_{max_1} = \frac{8\times 160\times 16}{\pi\times 1.6^3} \cong 1590 \text{ kgf/cm}^2$$

The inner spring also has a small pitch but for it $\frac{D_2}{d_2} = \frac{8}{1.2} \cong$ $\cong 6.67 < 10$, therefore τ_{max_2} is found by formula (160).

Since the correction factor is

$$k = \frac{6.67-0.25}{6.67-1} + \frac{0.615}{6.67} \cong 1.224$$

then

$$\tau_{max_2} = 1.224\,\frac{8\times 270\times 8}{\pi\times 1.2^3} \cong 3900 \text{ kgf/cm}^2$$

Problem 699. A spring is compressed until the clearance between the coils disappears. Determine the force P required and shearing stress τ_{max} developed if $D = 50$ mm, pitch $t = 15$ mm, $n = 10$, the side of the square section of the spring rod $a = 5$ mm and $G = 8 \times 10^4$ MN/m^2.

Problem 700. Two springs are compressed by equal forces P. The first spring is of round and the second of square cross section, with a side of the square equal to a.

Determine the ratios $\frac{\tau_{max_1}}{\tau_{max_2}}$ and $\frac{\delta_1}{\delta_2}$ if $D_1 = D_2$; $n_1 = n_2$; $\frac{\pi d_1^2}{4} = a^2$ and $G_1 = G_2$.

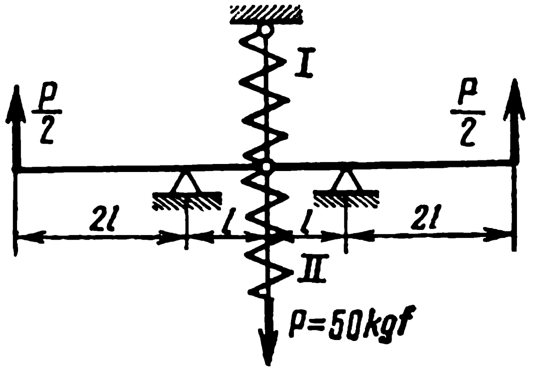
Problem 701. Let $D_1 = D_2 = 200$ mm; $d_1 = 2d_2 = 20$ mm; $n_1=8$; $n_2 = 5$; $G_1 = G_2 = 8 \times 10^5$ kgf/cm² and $l \gg \delta_1$.

Determine $\max \tau_1$, $\max \tau_2$ and $\delta = \delta_1 + \delta_2$.

Problem 702. A safety valve should open at a steam pressure $p = 5$ atm. Find τ_{max}, n and δ_0 for the spring, if $D_0 = 80$ mm; $D = 60$ mm; $d = 10$ mm; $t = 18$ mm; $G = 8 \times 10^5$ kgf/cm² and the deflection of the spring in the closed position (when the coils come into contact) is 40 mm.

It is assumed that at the maximum rise of the valve (with pressure increasing as the valve opens) there is a reserve of 20 mm to the closed position.

701



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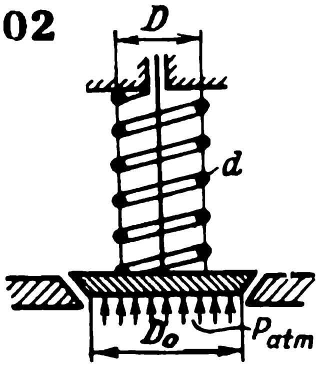
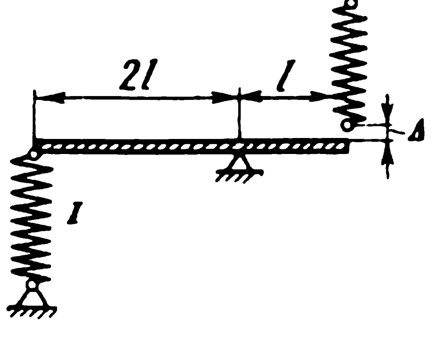


Problem 703. Determine the assembly stresses in the springs, if $\Delta = 5$ mm, $D_1 = 60$ mm, $d_1 = 10$ mm, $n_1 = 10$, $D_2 = 50$ mm, $d_2 = 8$ mm, $n_2 = 8$ and $G_1 = G_2 = 8 \times 10^5$ kgf/cm².

Problem 704. Determine the permissible force P, if for the bar ABC: $l = 500$ mm, $b = 750$ mm, $d_0 = 30$ mm, $[\sigma] = 1600$ kgf/cm², the design formula is to use the third strength theory; for the spring: $D = 50$ mm, $d = 10$ mm, $n = 10$, $G = 8 \times 10^5$ kgf/cm², $E = 2 \times 10^6$ kgf/cm² and $[\tau] = 4000$ kgf/cm².

703



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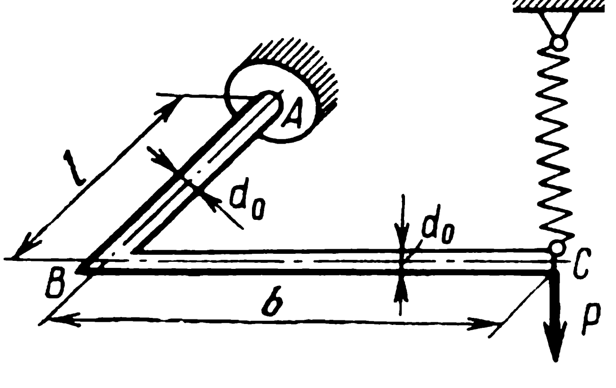


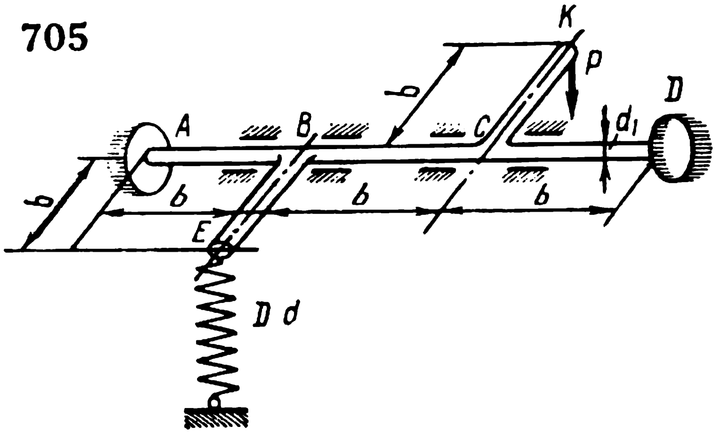
Problem 705. Let $D = 50$ mm, $d = 10$ mm, $n = 2$, $G = 8 \times 10^5$ kgf/cm², $d_1 = 50$ mm, $b = 500$ mm and $P = 100$ kgf. Bar *BE* is assumed to be perfectly rigid.

Determine by how many per cent the spring reduces the stress in shaft *AD*.

Problem 706. Let $P = 1.6$ kN, $D = 60$ mm, $d = 12$ mm, $n = 15$, $G = 8 \times 10^4$ M/m², $l = 500$ mm, $b = 50$ mm and $E = 2 \times 10^5$ MN/m².

Determine the force P of preliminary compression of the spring required for the stresses in the frame to be reduced by one half (after fitting the spring into the frame).

705



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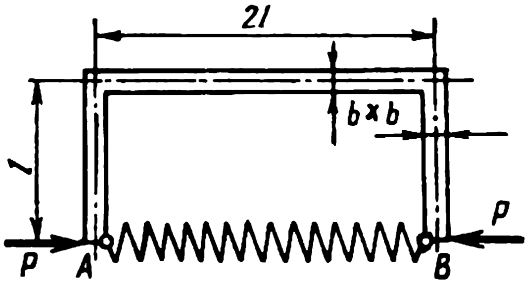
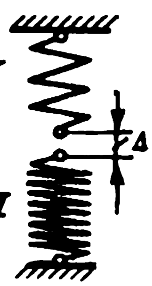


Problem 707. Let $P = 105$ kgf, $D = 100$ mm, $t = 182$ mm, $[\sigma] = 4800$ kgf/cm², $E = 2 \times 10^6$ kgf/cm², $G = 8 \times 10^5$ kgf/cm² and the allowable deflection (shortening) is $|\delta| = 100$ mm.

Determine d and n, using the fourth strength theory.

709



Problem 708. Let $D = 100$ mm, $a = 10$ mm, $t = 182$ mm and $[\sigma] = 4800$ kgf/cm², in which a is a side of the square section of the spring.

Determine P using the third strength theory.

Problem 709. Determine the axial force P in the springs after joining them together, if $\Delta = 60$ mm; spring I has a large pitch and for it $D_1 = 100$ mm, $d_1 = 10$ mm, $n_1 = 4$, $t_1 = 182$ mm; spring II has a small pitch and for it $D_2 = 80$ mm, $d_2 = 8$ mm, $n_2 = 8$, $E_1 = E_2 = 2 \times 10^6$ kgf/cm² and $G_1 = G_2 = 8 \times 10^5$ kgf/cm².

CHAPTER 10. LATERAL DEFLECTION AND BUCKLING OF COLUMNS

10.1.

Critical Force and Critical Stress

The critical force P_{cr} for a prismatic bar, or column, in axial compression within the proportional limit is determined by Euler's formula:

$$P_{cr}=\frac{\pi^2 EI}{l_{red}^2} \tag{163}$$

in which E = Young's modulus (modulus of elasticity) of the bar material

I = minimum moment of inertia of the cross-sectional area (gross) F of the bar

$l_{red} = \mu l$ = reduced (free) length of the bar (length at which a bar with hinged ends is equivalent in stability to the bar with the given end conditions)

l = true length of the bar

μ = length reduction factor depending on the end conditions and type of loading.

Figure 146 illustrates the simplest cases of compression of bars and the corresponding values of factor μ.

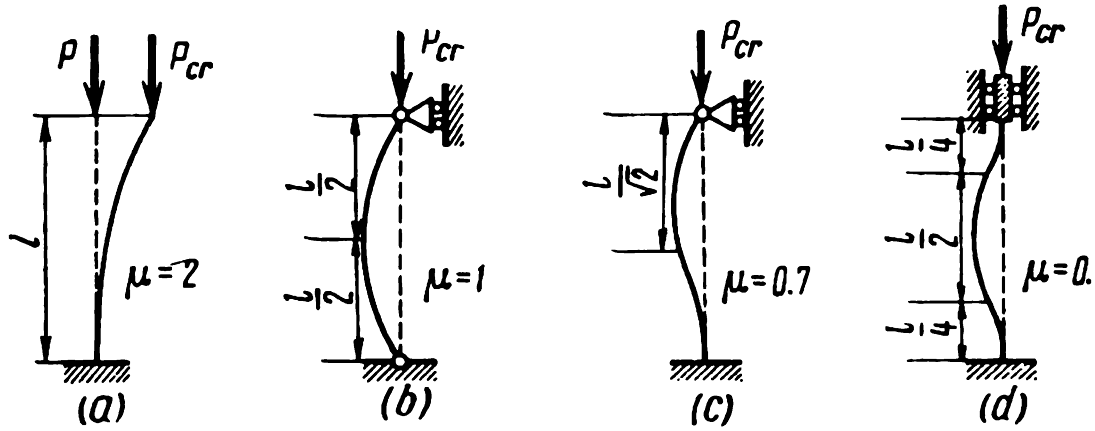


Fig. 146

The critical stress σ_{cr} is found from the formula:

$$\sigma_{cr}=\frac{P_{cr}}{F}=\frac{\pi^2 E}{\lambda^2} \tag{164}$$

in which $\lambda = \frac{l_{red}}{i}$ denotes the slenderness ratio of the bar (an abstract value, characterizing the tendency of the bar to buckle); $i=\sqrt{\frac{I}{F}}$ is the minimum radius of inertia of area F of the bar cross section.

Since the critical stress σ_{cr} must not exceed the proportional limit σ_p for the bar material, the slenderness ratio λ at which formulas (163) and (164) can be applied is determined from the inequality

$$\lambda \geqslant \pi \sqrt{\frac{E}{\sigma_p}} \tag{165}$$

For example, for steel grade Ст. 3 (USSR Std) $\lambda \geqslant 100$; for steel grade Ст. 5 $\lambda \geqslant 85$; for cast iron $\lambda \geqslant 80$; for wood $\lambda \geqslant 70$, etc.

If buckling occurs in a bar only beyond the proportional limit of its material, the critical stress can be calculated by Yasinsky's empirical formula

$$\sigma_{cr} = a - b\lambda + c\lambda^2 \tag{166}$$

in which a, b and c are empirical factors, depending on the material and having the dimensions of stress.

For steel Ст. 3: $a = 3100$ kgf/cm², $b = 11.4$ kgf/cm², $c = 0$.

For steel Ст. 5: $a = 4640$ kgf/cm², $b = 36.17$ kgf/cm², $c = 0$.

For cast iron: $a = 7760$ kgf/cm², $b = 120$ kgf/cm², $c = 0.53$ kgf/cm².

For wood: $a = 293$ kgf/cm², $b = 1.94$ kgf/cm², $c = 0$.

Formula (166) can be applied if σ_{cr} is less than σ_y for a ductile material and less than σ_u for a brittle material.

Example 78. Determine P_{cr} and σ_{cr} for a duralumin bar with $E = 0.71 \times 10^5$ MN/m², $\sigma_p = 180$ MN/m², $l = 1.2$ m, $D = 4$ cm and $d = 3$ cm (Fig. 147).

Solution. The moment of inertia of an annular section of the bar is

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(256 - 81) = \frac{175}{64}\pi \text{ cm}^4$$

The cross-sectional area of the bar is

$$F = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}(16 - 9) = \frac{7}{4}\pi \text{ cm}^2$$

and the radius of inertia of the cross section is

$$i = \sqrt{\frac{I}{F}} = \sqrt{\frac{175 \times 4}{64 \times 7}} = \frac{5}{4} \text{ cm}$$

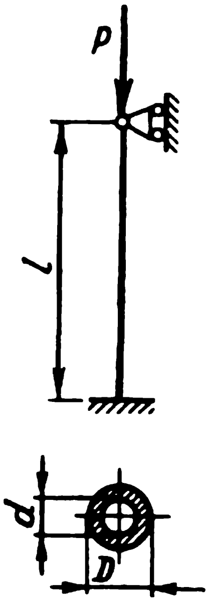


Fig. 147

With the given bar end conditions the length reduction factor is $\mu = 0.7$.

Since the slenderness ratio of the bar is

$$\lambda = \frac{\mu l}{i} = \frac{0.7 \times 120 \times 4}{5} = 67.2 > \pi \sqrt{\frac{E}{\sigma_p}} \cong 3.14 \sqrt{\frac{0.71 \times 10^5}{180}} \cong 62$$

the critical force can be found by Euler's formula

$$P_{cr} = \frac{\pi^2 EI}{(\mu l)^2} = \frac{\pi^2 \times 0.71 \times 10^5 \times 10^6 \times 175\pi \times 10^{-8}}{64\,(0.7 \times 1.2)^2}$$
$$= 85.3 \times 10^3 \text{ N} = 85.3 \text{ kN}$$

The critical stress will be

$$\sigma_{cr}=\frac{P_{cr}}{F}=\frac{85.3\times10^3\times4\times10^6}{7\pi}=155\times10^6\ \text{N/m}^2=155\ \text{MN/m}^2$$

Example 79. Determine P_{cr} and σ_{cr} for a cast iron bar with $l=1.6$ m, $d=6$ cm and $t=1$ cm (Fig. 148).

Solution. The moment of inertia for the cross-shaped section of the bar is

$$I=\frac{ta^3}{12}+\frac{(a-t)\,t^3}{12}=\frac{1\times6^3}{12}+\frac{5\times1}{12}=\frac{221}{12}\ \text{cm}^4$$

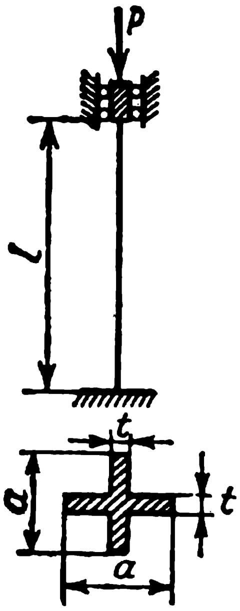


Fig. 148

The cross-sectional area of the bar is

$$F=at+(a-t)\,t=6+5=11\ \text{cm}^2$$

and the radius of inertia of the cross section is

$$i=\sqrt{\frac{221}{12\times11}}\cong1.294\ \text{cm}$$

With the given end conditions $\mu=0.5$.

Since the slenderness ratio of the bar $\lambda=\frac{\mu l}{i}=\frac{160}{2\times1\times294}\cong61.8<80$ the critical stress should be calculated by the empirical formula

$$\sigma_{cr}=7760-120\lambda+0.53\lambda^2=7760-120\times61.8+0.53\times61.8^2\cong2370\ \text{kgf/cm}^2$$

The critical force is

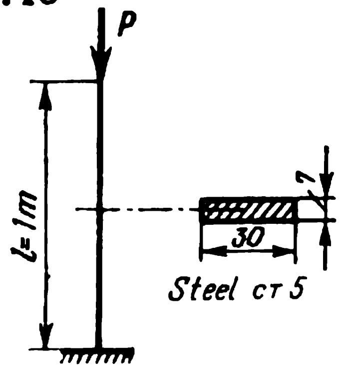
$$P_{cr}=\sigma_{cr}F=2370\times11=26{,}100\ \text{kgf}$$

Problems 710 through 715. Determine the critical forces P_{cr} and critical stresses σ_{cr} for the compressed bars.

Take the following rounded-off values for the moduli of elasticity E (Young's moduli) and proportional limits σ_p of the materials:

Material	Ст. 3	Ст. 5	Cast iron	Duralumin	Wood (pine)
E, kgf/cm²	2×10^6	2×10^6	1.2×10^6	0.7×10^6	1×10^5
σ_p, kgf/cm²	2000	2400	1800	1700	200

710



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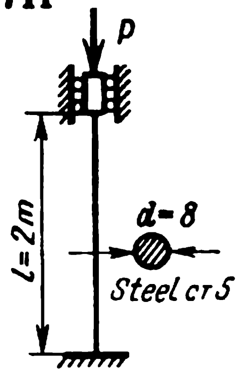

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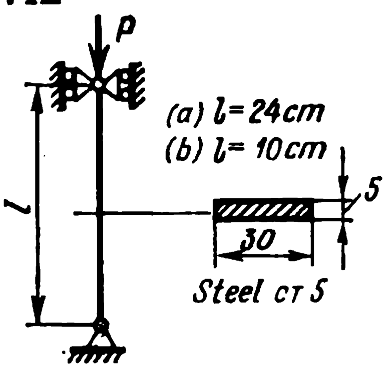

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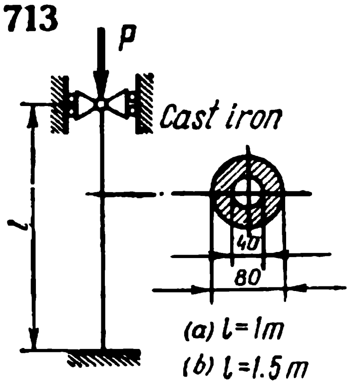

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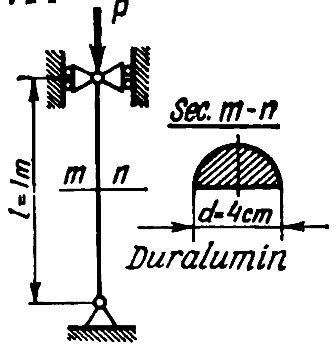

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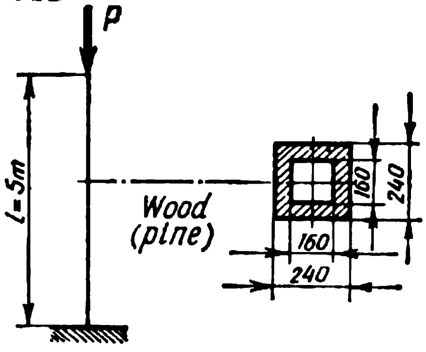


10.2.

Stability Calculations for Compressed Bars (Columns)

In designing compressed bars (columns) stability conditions shouid be complied with in addition to strength conditions, i.e.:

$$P \leqslant \frac{P_{cr}}{[n_{st}]} \tag{167}$$

or

$$\sigma = \frac{P}{F} \leqslant [\sigma_{st}] \tag{167'}$$

Here P_{cr} is the critical force determined, depending on the slenderness ratio of the material, either by Euler's formula (163) or Yasinsky's formula (166), i.e. by the expression $P_{cr} = \sigma_{cr}F = (a - b\lambda + c\lambda^2)\,F$; $[\sigma_{st}] = \frac{[\sigma_{cr}]}{[n_{st}]}$ is the allowable stress ensuring stability;

$[n_{st}]$ is the stability factor of safety. This factor is always somewhat greater than the principal (ultimate) factor of safety, since in designing centrally compressed bars (columns), for additional safety against buckling, certain practically inevitable factors should be taken into account (eccentricity of application of the compressive forces, initial curvature of the bar and nonhomogeneity of its material) which may contribute to buckling.

A large eccentricity and initial curvature are specially provided for in design or analysis; whereas small ones, which cannot be accounted for and depend on the slenderness ratio of the bar, are taken into consideration by additionally increasing the factor of safety, i.e. the stability factor of safety $[n_{st}]$ which is 1.8 to 3 for steel, 5 to 5.5 for cast iron, and 2.8 to 3.2 for wood.

The ratio $\frac{[\sigma_{st}]}{[\sigma_c]} = \varphi$ is the *allowable-stress reduction factor*.

Values of φ versus λ are given for different materials, either in the form of standard curves or tables (Appendix 3).

Stability analysis can be performed by two methods:

1. With a given stability factor of safety $[n_{st}]$.
2. With the aid of allowable-stress reduction factor tables for φ (λ).

The first method is not sufficiently accurate, because, strictly speaking, the stability factor of safety $[n_{st}]$ cannot be reliably assumed in advance, since it depends on the slenderness ratio λ of the bar. Therefore, this method is used only for a rough analysis, as well as in cases when the required tables and graphs for φ (λ) are not available (for example for new materials or if bars are used that have a slenderness ratio above the recommended values).

The second method is in general use, being the principal one for analysing the stability of bars. In this case, use is made of equation

$$\sigma = \frac{P}{F} \leqslant \varphi [\sigma_c] \tag{167''}$$

and tables of factor φ. Here neither Euler's formula nor empirical formulas of Yasinsky are required.

DETERMINING THE PERMISSIBLE FORCE

If length l and the end conditions (μ), as well as the shape and size of the cross section (F, I and i) and the material (E, $[\sigma_c]$) are known for the bar being designed, the permissible compressive force $[P]$ is calculated by the use of one of the following methods.

First method (with a given $[n_{st}]$). (1) Determine the slenderness ratio of the bar $\lambda = \frac{\mu l}{i}$. (2) Find the critical force P_{cr} ($_cP_r = \sigma_{cr}F$) by Euler's formula (163), or by Yasinsky's formula (166) if condition (165) is not complied with. (3) Using formula (167) find the allowable force $[P] = \frac{P_{cr}}{[n_{st}]}$.

We should note that if $[n_{st}]$ is not given, it can be taken approximately in accordance with the material of the bar, its purpose in the structure and its slenderness ratio λ.

Second method (with the use of tables for $\varphi(\lambda)$). (1) Find $\lambda = \frac{\mu l}{i}$. (2) With the aid of either a standard curve or tables for $\varphi(\lambda)$ and by interpolation if necessary, find the allowable-stress reduction factor φ. (3) Find the permissible compressive force $[P] = [\sigma_{st}]\,F = \varphi\,[\sigma_c]\,F$.

CHECKING THE STABILITY OF BARS (COLUMNS)

The procedure for stability analysis also has two versions, similar to the previous procedure (i.e. to the ones for determining the permissible force), and is based on equations (167′) and (167″).

SELECTING CROSS SECTION

If the compressive force P, length l, the end conditions (μ), material (E, $[\sigma_c]$) and the shape of the cross section are known for the bar being designed, the stability condition (167″) is indeterminate since it is impossible to find λ and hence φ without knowing the cross section dimensions. It is also impossible (with a given stability factor of safety $[n_{st}]$ to make use of equation (167), since we do not know which formula, (163) or (166), should be applied in determining P_{cr}. The section is selected by the trial-and-error method with subsequent checking (using one of three versions).

The first bar design version (with a given $[n_{st}]$) is rarely used because it is insufficiently accurate due to the fact that the value $[n_{st}]$ is assigned irrespective of value λ (which is yet unknown). In this case calculations begin from the tentative supposition that Euler's formula (163) is applicable.

First we find the moment of inertia I, then F, i and λ. If λ is greater than λ_0, the calculation is considered to be completed, if it is less, new calculations are based on Yasinsky's formula (166).

The second bar design version [with the use of tables and graphs for $\varphi(\lambda)$] is carried out in the following order:

1. Assume that factor $\varphi = 0.6$ to 0.8.
2. Determine $[\sigma_{st}]$, $F = \frac{P}{[\sigma_{st}]}$ and then select the size of the cross section or the section size No. (if the bar is a standard rolled steel shape).
3. Find I, i and λ.
4. Determine the new value of φ_1. If φ_1 differs substantially from φ, then, as a second trial, assume that $\varphi_2 = \frac{1}{2}(\varphi + \varphi_1)$ and repeat the above calculations.

The section is considered to be satisfactory, if σ and $[\sigma_{st}]$ differ by not more than 5%.

For standard rolled steel bars the understress may exceed 5%.

The third (combined) *bar design version.* In this method the first trial is made using Euler's formula and assigning an approximate factor n_{st}. The final selection is based on the stability condition (167″).

The procedure is as follows:

1. Assign factor n_{st} in accordance with the material of the bar ($n_{st} \cong 2$ for steel, $n_{st} = 5$ for cast iron and $n_{st} = 3$ for wood).
2. Find the minimum moment of inertia for the cross section by formula (163).
3. Select the cross section or find the section size No. (for rolled steel), as well as F, i and λ.
4. Determine the factor φ and then $[\sigma_{st}]$.
5. Formulate the stability condition equation (167″).
6. If condition (167″) is not complied with, continue the selection, either varying factor φ (see the second design version) or varying the dimensions of the cross section (changing the section size No.).

In practical stability design it is not advisable to take bars whose slenderness ratio exceeds the maximum values given in the standards for factor φ. If however, it is necessary to determine the permissible force or to select a section for bars with a slenderness ratio beyond those recommended in the standards, calculations should be based on Euler's formula, selecting an appropriate stability factor of safety.

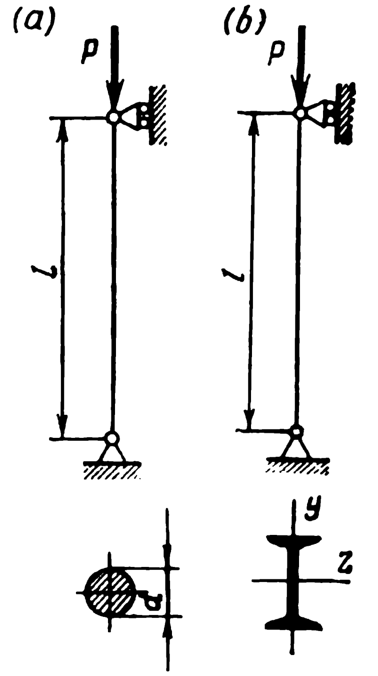


Fig. 149

Example 80. Find P for a wooden (pine) strut (with the grain), if $[\sigma] = 100$ kgf/cm², $l = 2$ m, $\mu = 1$ and $d = 10$ cm (Fig. 149*a*).

Solution. For a round strut

$$i = \frac{d}{4} = \frac{10}{4} = 2.5 \text{ cm}$$

The slenderness ratio is

$$\lambda = \frac{l}{i} = \frac{200}{2.5} = 80$$

For wood with $\lambda = 80$, $\varphi = 0.48$ (tabulated data).

Therefore $[\sigma_{st}] = \varphi\,[\sigma] = 0.48 \times 100 = 48$ kgf/cm² and the permissible force

$$P = [\sigma_{st}]\,F = 48\,\frac{\pi \times 10^2}{4} = 3770 \text{ kgf}$$

Next we find the stability factor of safety for the strut.

Since $\lambda = 80 > 70$, formula (164) will give

$$\sigma_{cr} = \frac{10 \times 1 \times 10^5}{80^2} = 156 \text{ kgf/cm}^2 \quad \text{and} \quad n_{st} = \frac{\sigma_{cr}}{[\sigma_{st}]} = \frac{156}{48} = 3.25$$

Example 81. Find the size of the I-beam to use for a rolled steel strut with $[\sigma] = 1600$ kgf/cm², $P = 40$ tnf and $l = 2$ m (Fig. 149*b*).

Solution. Assume that $\varphi = 0.6$.

Then

$$[\sigma_{st}] = \varphi[\sigma] = 0.6 \times 1600 = 960 \text{ kgf/cm}^2$$

and

$$F = \frac{P}{[\sigma_{st}]} = \frac{40 \times 10^3}{960} \cong 41{,}7 \text{ cm}^2$$

The nearest I-beam section from the data for standard rolled steel shapes is No. 27 for which $F = 40.2$ cm² and $i_{st} = 2.54$ cm.

The slenderness ratio of the strut is $\lambda = \frac{\mu l}{i_{st}} = \frac{1 \times 200}{2.54} = 78.7$.

In the table for steel, grade Ст.3: $\varphi = 0.81$ for $\lambda = 70$ and $\varphi = 0.75$ for $\lambda = 80$, therefore, for $\lambda = 78.7$

$$\varphi_1 = 0.75 + 0.006 \times 1.3 = 0.758$$

We take

$$\varphi_2 = \frac{\varphi + \varphi_1}{2} = \frac{0.6 + 0.758}{2} = 0.679$$

Then

$$[\sigma_{st}] = 0.679 \times 1600 = 1086 \text{ kgf/cm}^2 \quad \text{and} \quad F = \frac{40 \times 10^3}{1086} \cong 36.8 \text{ cm}^2$$

From the data for rolled steel shapes the nearest I-beam section is No. 24 for which $F = 34.8$ cm² and $i_{st} = 2.37$ cm.

The slenderness ratio of the strut is $\lambda = \frac{200}{2.37} \cong 84.5$.

In the table for steel, grade Ст. 3: $\varphi = 0.75$ for $\lambda = 80$ and $\varphi = 0.69$ for $\lambda = 90$. For $\lambda = 84.5$

$$\varphi_3 = 0.69 + 0.006 \times 5.5 = 0.723$$

The permissible stress is

$$[\sigma_{st}] = 0.723 \times 1600 = 1157 \text{ kgf/cm}^2$$

The actual stress in the strut is

$$\sigma = \frac{40 \times 10^3}{34.8} \cong 1150 \text{ kgf/cm}^2$$

and the overstress is

$$\frac{[\sigma] - [\sigma_{st}]}{[\sigma_{st}]} \times 100 = \frac{70 \times 100}{1157} \cong 0.8\%$$

Now we can find the stability factor of safety for the designed strut. Since $\lambda = 84.5 < 100$, then by formula (166)

$$\sigma_{cr} = 3100 - 11.4 \times 84.5 = 2137 \text{ kgf/cm}^2$$

$$n_{st} = \frac{2137}{1150} \cong 1.86$$

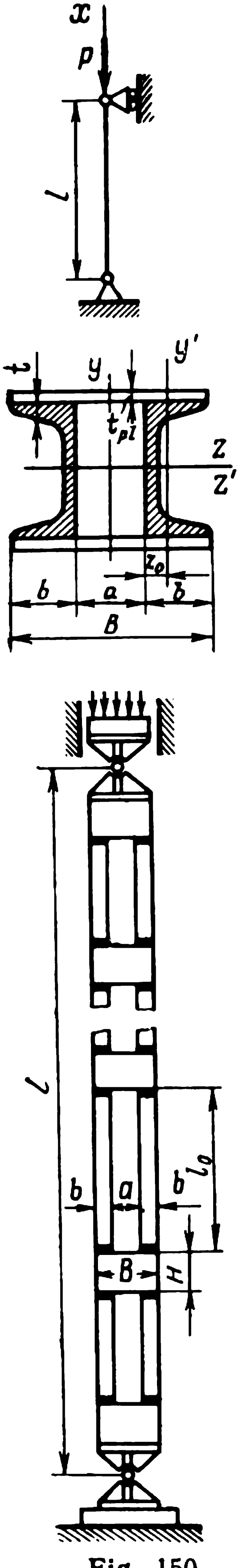


Fig. 150

Example 82. Determine the required size of the channel, B and l_0 for a built-up column welded of two channels with cross plates, if $P = 35$ tnf, $l = 6$ m and $[\sigma] = 1600$ kgf/cm² (Fig. 150).

Solution. With an increase in the distance a between the channels the moment of inertia I_y of a section of a column about the y-axis, normal to the plane of the plates, will also increase, whereas the moment of inertia I_z of a section about the z-axis will not change and will remain equal to $I_z = 2I_{z'}$ in which $I_{z'}$ is the moment of inertia of the section of one leg of the column about the central axis z' of the channel coinciding with axis z. Therefore I_z should be taken as the minimum moment of inertia of the column cross section.

We assign $n_{st} = 2$, then by Euler's formula

$$I_{z'} = \frac{Pl^2 n_{st}}{2\pi^2 E} = \frac{35\times 10^3 \times 36 \times 10^4 \times 2}{2 \times 10 \times 2 \times 10^6} = 630 \text{ cm}^4$$

The nearest smaller channel in the data for standard rolled steel shapes is No. 14a for which $I_{z'} = 545$ cm⁴, $i_{z'} = 5.66$ cm, $F = 17.0$ cm².

The slenderness ratio of the column is

$$\lambda = \frac{l}{iz'} = \frac{600}{5.66} \cong 106$$

From the table for steel, grade Ст. 3: $\varphi = 0.60$ for $\lambda = 100$ and $\varphi = 0.52$ for $\lambda = 110$.

For $\lambda = 106$

$$\varphi = 0.52 + 0.008 \times 4 = 0.552$$

The permissible stress is

$$[\sigma_{st}] = 0.552 \times 1600 = 883 \text{ kgf/cm}^2$$

The working stress is

$$\sigma = \frac{P}{2F} = \frac{35 \times 10^3}{2 \times 17.0} = 1029 \text{ kgf/cm}^2$$

The overstress is

$$\frac{1029 - 883}{1029} \times 100\% = 14.2\% > 5\%$$

which is inadmissible.

Next we try channel No. 16, for which: $I_{z'} = 747$ cm⁴, $i_{z'} = 6.42$ cm, $F = 18.1$ cm²,

$I_{y'} = 63.3$ cm^4, $i_{y'} = 1.87$ cm, $b = 6.4$ cm, $t = 0.84$ cm and $z_0 = 1.80$ cm.

The slenderness ratio of the column is

$$\lambda = \frac{600}{6.42} \cong 93.5$$

In the table for steel, grade Cт. 3: $\varphi = 0.69$ for $\lambda = 90$ and $\varphi = 0.60$ for $\lambda = 100$.

For $\lambda = 93.5$

$$\varphi = 0.60 + 0.009 \times 6.5 = 0.658$$

The permissible stress

$$[\sigma_{st}] = 0.658 \times 1600 \cong 1051 \text{ kgf/cm}^2$$

The working stress is

$$\sigma = \frac{P}{2F} = \frac{35 \times 10^3}{2 \times 18.1} \cong 967 \text{ kgf/cm}^2$$

The understress is

$$\frac{1051 - 967}{1051} \times 100\% = 7.99\%$$

Hence we use channel No. 16.

Since $\lambda = 93.5 < 100$, then

$$\sigma_{cr} = 3100 - 11.4\lambda \cong 2034 \text{ kgf/cm}^2$$

and the column will have stability factor of safety equal to

$$n_{st} = \frac{2034}{967} \cong 2.1$$

The rational distance a between the legs is established from the con dition of equal stability of the column in the principal planes of inertia xy and xz.

Since the legs of the column are not joined by perfectly rigid plates, it is advisable to take $I_y = (1.15 \text{ to } 1.2)\, I_z$.

Let us take $I_y = 1.15\, I_z$, then

$$1.15 I_{z'} = I_{y'} + \left(z_0 + \frac{a}{2}\right)^2 F;$$

$$a = 2\left(\sqrt{\frac{1.15 I_{z'} - I_{y'}}{F}} - z_0\right) = 2\left(\sqrt{\frac{1.15 \times 747 - 63.3}{18.1}} - 1.80\right)$$

$$= 9.66 \cong 10 \text{ cm}$$

and the sought-for length of the cross plate is

$$B = a + 2b = 10 + 2 \times 6.4 = 22.8 \text{ cm}$$

To provide sufficient stability of each separate leg in the plane of its minimum rigidity xz', the connection plates should be spaced, theoretically, at distances l_{sp} such that the slenderness ratio of the entire column is equal to that of the portion of a leg between adjacent plates, i.e.

$$\frac{l}{i_z}=\frac{l_{sp}}{i_{y'}}$$

From this condition, the theoretical length of the portion of the leg is

$$l_{sp}=\frac{l}{i_z}i_{y'}=\lambda i_{y'}=93.5\times 1.87=175\ \text{cm}$$

It was previously recommended in engineering practice to take $l_0=(1/2$ to $1/3)\ l_{sp}$. In accordance with new standards (see USSR Building Code, Part II, Section B, Chapter 3, 1962), the slenderness ratio λ_l of the separate legs in the portion between adjacent plates should not exceed 40, i.e. $l_0\leqslant 40i_{y'}$. In our case

$$l_0=40\times 1.87=74.8\ \text{cm}$$

We can take $l_0=75$ cm. Then the length of the column l will be divided by the plates into six equal portions and 24 cm will be left for the end supports.

From design considerations we can take

$$H=(0.6+0.8)\,B;\quad l_0-H=(3\text{ to }4)\,H\ \text{and}\ t_{pl}=(0.8\text{ to }1)\,t$$

For our case

$$H=0.8B=0.8\times 22.8=18\ \text{cm and}\ t_{pl}\cong t\cong 0.8\ \text{cm}$$

Usually a column with cross plates is checked to find its design slenderness ratio:

$$\lambda_d=\sqrt{\lambda_c^2+\lambda_l^2}$$

for a column made up of two channels or two I-beams and

$$\lambda_d=\sqrt{\lambda_c^2+\lambda_{l_1}^2+\lambda_{l_2}^2}$$

for a column made up of four angles; in which λ_c is the slenderness ratio of the column with respect to the axis perpendicular to the plane of the plates; λ_{l_1} and λ_{l_2} are the slenderness ratios of the legs with respect to their own centroidal axes perpendicular to the planes of the plates.

We shall check the slenderness ratio for our case: for the entire column

$$i_y=\sqrt{\frac{I_{y'}}{F}+\left(z_0+\frac{a}{2}\right)^2}=\sqrt{1.87^2+(1.8+5)^2}\cong 7.05\ \text{cm};$$

$$\lambda_c = \frac{600}{7.05} = 85.1$$

for a leg

$$\lambda_l = \frac{l_0}{i_{y'}} = \frac{75}{1.87} = 40$$

Therefore

$$\lambda_d = \sqrt{85.1^2 + 40^2} = 94.1$$

In the table for steel, grade Ст. 3 (after interpolation) find for $\lambda = 94.1$:

$$\varphi = 0.60 + 0.009 \times 4.1 = 0.637$$

The permissible stress

$$[\sigma_{st}] = 0.637 \times 1600 = 1019 \text{ kgf/cm}^2$$

The understress is

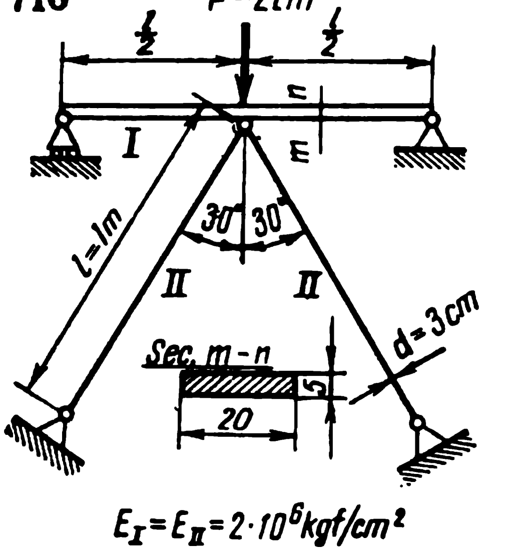
$$\frac{1019 - 967}{1019} \times 100\% = 5.1\%$$

Consequently the column is also stable with respect to the principal axis perpendicular to the plates.

It should be noted that if the conditions $I_y = 1.2 I_z$ and $\lambda_l = 40$ are complied with, values λ_d and λ will always be very close to each other.

Problems 716 and 717. Determine the stability factors of safety n_{st} for the compressed bars in the systems.

716



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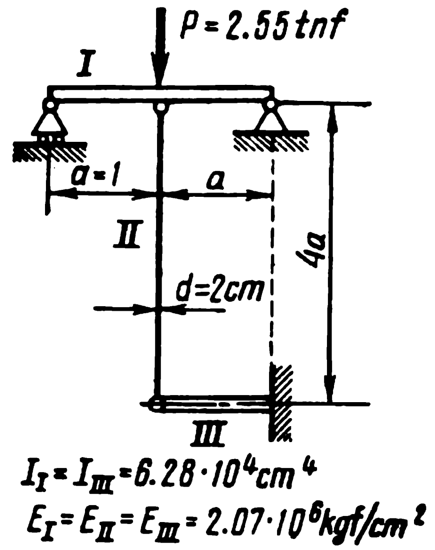
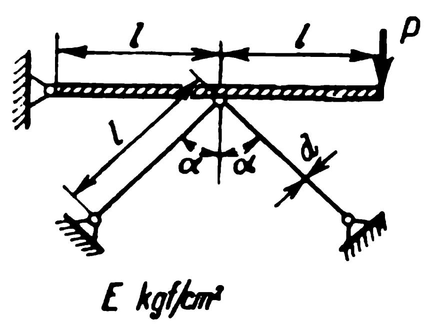


Problems 718, 719 and 720. Determine at what length l the systems will lose their stability.

Assume that for the given load and the given cross-sectional dimensions the strain is within the proportional limit. In Problem 720 the

compressive strain of bar *II* is to be neglected in calculations involving the statically indeterminate beam.

718



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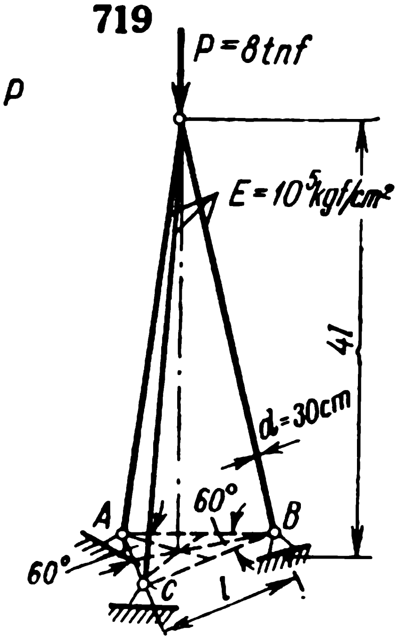

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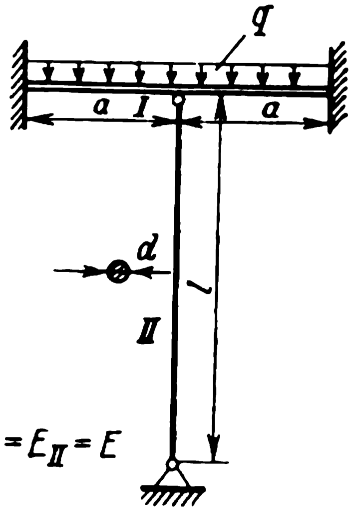


Problem 721. Determine length l at which a compressed strut made of round steel stock, grade Ст. 3, of diameter d with hinged ends will lose its stability.

Assume that: $E = 2 \times 10^6$ kgf/cm², $\sigma_p = 1900$ kgf/cm², $\sigma_y = 2400$ kgf/cm².

(a) $d = 1$ cm, $P = 1$ tnf; (b) $d = 1$ cm, $P = 1.8$ tnf.

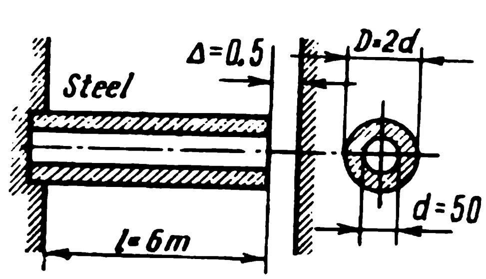
Problems 722, 723 and 724. Determine at what increase in temperature (Δt°C) the compressed members of the systems will lose their stability. Assume that: for steel: $E = 2 \times 10^6$ kgf/cm²; $\alpha = 12.5 \times 10^{-6}$ and

$$\sigma_p = 2000 \text{ kgf/cm}^2;$$

for copper: $E = 1 \times 10^6$ kgf/cm²; $\alpha = 16.5 \times 10^{-6}$ and

$$\sigma_p = 1000 \text{ kgf/cm}^2.$$

722



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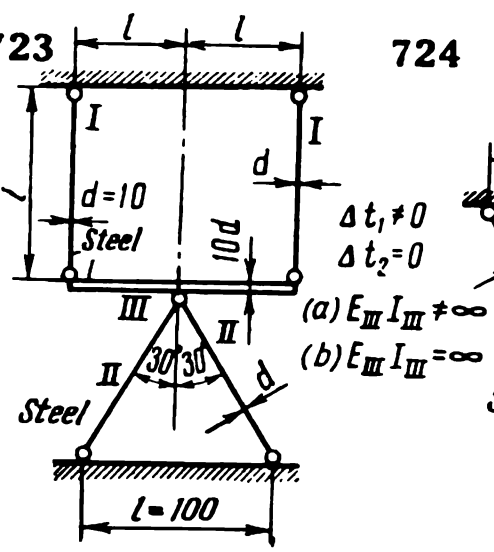

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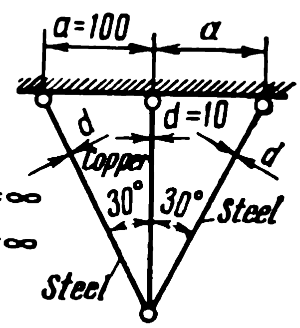
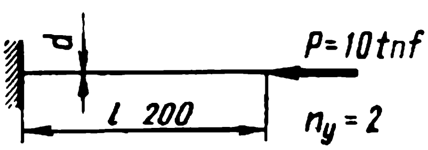


Problems 725 through 728. Select the cross sections of the compressed elements in the systems on the basis of a given stability factor of safety n_{st}.

All the compressed members are made of steel, grade Ст. 3, for which $E = 2 \times 10^6$ kgf/cm², $\sigma_p = 2000$ kgf/cm².

In Problem 726 the indicated position of the connecting rod is to be considered the most dangerous one.

725



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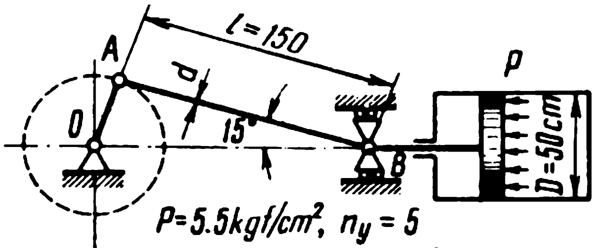

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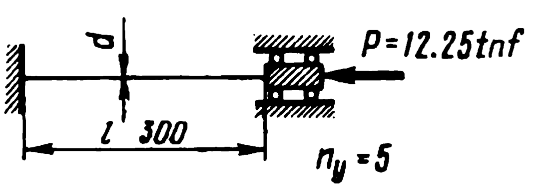

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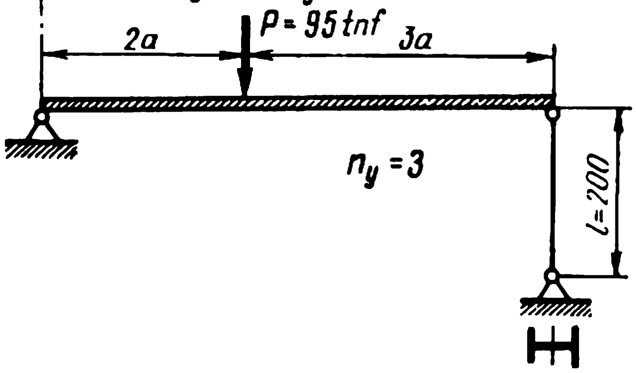
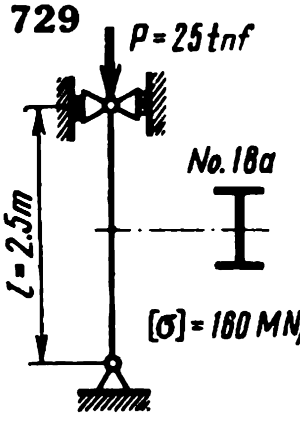


Problems 729 through 733. Check the stability of the compression members and find by how many per cent they are over- or understressed. Here and in the following problems the possible torsional strains of the compressed members are to be neglected.

729



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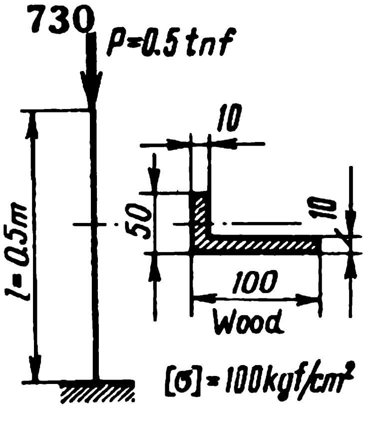

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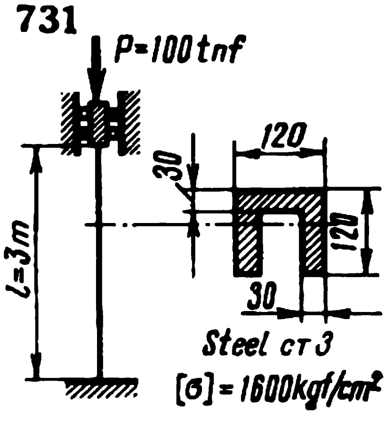

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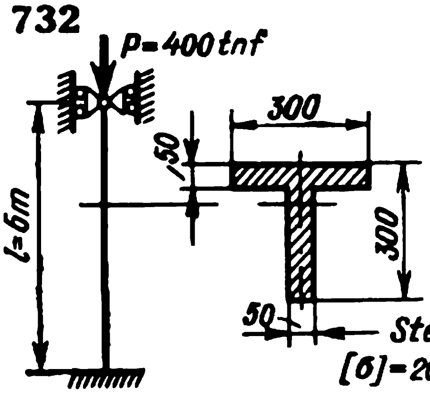

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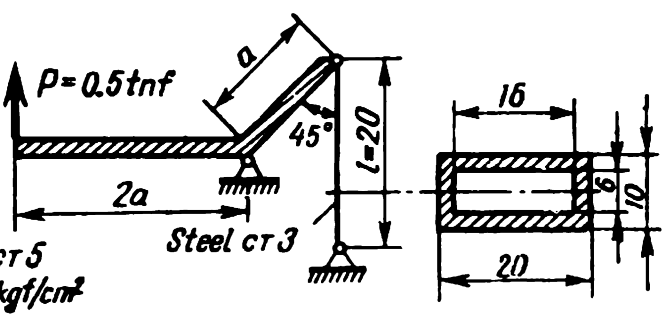
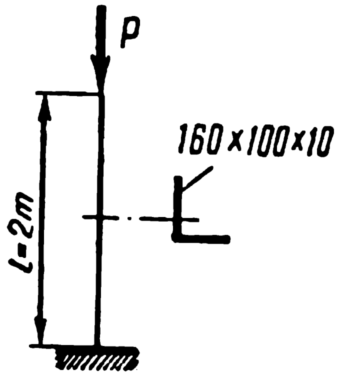


Problems 734, 735 and 736. Determine the load-carrying capacity of the columns. They are made of steel, grade Cт. 3 with $[\sigma] = 1600$ kgf/cm².

734



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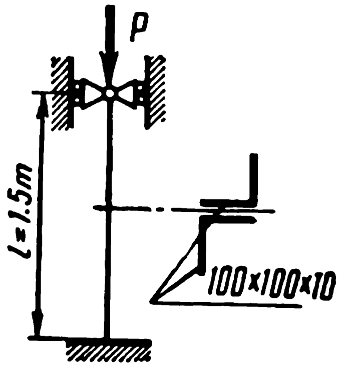

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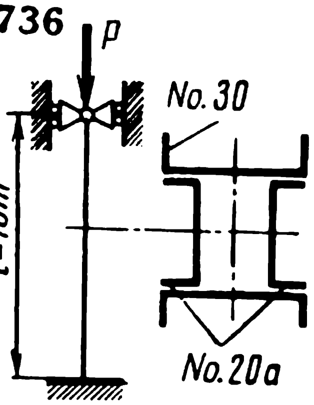
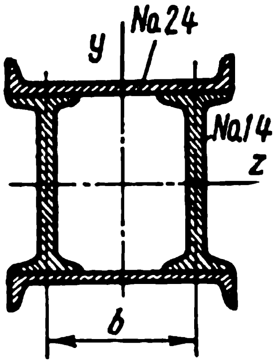


Problems 737, 738 and 739. Determine dimension b in the column sections from the condition of their equal stability with respect to axes z and y.

737



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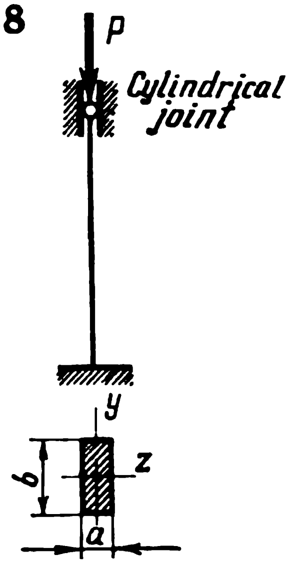

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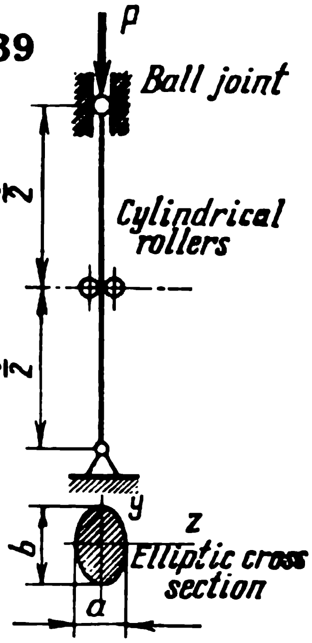
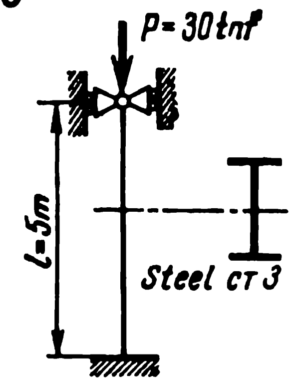


Problems 740 through 744. Determine the cross-sectional dimensions of the columns and compressed members of the systems.

740



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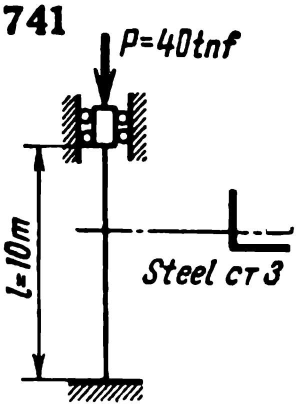

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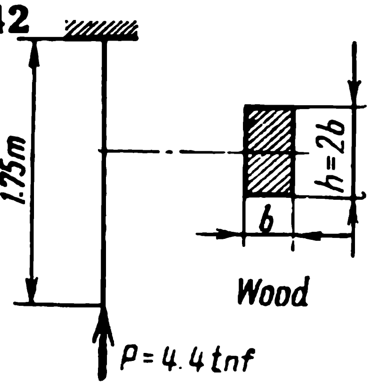

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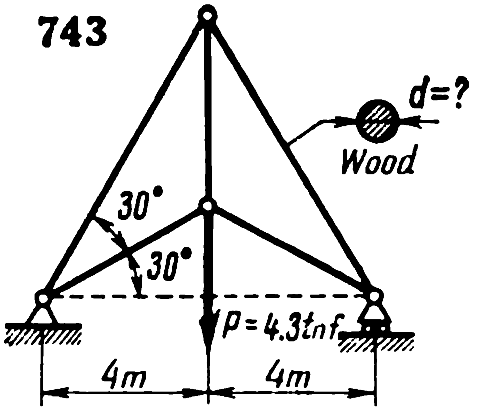

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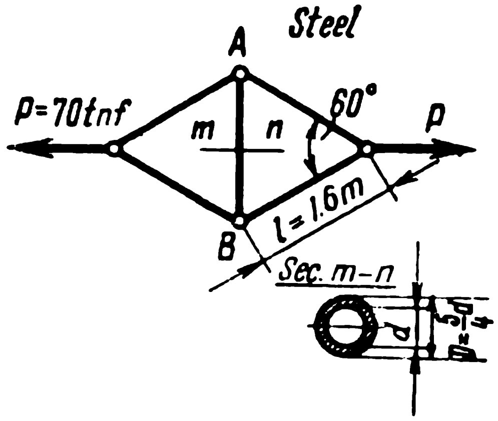
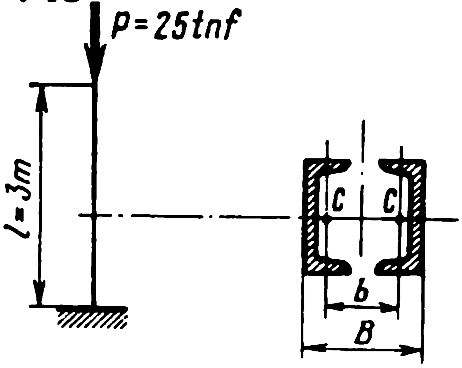


Assume that for steel, grade Ст. 3, $[\sigma] = 1600$ kgf/cm² and for wood, $[\sigma] = 100$ kgf/cm².

Problems 745 through 748. Find for the built-up columns:
(1) size of the column leg section
(2) width b of the column section
(3) distance l_0 (clear) between the cross plates.

Assume that for the material of the column legs $E = 2 \times 10^6$ kgf/cm and $[\sigma] = 1600$ kgf/cm².

745



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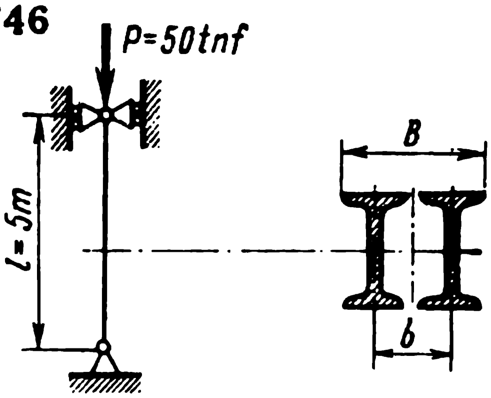

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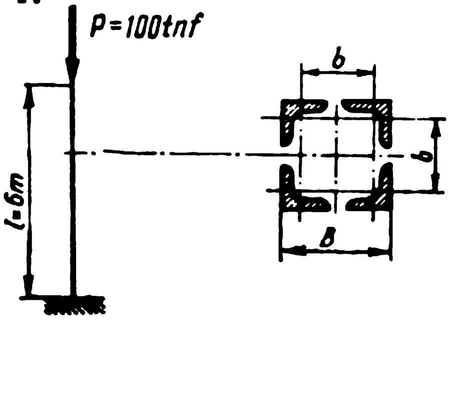

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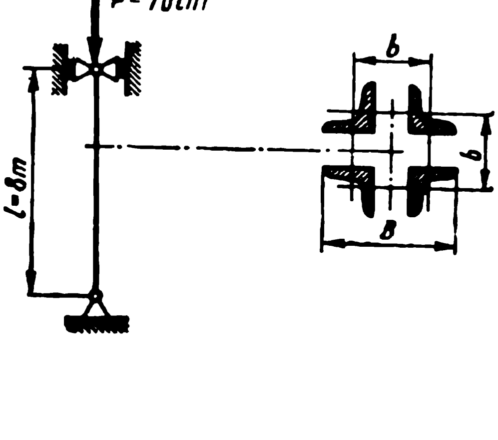


10.3.

Bending Induced by Axial Loading

Bars subject to axial and transverse forces and moments are designed or analysed by approximate calculations, assuming that the elastic line of the bar is close to a sine curve.

For beams with hinged ends (Fig. 151), which are the type mainly dealt with in this section, this sine curve is described by the equation

$$f_x = f \sin \frac{\pi x}{l} \tag{168}$$

In such calculations (intended to be used in solving the problems) more accurate results are obtained for bars with hinged ends and with

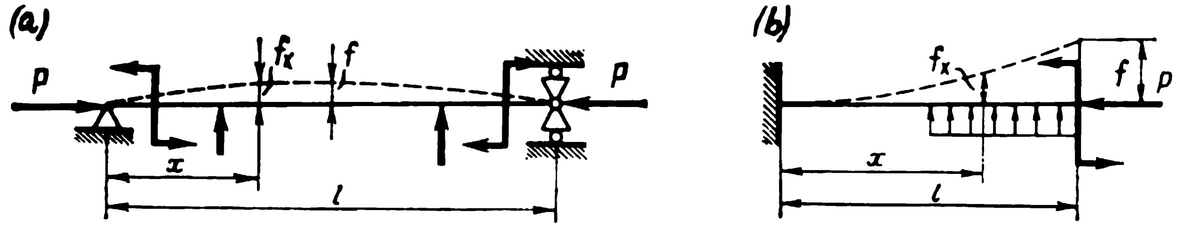


Fig. 151

transverse loads acting to one side and symmetrical with respect to the middle of the bar.

For such bars, the elastic line due to transverse bending has no inflection points, i.e. has a one-directional curvature and therefore can be represented as the half-wave of a sine curve (168).

In this case the maximum deflection is at the middle of the bar:

$$f = \frac{f_{tl}}{1 - \frac{P}{P_E}} \tag{168a}$$

The maximum bending moment is

$$M_{max} = M_{tl} + Pf = M_{tl} + \frac{Pf_{tl}}{1 - \frac{P}{P_E}} \tag{169}$$

The maximum compressive stress is

$$\max|\sigma| = \frac{P}{F} + \frac{M_{max}}{W} = \frac{P}{F} + \frac{M_{tl}}{W} + \frac{Pf_{tl}}{W\left(1 - \frac{P}{P_E}\right)} \tag{170}$$

in which P = axial compressive force

$P_E = \frac{\pi^2 EI}{l^2}$ = Euler force calculated for any slenderness ratio of the bar on the basis of the principal centroidal moment of inertia I of area F of the cross section about the axis normal to the plane of action of the transverse force

W = axial section modulus of the bar cross section about the given axis

f_{tl} and M_{tl} = deflection and bending moment in the middle of length l of the bar due only to the transverse load.

It is evident from formula (170) that the principle of superposition is unacceptable here and that stresses due to the increase of external forces grow much more rapidly than the external forces themselves.

This necessitates a change from strength calculations based on the allowable stresses to ones based on the allowable loads.

We assume that the beam system under consideration, subject to bending induced by axial loading, has a given factor of safety n if for an n-fold increase of all the external forces it reaches a dangerous state which for ductile materials is identified with the maximum (in absolute value) normal stress of the yield point σ_y, i.e.

$$\max|\sigma| = \frac{(nP)}{F} + \frac{(nM_{tl})}{W} + \frac{(nP)(nf_{tl})}{W\left(1 - \frac{nP}{P_E}\right)} = \sigma_y \tag{171}$$

Equating n to the allowable factor of safety $[n]$, we obtain the strength design formula

$$\max|\sigma| = \frac{[n]P}{F} + \frac{[n]M_{tl}}{W} + \frac{[n]P[n]f_{tl}}{W\left(1 - \frac{[n]P}{P_E}\right)} = \sigma_y \tag{171a}$$

In computations using the method of allowable loads we take the allowable factor of safety $[n]$ corresponding to that used for the allowable stress with respect to the yield point, i.e.

$$[n] = n_y$$

in which

$$n_y = \frac{\sigma_y}{[\sigma]}$$

Then the following modified design formula can be obtained from formula (171a):

$$\frac{P}{F} + \frac{M_{tl}}{W} + \frac{[n]Pf_{tl}}{W\left(1 - \frac{[n]P}{P_E}\right)} \leqslant [\sigma] \tag{171b}$$

With somewhat less accuracy formulas (168 through 171) can also be applied for a non-symmetrical transverse load, if its asymmetry is not very close to the case of the skew-symmetry.

Design calculations for bending induced by axial loading in bars with other types of supports are performed in a similar way, except that equation (168) should be modified for each particular case. Thus, for a beam fixed at one end (Fig. 151*b*) the elastic line can be approximated by the function

$$f = f_{\max}\left(1 - \cos\frac{\pi x}{2l}\right)$$

Nevertheless, design formulas (169, 170 and 171) are applicable, except that the magnitude of the Euler force P_E changes depending on the end conditions of the bar in accordance with formula (163).

If the transverse load is applied in the plane of the maximum rigidity of the bar, the bar should also be checked for stability in the plane of minimum rigidity.

Example 83. Let $P = 800$ kgf; $P_1 = 100$ kgf; $l = 2$ m; $b = 2$ cm; $h = 4$ cm; $E = 2 \times 10^6$ kgf/cm² and $\sigma_y = 2400$ kgf/cm² (Fig. 152).

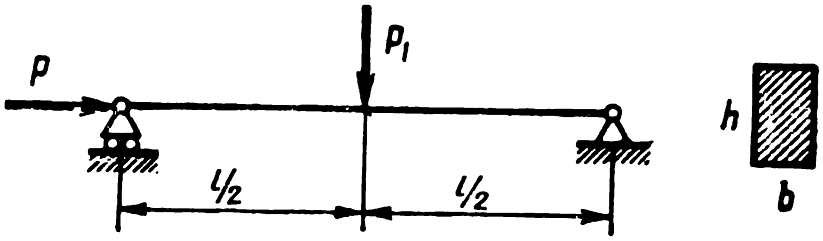


Fig. 152

Determine f, σ_{max}, n and n_{st}.

Solution. Since

$$I = \frac{bh^3}{12} = \frac{2 \times 64}{12} = \frac{32}{3} \text{ cm}^3$$

the deflection in the middle of the beam due to force P_1 is

$$f_{tl} = \frac{P_1 l^3}{48EI} = \frac{100 \times 8 \times 10^6 \times 3}{48 \times 2 \times 10^6 \times 32} = \frac{25}{32} = 0.781 \text{ cm}$$

The Euler force is

$$P_E = \frac{\pi^2 EI}{l^2} = \frac{10 \times 2 \times 10^6 \times 32}{4 \times 10^4 \times 3} = \frac{16}{3} \times 10^3 \text{ kgf}$$

The ratio

$$\frac{P}{P_E} = \frac{800 \times 3}{16 \times 10^3} = \frac{3}{20} \quad \text{and} \quad 1 - \frac{P}{P_E} = 1 - \frac{3}{20} = \frac{17}{20}$$

Using formula (168) we find the sought-for deflection $f = \frac{25}{32} \times \times \frac{20}{17} = 0.919$ cm.

Thus f_{tl} is equal to $\frac{17}{20} \times 100\% = 85\%$ of f.

Since

$$M_{tl} = \frac{P_1 l}{4} = \frac{100 \times 200}{4} = 5 \times 10^3 \text{ kgf-cm}$$

$$M_{max} = 5 \times 10^3 + 800 \times 0.919 = 5735 \text{ kgf-cm}$$

$$F = bh = 8 \text{ cm}^2$$

$$W = \frac{bh^2}{6} = \frac{2 \times 16}{6} = \frac{16}{3} \text{ cm}^3$$

then from formula (170)

$$\sigma_{max} = \frac{800}{8} + \frac{5735}{16} \times 3 = 1175 \text{ kgf/cm}^2$$

Neglecting bending due to the axial force

$$\sigma'_{max} = \frac{P}{F} + \frac{M_{tl}}{W} = 100 + \frac{5 \times 10^3}{16} \times 3 = 1038 \text{ kgf/cm}^2$$

which is $\frac{1038}{1175} \times 100\% = 88\%$ of σ_{max}.

The factor of safety n (with respect to the yield point) for the loaded beam is found from formula (171)

$$\max|\sigma| = \frac{nP}{F} + \frac{nM_{tl}}{W} + \frac{nP}{W} \times \frac{nf_{tl}}{\left(1 - \frac{nP}{P_E}\right)} = \sigma_y$$

or

$$\frac{n \times 800}{8} + \frac{n \times 15 \times 10^3 \times 3}{16} + \frac{n \times 800 \times 3 \times n \times 0.781}{16\left(1 - \frac{n \times 800 \times 3}{16 \times 10^2}\right)} = 2400$$

from which

$$\frac{n^2 - 35.9n + 61.6}{(1 - 0.15n)} = 0$$

This equation has two solutions for n:
$n_1 = 1.85$ and $n_2 = 34.05$

The value $n_2 = 34.05$ is unacceptable, because even at $n = 6.67$ ($P_{sp} = nP = 5320$ kgf), $\max|\sigma| = \infty$, since the binomial $1 - \frac{nP}{P_E} = (1 - 0.15)$ becomes zero.

Next we check the stability of the beam in the plane of minimum rigidity. Since

$$i_y^2 = \frac{b^2}{16} = \frac{4}{12} = \frac{1}{3} \text{ cm}^2 \quad \text{and} \quad \lambda = \frac{l}{i} = 200\sqrt{3}$$

then from formula (164)

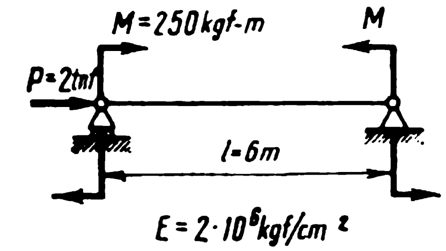
$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} = \frac{10 \times 2 \times 10^6}{3 \times 4 \times 10^4} = 167 \text{ kgf/cm}^2$$

due to force P alone, the compressive stress is $\sigma = \frac{800}{8} = 100$ kgf/cm^2. The stability factor of safety is

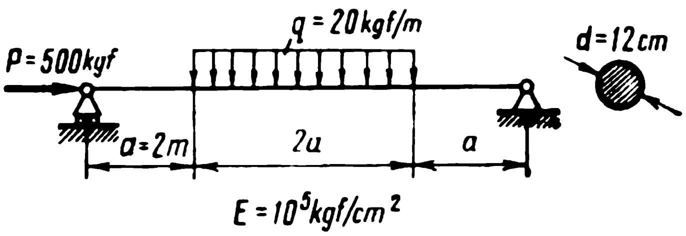
$$n_{st} = \frac{\sigma_{cr}}{\sigma} = \frac{167}{100} = 1.67$$

Problems 749, 750 and 751. Determine the maximum deflection f and the maximum compressive stress $\max|\sigma_c|$ for the beams shown.

749



750



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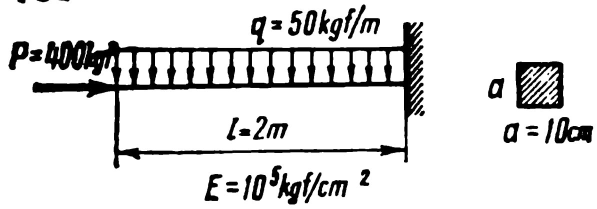
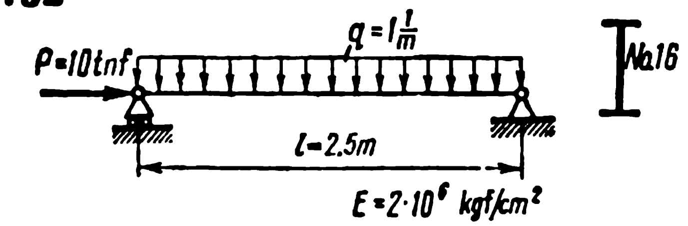


Problems 752 and 753. Determine f, max $|\sigma_c|$, factors of safety n and stability factors of safety n_{st} of the beams shown.

752



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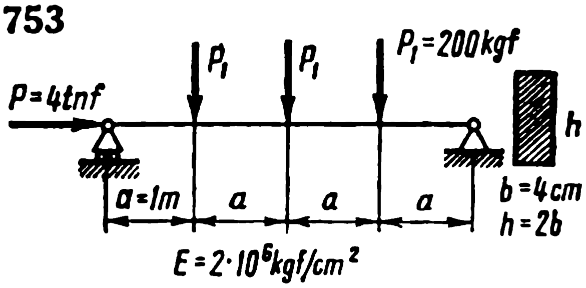

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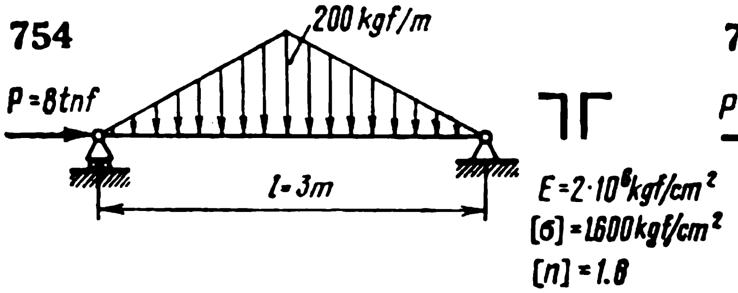

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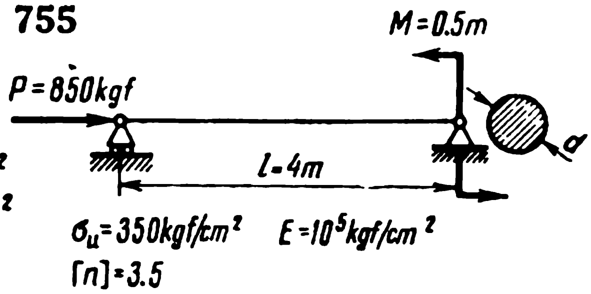

Problems 754 and 755. Select the cross-sectional dimensions of the beams.

CHAPTER 11. PLANE CURVED BEAMS

11.1.

Axial Force, Transverse (Shearing) Force and Bending Moment

A *curved beam* is a beam whose geometric axis is curvilinear.

Here we shall consider curved beams in which (1) the geometric axis is a plane curve; (2) the plane of curvature is the plane of symmetry; (3) the acting forces are in the plane of curvature; (4) the material obeys Hooke's law; and (5) the rigidity is sufficient for the principle of superposition to be applied.

The internal forces in a cross section of a curved beam are determined by the method of sections. The forces are reduced to the axial force N, transverse (shearing) force Q and the bending moment M.

We shall consider the following to be positive: a stretching (tensile) force N, a transverse (shearing) force Q whose direction coincides with the direction of the tensile force N if it is revolved through 90° clockwise, and a bending moment M which increases the curvature of the beam (Fig. 153).

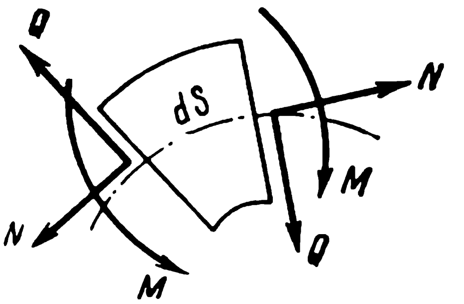

Fig. 153

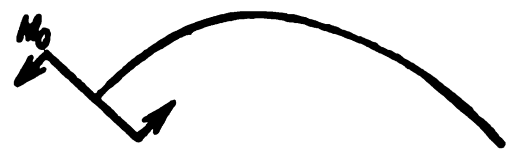

Fig. 154

We shall plot the positive magnitudes of N, Q and M on the diagrams normal to the geometric axis of the beam and away from its centre of curvature, while the negative values will be plotted towards its centre of curvature. For beams consisting of curved and straight portions, it is more convenient to arrange the positive and negative diagrams for the straight portion in the same directions from the geometric axis as for the curved portion.

Irrespective of the shape of the curved beam the values of N, Q and M in a cross section determined by the coordinates x and y and the angle $\beta = \arctan\left(\frac{dy}{dx}\right)$ are found using the same method.

We shall first deal with cases when various loads are applied at one side of the section. Thus we apply:

1. A concentrated force couple (Fig. 154):

$$N = 0; \quad Q = 0; \quad M = M_0$$

2. A concentrated force (Fig. 155*a*).

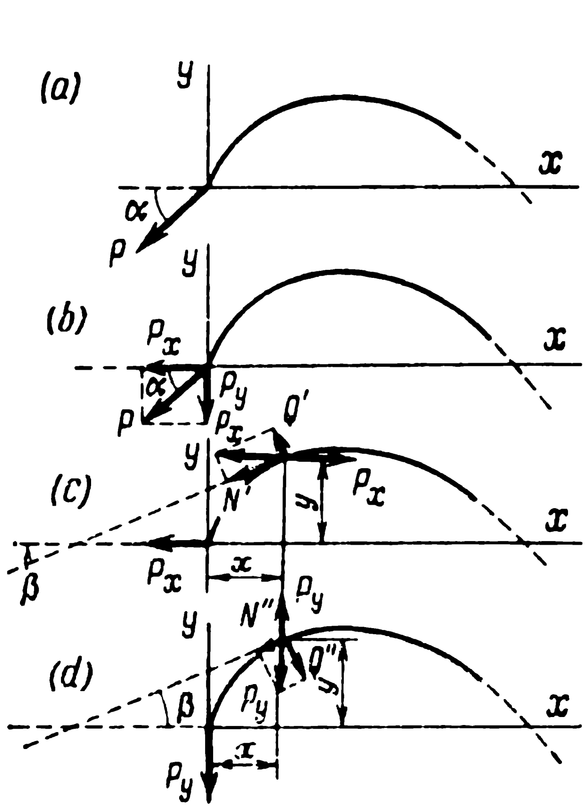

Fig. 155

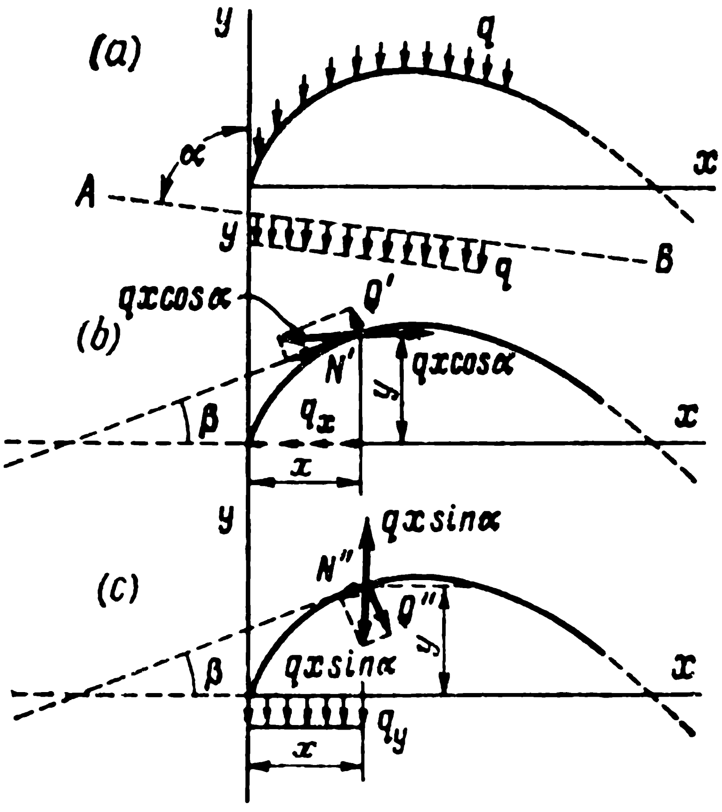

Fig. 156

The components of the force P along the x- and y-axes (Fig. 155*b*) equal:

$$P_x = P \cos \alpha; \quad P_y = P \sin \alpha$$

From the component P_x (Fig. 155*c*):

$$N' = P \cos \alpha \cos \beta; \quad Q' = P \cos \alpha \sin \beta; \; M' = -Py \cos \alpha$$

From the component P_y (Fig. 155*d*):

$$N'' = P \sin \alpha \sin \beta; \quad Q'' = -P \sin \alpha \cos \beta; \quad M'' = Px \sin \alpha$$

The resultant internal forces due to force P are:

$$N = P(\cos \alpha \cos \beta + \sin \alpha \sin \beta) = P \cos(\alpha - \beta);$$
$$Q = P(\cos \alpha \sin \beta - \sin \alpha \cos \beta) = -P \sin(\alpha - \beta);$$
$$M = P(x \sin \alpha - y \cos \alpha)$$

3. A load uniformly distributed along straight line AB and normal to it (Fig. 156*a*).

The component loads q acting along axes x and y (similar to the preceding case) are:

$$q_x = q\cos\alpha; \qquad q_y = q\sin\alpha$$

From the component q_x (Fig. 156*b*) we find

$$N' = qx\cos\alpha\cos\beta; \quad Q' = qx\cos\alpha\sin\beta; \quad M' = -qxy\cos\alpha$$

From the component q_y (Fig. 156*c*):

$$N'' = qx\sin\alpha\sin\beta; \quad Q'' = -qx\sin\alpha\cos\beta;$$

$$M'' = q\frac{x^2}{2}\sin\alpha$$

The resultant internal forces due to load q are

$$N = qx(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = qx\cos(\alpha - \beta);$$

$$Q = qx(\cos\alpha\sin\beta - \sin\alpha\cos\beta) = -qx\sin(\alpha - \beta);$$

$$M = qx\left(\frac{x}{2}\sin\alpha - y\cos\alpha\right)$$

4. A transverse load uniformly distributed along the geometric axis of the beam (Fig. 157*a*).

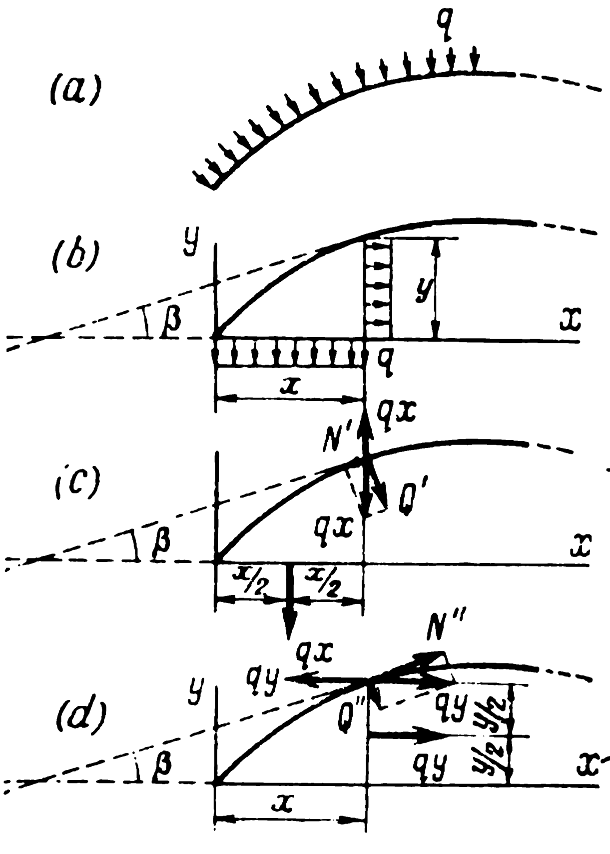

Fig. 157

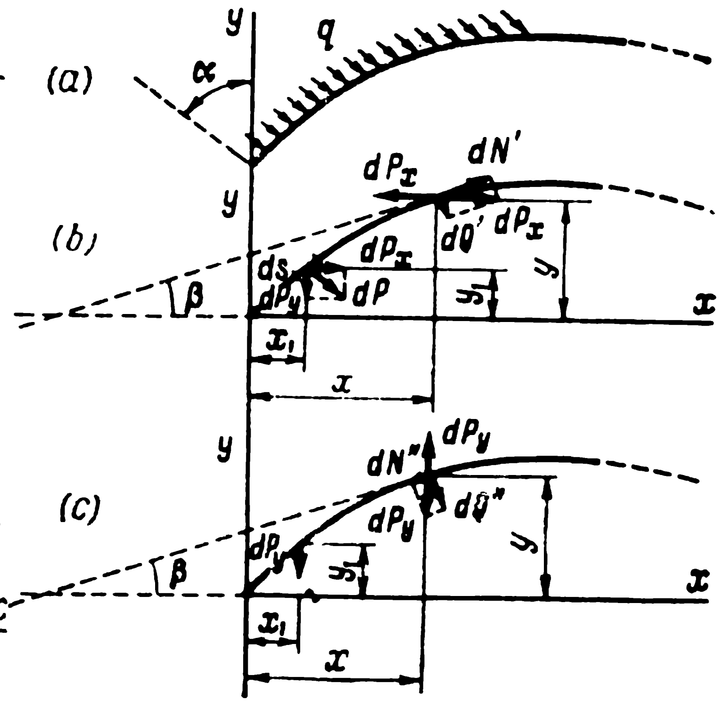

Fig. 158

From the action of the projection of the cut-off load on the y-axis (Fig. 157*b* and *c*)

$$N' = qx\sin\beta; \quad Q' = -qx\cos\beta; \quad M' = q\frac{x^2}{2}$$

and the action of the projection of the cut-off load on the x-axis (Fig. 157d)

$$N'' = -qy\cos\beta; \quad Q'' = -qy\sin\beta; \quad M'' = q\frac{y^2}{2}$$

From the combined action of q_x and q_y

$$N = q\,(x\sin\beta - y\cos\beta); \quad Q = -q\,(x\cos\beta + y\sin\beta);$$

$$M = \frac{q}{2}\,(x^2 + y^2)$$

5. A parallel load uniformly distributed along the geometric axis of the beam (Fig. 158a).

The element of force in the section with the coordinates x_1y_1, acting on element ds of the arc of the geometric axis on the beam (Fig. 158b) is

$$dP = q\,ds$$

The projections of force dP on the x- and y-axes are: $dP_x = dP\sin\alpha = q\,ds\sin\alpha$; $dP_y = dP\cos\alpha = q\,ds\cos\alpha$.

The elements of internal force due to force dP_x in the section with coordinates x and y and angle β (Fig. 158b) are

$$dN' = -dP_x\cos\beta = -q\sin\alpha\cos\beta\,ds; \quad dQ' = -dP_x\sin\beta$$
$$= -q\sin\alpha\sin\beta\,ds; \quad dM' = dP_x(y - y_1) = q\,(y - y_1)\sin\alpha\,ds$$

The elements of internal force due to force dP_y in the same cross section (Fig. 158c) are

$$dN'' = dP_y\sin\beta = q\cos\alpha\sin\beta\,ds;$$
$$dQ'' = -dP_y\cos\beta = -q\cos\alpha\cos\beta\,ds;$$
$$dM'' = dP_y\,(x - x_1) = q\,(x - x_1)\cos\alpha\,ds$$

The elements of internal force due to the combined action of forces dP_x and dP_y in the section being considered are

$$dN = -q\,(\sin\alpha\cos\beta - \cos\alpha\sin\beta)\,ds = -q\sin(\alpha - \beta)\,ds;$$
$$dQ = -q\,(\sin\alpha\sin\beta + \cos\alpha\cos\beta)\,ds = -q\cos(\alpha - \beta)\,ds;$$
$$dM = q\,[(y - y_1)\sin\alpha + (x - x_1)\cos\alpha]\,ds$$

The total internal forces in the section being considered, which cuts off an arc s from the geometric axis of the beam, are

$$N = -q\sin(\alpha - \beta)\int_0^s ds = -qs\sin(\alpha - \beta);$$

$$Q = -q\cos(\alpha - \beta)\int_0^s ds = -qs\cos(\alpha - \beta);$$

$$M = q\left[\sin\alpha\left(y\int_0^s ds - \int_0^s y_1\,ds\right) + \cos\alpha\left(x\int_0^s ds - \int_0^s x_1\,ds\right)\right] = q\,[\sin\alpha\,(ys - S_x) + \cos\alpha\,(xs - S_y)]$$

$$= qs\,[(y - y_c)\sin\alpha + (x - x_c)\cos\alpha]$$

in which $S_x = \int_0^s y_1\,ds$ and $S_y = \int_0^s x_1\,ds$ = statical moments of the arc s with respect to axes x and y.

$y_c = \frac{S_x}{s}$ and $x_c = \frac{S_y}{s}$ = coordinates of the centre of gravity of arc s.

6. A load uniformly distributed along the geometric axis of the beam and tangential to this axis (Fig. 159*a*).

An element of force in the cross section with coordinates $x_1 y_1$ and angle β_1 (Fig. 159*b*) and acting tangential to the geometric axis of the beam on the element of arc ds is

$$dP = q\,ds$$

The elements of internal force due to the action of force dP in the cross section with coordinates x, y and angle β (Fig. 159*b*) are

$$dN = -dP\cos(\beta_1 - \beta) = -q\cos(\beta_1 - \beta)\,ds;$$

$$dQ = dP\sin(\beta_1 - \beta) = q\sin(\beta_1 - \beta)\,ds;$$

$$dM = -dP\,[(x - x_1)\sin\beta_1 - (y - y_1)\cos\beta_1] = q\,[(y - y_1)\cos\beta_1 - (x - x_1)\sin\beta_1]\,ds$$

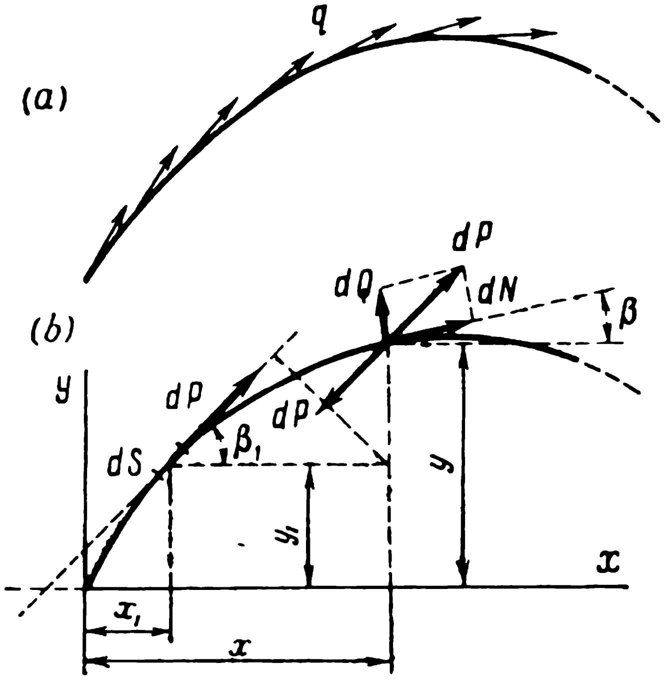

Fig. 159

The total internal forces in the section being considered cutting off an arc s of the geometric axis of the beam are

$$N = -q\int_0^s \cos(\beta_1 - \beta)\,ds; \qquad Q = q\int_0^s \sin(\beta_1 - \beta)\,ds;$$

$$M = q\left[\int_0^s (y - y_1)\cos\beta_1\,ds - \int_0^s (x - x_1)\sin\beta_1\,ds\right]$$

Example 84. Given: P and ρ (Fig. 160*a*). Plot the diagrams for N, Q and M.

Solution. First we find N, Q and M on the portions of the beam (Fig. 160*b*).

For the first portion: $0 \leqslant \varphi_1 \leqslant \frac{\pi}{2}$;

$$N_{\varphi_1} = P\cos\varphi_1; \quad Q_{\varphi_1} = P\sin\varphi_1; \quad M_{\varphi_1} = P\rho(1-\cos\varphi_1);$$

$$N_{\varphi_1=0} = P; \quad N_{\varphi_1=\frac{\pi}{4}} \cong 0.707P;$$

$$N_{\varphi_1=\frac{\pi}{2}} = 0; \qquad Q_{\varphi_1=0} = 0;$$

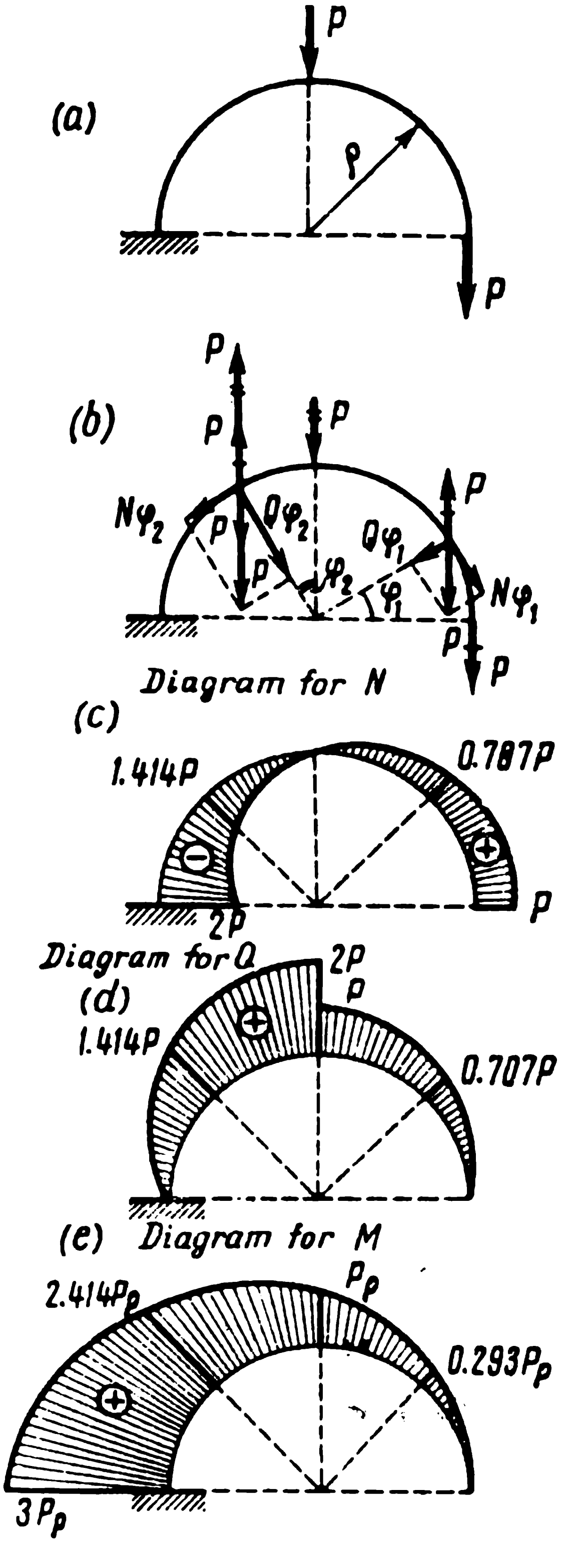

Fig. 160

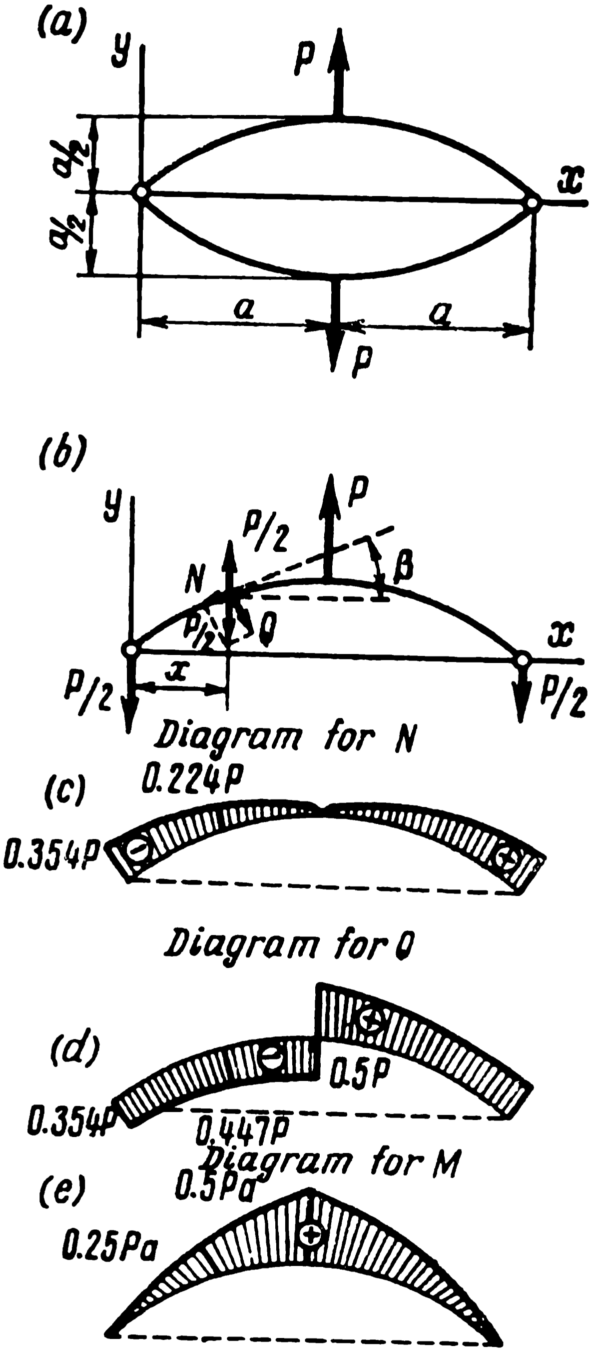

Fig. 161

$$Q_{\varphi_1=\frac{\pi}{2}} \cong 0.707P; \quad Q_{\varphi_1=\frac{\pi}{2}} = P;$$

$$M_{\varphi_1=0} = 0; \quad M_{\varphi_1=\frac{\pi}{4}} \cong 0.293P\rho; \quad M_{\varphi_1=\frac{\pi}{2}} = P\rho$$

For the second portion: $0 \leqslant \varphi_2 \leqslant \frac{\pi}{2}$;

$$N_{\varphi_2} = -2P\sin\varphi_2; \quad Q_{\varphi_2} = 2P\cos\varphi_2; \quad M_{\varphi_2} = P\rho(1+2\sin\varphi_2);$$

$$N_{\varphi_2=0} = 0; \quad N_{\varphi_2=\frac{\pi}{4}} \cong -1.414P; \quad N_{\varphi_2=\frac{\pi}{2}} = -2P; \quad Q_{\varphi_2=0} = 2P;$$

$$Q_{\varphi_2=\frac{\pi}{4}} \cong 1.414P; \quad Q_{\varphi_2=\frac{\pi}{2}} = 0;$$

$$M_{\varphi_2=0} = P\rho; \quad M_{\varphi_2=\frac{\pi}{4}} \cong 2.414P\rho; \quad M_{\varphi_2=\frac{\pi}{2}} = 3P\rho$$

The diagrams for N, Q and M are shown in Figs. 160*c*, *d* and *e*.

Example 85. Given: P, a, and equations for the geometric axes of the upper and lower branches of the spring which are parabolas with $y = \pm x\left(1-\frac{x}{2a}\right)$ (Fig. 161*a*).

Plot the diagrams for N, Q and M.

Solution. Since the system is symmetrical about two mutually perpendicular axes, we shall consider only half of one branch of the spring (Fig. 161*b*). For an arbitrary cross section at a distance x from the left end

$$\tan\beta = \frac{dy}{dx} = \frac{a-x}{a};$$

$$\sin\beta = \frac{\tan\beta}{\sqrt{1+\tan^2\beta}} = \frac{a-x}{\sqrt{2a^2-2ax+x^2}};$$

$$\cos\beta = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{a}{\sqrt{2a^2-2ax+x^2}}$$

Therefore

$$N = \frac{P}{2}\sin\beta = \frac{P(a-x)}{2\sqrt{2a^2-2ax+x^2}};$$

$$Q = -\frac{P}{2}\cos\beta = -\frac{Pa}{2\sqrt{2a^2-2ax+x^2}}$$

$$M = \frac{P}{2}x;$$

$$N_{x=0} = \frac{\sqrt{2}}{4}P \cong 0.354P; \quad N_{x=\frac{a}{2}} = \frac{\sqrt{5}}{10}P \cong 0.224P; \quad N_{x=a} = 0;$$

$$Q_{x=0}=-\frac{\sqrt{2}}{4}P\cong-0.354P;\quad Q_{x=\frac{a}{2}}=-\frac{\sqrt{5}}{5}P\cong-0.447P;$$

$$Q_{x=a}=-0.5P;$$

$$M_{x=0}=0;\quad M_{x=\frac{a}{2}}=0.25Pa;\qquad M_{x=a}=0.5Pa$$

The diagrams for N, Q, M are shown in Figs. 161*c*, *d* and *e*.

Example 86. Given: q, a, b, the ring is bent into an arc of the ellipse $y=\frac{b}{a}\sqrt{2ax-x^2}$ and is cut through at the origin of coordinates (Fig. 162*a*).

Find N, Q and M.

Solution. Since the beam is symmetrical about the x-axis we shall consider only its upper half.

Using the principle of superposition, we find the internal forces in the cross section with coordinates x, y and angle β (Fig. 162*c* and *d*), due to the action of the projections of load q_x and q_y (Fig. 162*b*)

$$N=qy\cos\beta-qx\sin\beta;$$

$$Q=qx\cos\beta+qy\sin\beta;$$

$$M=-\frac{q}{2}(x^2+y^2)$$

Since

$$\tan\beta=\frac{dy}{dx}=\frac{b}{a}\times\frac{a-x}{\sqrt{2ax-x^2}}$$

then

$$\sin\beta=\frac{a-x}{\sqrt{\frac{a^2}{b^2}(2ax-x^2)+(a-x)^2}};$$

$$\cos\beta=\frac{\frac{a}{b}\sqrt{2ax-x^2}}{\sqrt{\frac{a^2}{b^2}(2ax-x^2)+(a-x)^2}}$$

and

$$N=\frac{qax}{\sqrt{\frac{a^2}{b^2}(2ax-x^2)+(a-x)^2}};$$

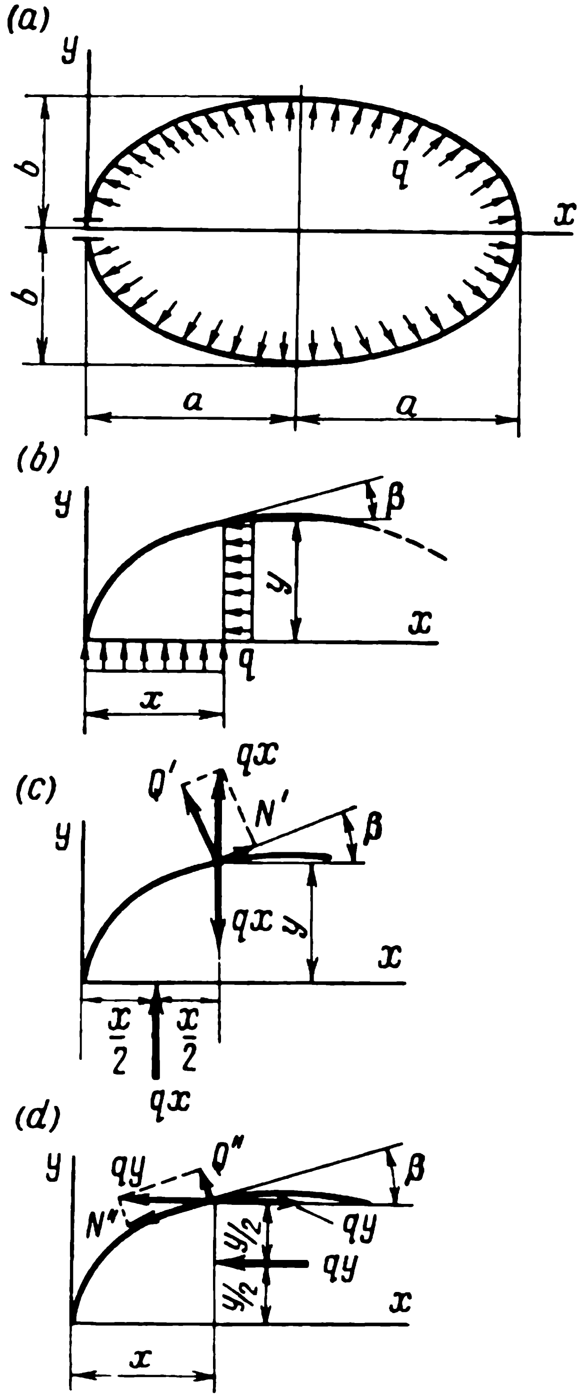

Fig. 162

$$Q=\frac{q\left[\frac{a}{b}x+\frac{b}{a}(a-x)\right]\sqrt{2ax-x^2}}{\sqrt{\frac{a^2}{b^2}(2ax-x^2)+(a-x)^2}};$$

$$M=-\frac{q}{2}\left[\frac{b^2}{a^2}(2ax-x^2)+x^2\right]$$

If, for example, $a=b$, then $N=qx$; $Q=q\sqrt{2\,ax-x^2}$ and $M=-qax$. The diagrams for N, Q and M are shown in Figs. 163*a*, *b* and *c*.

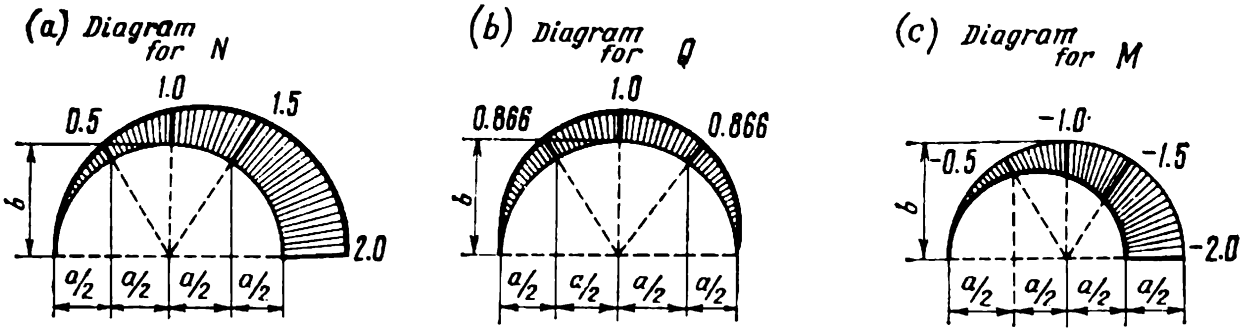

Fig. 163

Example 87. Given: q, ρ and that the vertical load of intensity q is uniformly distributed over the arc of a quarter of a circle of radius ρ (Fig. 164*a*).

Plot the diagrams for N, Q and M.

Solution. Internal forces in the cross section at an angle φ to the vertical due to the element of force $dP=q\,ds=q\rho\,d\alpha$ are (Fig. 164*b*)

$$dN=-dP\sin\varphi=q\rho\sin\varphi\,d\varphi;$$

$$dQ=dP\cos\varphi=q\rho\cos\varphi\,d\varphi;$$

$$dM=dP\rho\,(\sin\varphi-\sin\alpha)=q\rho^2(\sin\varphi-\sin\alpha)\,d\alpha$$

Internal forces due to the load acting on the cut-off section of the beam are

$$N=-q\rho\sin\varphi\int_0^{\varphi}d\alpha=-q\rho\varphi\sin\varphi;$$

$$Q=q\rho\cos\varphi\int_0^{\varphi}d\alpha=q\rho\varphi\cos\varphi;$$

$$M=q\rho^2\int_0^{\varphi}(\sin\varphi-\sin\alpha)\,d\alpha=q\rho^2(\varphi\sin\varphi+\cos\varphi-1);$$

$$N_{\varphi=0}=0;\quad N_{\varphi=\frac{\pi}{4}}\cong-q\rho\,\frac{\pi}{4}\,0.707=-0.555q\rho;$$

$$N_{\varphi=\frac{\pi}{2}} = -q\rho\frac{\pi}{2} \cong -1.571q\rho;$$

$$Q_{\varphi=0} = 0; \quad Q_{\varphi=\frac{\pi}{4}} \cong 0.555q\rho; \quad Q_{\varphi=\frac{\pi}{2}} = 0;$$

$$M_{\varphi=0} = 0; \quad M_{\varphi=\frac{\pi}{4}} \cong q\rho^2\left(\frac{\pi}{4}0.707+0.707-1\right) \cong 0.262q\rho^2;$$

$$M_{\varphi=\frac{\pi}{2}} = q\rho^2\left(\frac{\pi}{2}-1\right) \cong 0.571q\rho^2$$

The diagrams for N, Q and M are shown in Figs. 164*c*, *d* and *e*.

Example 88. Given: q, ρ, and that the load of intensity q is evenly distributed over the arc of a half circle of radius ρ and is tangential to the geometric axis of the beam (Fig. 165*a*).

Plot the diagrams for N, Q and M.

Solution. An element of force in the cross section at an angle α to the horizontal is

$$dP = q\,ds = q\rho\,d\alpha.$$

The elements of internal forces in the cross section of the beam at an angle φ to the horizontal and due to force dP are (Fig. 165*b*)

$$dN = -dP\cos(\varphi-\alpha) = -q\rho\cos(\varphi-\alpha)\,d\alpha;$$
$$dQ = dP\sin(\varphi-\alpha) = q\rho\sin(\varphi-\alpha)\,d\alpha;$$
$$dM = -dP\,[\rho-\rho\cos(\varphi-\alpha)] = -q\rho^2\,[1-\cos(\varphi-\alpha)]\,d\alpha$$

The total internal forces due to the load acting on the cut-off portion of the beam are

$$N = -q\rho\int_0^{\varphi}\cos(\varphi-\alpha)\,d\alpha = -q\rho\sin\varphi;$$

$$Q = q\rho\int_0^{\varphi}\sin(\varphi-\alpha)\,d\alpha = q\rho(1-\cos\varphi);$$

$$M = -q\rho^2\left[\int_0^{\varphi}d\alpha-\int_0^{\varphi}\cos(\varphi-\alpha)\,d\alpha\right] = q\rho^2(\sin\varphi-\varphi);$$

$$N_{\varphi=0} = 0; \; N_{\varphi=\frac{\pi}{4}} \cong -0.707q\rho; \; N_{\varphi=\frac{\pi}{2}} = -q\rho; \; N_{\varphi=\frac{3}{4}\pi} \cong 0.707q\rho;$$

$$N_{\varphi=\pi} = 0;$$

$$Q_{\varphi=0} = 0; \quad Q_{\varphi=\frac{\pi}{4}} \cong 0.293q\rho; \quad Q_{\varphi=\frac{\pi}{2}} = q\rho; \quad Q_{\varphi=\frac{3}{4}\pi} \cong 1.707q\rho;$$

$$Q_{\varphi=\pi} = 2q\rho;$$

$$M_{\varphi=0}=0;\quad M_{\varphi=\frac{\pi}{4}}=-0.078q\rho^2;\quad M_{\varphi=\frac{\pi}{2}}\cong-0.571q\rho^2;$$

$$M_{\varphi=\frac{3}{4}\pi}=-1.649q\rho^2;\quad M_{\varphi=\pi}=-3.142q\rho^2$$

The diagrams for N, Q and M are shown in Figs. 165*c*, *d* and *e*.

Problems 756 through 765. Plot the diagrams of the axial internal force N, transverse force Q and bending moment M.

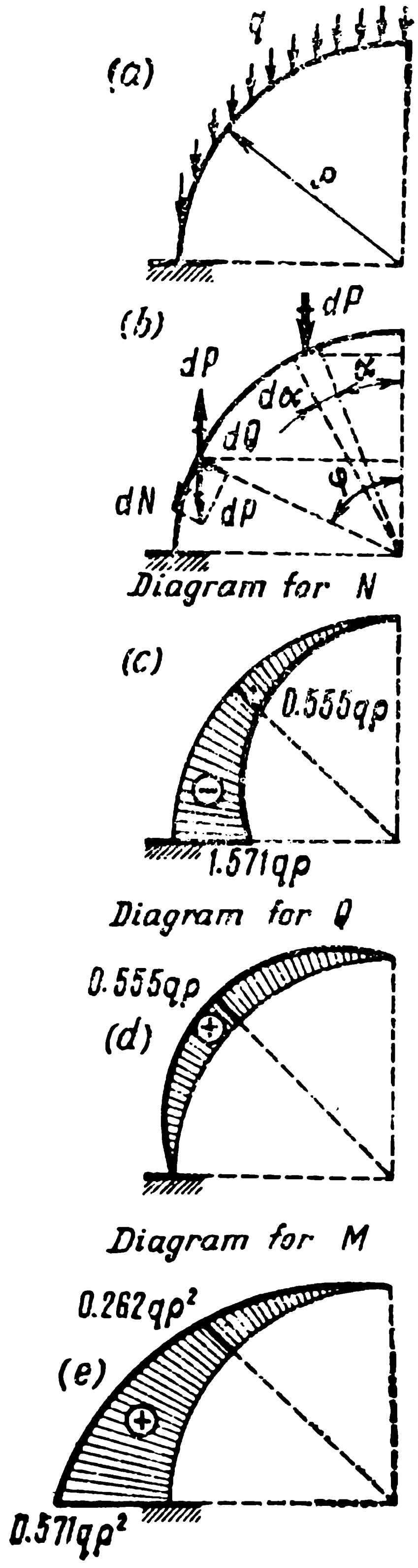

Fig. 164

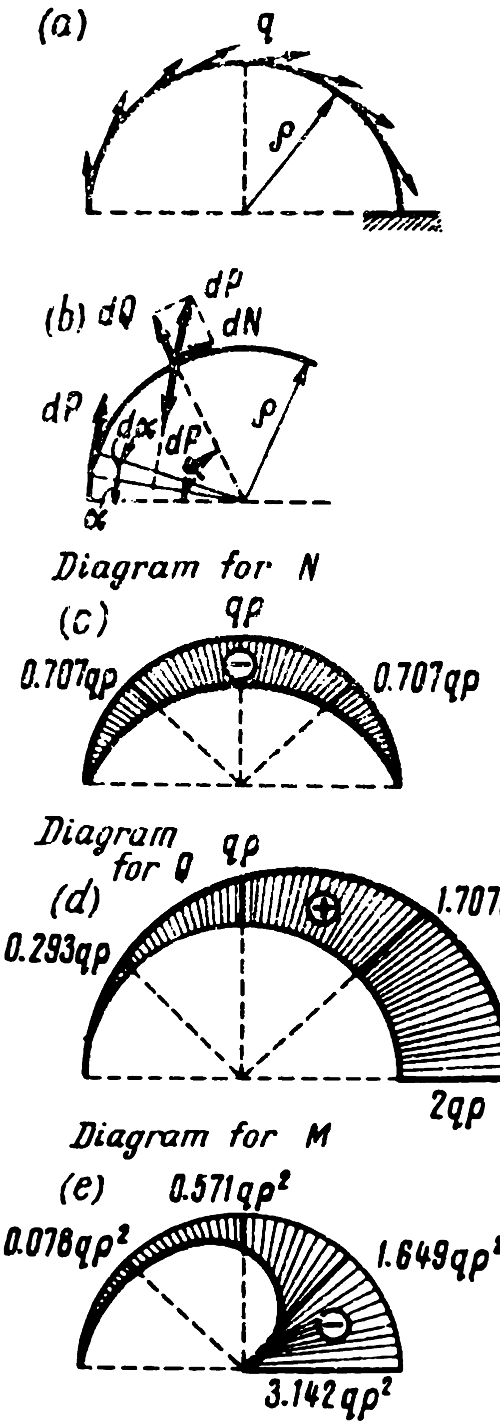

Fig. 165

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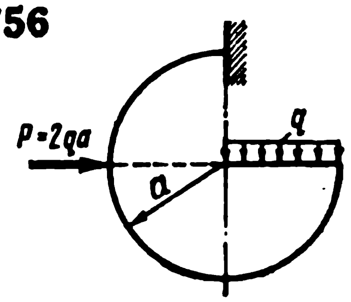

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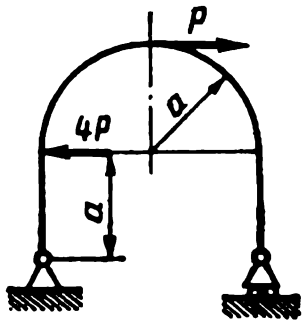

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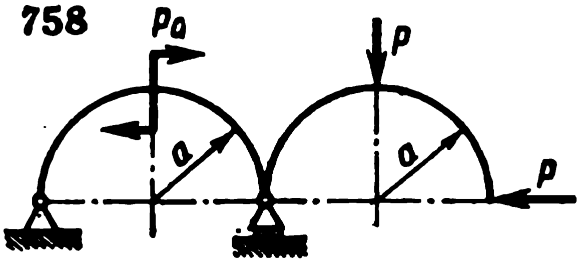

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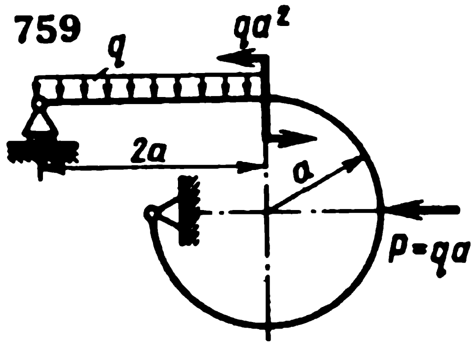

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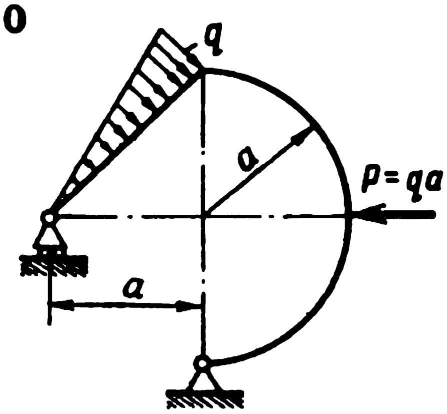

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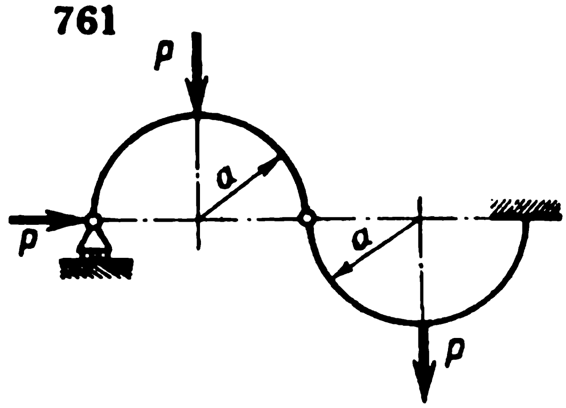

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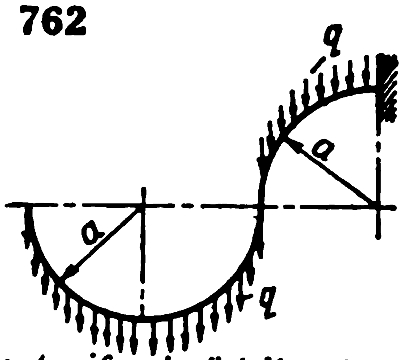
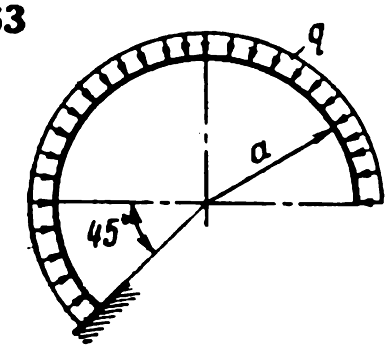


Load uniformly distributed along a horizontal projection

763



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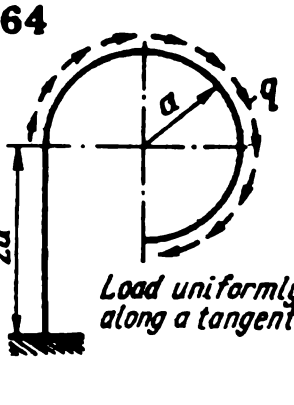
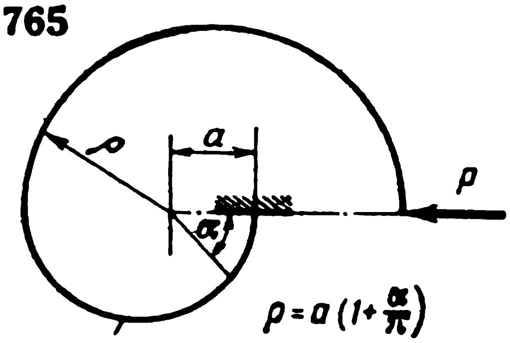


Load uniformly distributed along a tangent to the arc

765



$\rho = a\left(1+\frac{\alpha}{\pi}\right)$

Archimedean spiral

11.2.

Stresses

The axial internal force N and bending moment M determine the normal stresses σ_N and σ_M and the transverse (shearing) force Q determines the shearing stresses τ developed at points of a cross section of a curved bar.

Stresses σ_N are assumed to be uniformly distributed over the cross-sectional area F, and stresses σ_M according to a hyperbolic law; they are calculated from the formulas:

$$\sigma_N = \frac{N}{F}; \tag{172}$$

$$\sigma_M = \frac{M}{S} \times \frac{y}{r+y} \tag{173}$$

in which S is the static moment of area F about the neutral axis z which does not pass through the centre of gravity O_0 of the cross secti-

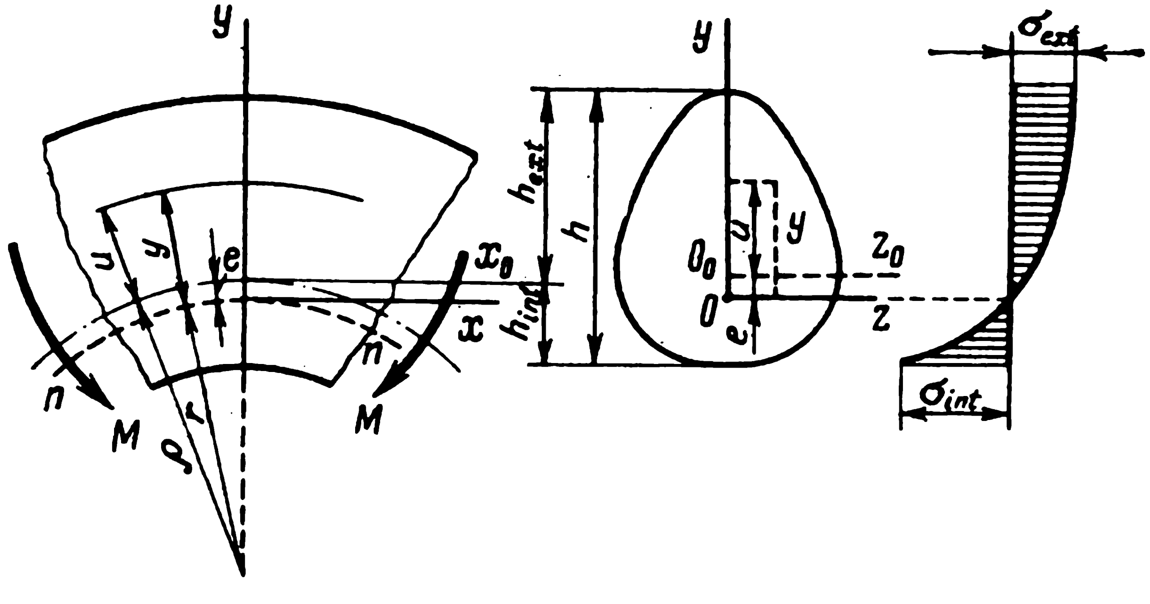


Fig. 166

on; r is the radius of curvature of the neutral line nn and y is the coordinate of the point being considered from the z-axis (Fig. 166).

The neutral line nn is displaced with respect to the geometric axis of the beam towards the centre of its curvature by the distance $e = \rho - r$, in which ρ is the radius of curvature of the geometric axis of the beam.

The radius of curvature of the neutral line of the beam for each shape of its cross section is found from the equation

$$r = \frac{F}{\int\limits_F \frac{dF}{\rho + u}} \tag{174}$$

in which u is the coordinate of the point being considered in the cross section from the centroidal axis z_0 (Fig. 166).

For a rectangular cross section

$$r=\frac{h}{\ln\frac{r_{ext}}{r_{int}}}$$

for a round cross section

$$r=\frac{1}{2}\left[\rho+\sqrt{\rho^2-\left(\frac{d}{2}\right)^2}\right]$$

and for a trapezoidal cross section

$$r=\frac{F}{\left(b_{ext}+r_{ext}\frac{b_{int}-b_{ext}}{h}\right)\ln\frac{r_{ext}}{r_{int}}-(b_{int}-b_{ext})}$$

in which r_{ext}, r_{int}, b_{ext} and b_{int} are the radii of curvature and the widths of the external and internal fibres of the cross section, respectively.

For certain other shapes of cross sections, the values of r are given in the respective handbooks and textbooks on the strength of materials.

For beams of not very large curvature, the value of e can also be found by using the approximate formula

$$e\cong\frac{i^2}{\rho} \tag{175}$$

in which $i=\sqrt{\frac{I}{F}}$ is the radius of inertia of the beam cross section about the centroidal axis z_0.

The closest approximation to an accurate solution is given by formula (175) for beams with cross sections symmetrical with respect to axis z_0.

Since

$$S=Fe\cong\frac{I}{\rho};\quad y=u+e=u\left(1+\frac{i^2}{\rho u}\right)$$

and

$$r+y=\rho+u=\rho\left(1-\frac{u}{\rho}\right)$$

in which u is the distance of the point being considered in the cross section from the centroidal axis z_0, then formula (173) can be written as

$$\sigma_M=\frac{Mu}{I}\times\frac{1+\frac{i^2}{\rho u}}{1+\frac{u}{\rho}}=\frac{Mu}{I}\alpha \tag{176}$$

Here

$$\alpha=\frac{1+\frac{i^2}{\rho u}}{1+\frac{u}{\rho}} \tag{177}$$

is an abstract function of coordinate u, characterizes the non-linear law of distribution of σ_M over the cross section and is dependent on the shape of the latter and on the initial curvature of the beam. If $\frac{h}{\rho} \leqslant \leqslant \frac{1}{10}$, then α differs only slightly from unity and σ_M can be calculated from the formula for a straight beam, viz. $\sigma = \frac{Mu}{I}$.

The maximum and minimum normal stresses are obtained in the extreme fibres of the beam at $u = h_{ext}$ and $u = -h_{int}$. They are equal to

$$\sigma_{int}^{ext} = \pm \frac{M}{W_{int}^{ext}}\alpha_{int}^{ext} \tag{178}$$

in which W is the equatorial section modulus of the cross section which is $W_{ext} = \frac{I}{h_{ext}}$ for the external fibre and $W_{int} = \frac{I}{h_{int}}$ for the internal fibre and

$$\alpha_{int}^{ext} = \alpha_{u=\pm h_{int}^{ext}} = \frac{1 \pm \frac{i^2}{\rho h_{int}^{ext}}}{1 \pm \frac{h_{int}^{ext}}{\rho}} \tag{179}$$

The magnitudes of the coefficients α_{int}^{ext} for certain cross-sectional shapes are given in the tables of Appendix 4.

The resultant normal stress σ at an arbitrary point of a cross section of a curved beam can be found by the formula

$$\sigma = \frac{N}{F} + \frac{Mu}{I}\alpha \tag{180}$$

in which internal force N is written with its own sign In most cases σ_N is small as compared to σ_M.

The shearing stresses τ determined by the transverse (shearing) force Q are less important and usually neglected.

They can be calculated approximately, as for straight beams, by formula (99)

$$\tau = \frac{QS}{bI}$$

Problem 89. Let $P = 1$ tnf; $\rho = 50$ cm; $M = P\rho$ and $a = 10$ cm (Fig. 167*a*).

Determine max σ_{ext}, min σ_{int}, τ_{max} and e.

Solution. In accordance with the diagrams for N, Q and M (Fig. 167*b*, *c* and *d*) the maximum normal stresses are in the fixed section of the beam in which $N = -P = -1$ tnf and $M = 2P\rho = 2 \times 1 \times 50 = 100$ tnf-cm, and the maximum shearing stresses are in the section at the free end of the beam, in which $Q = P = 1$ tnf.

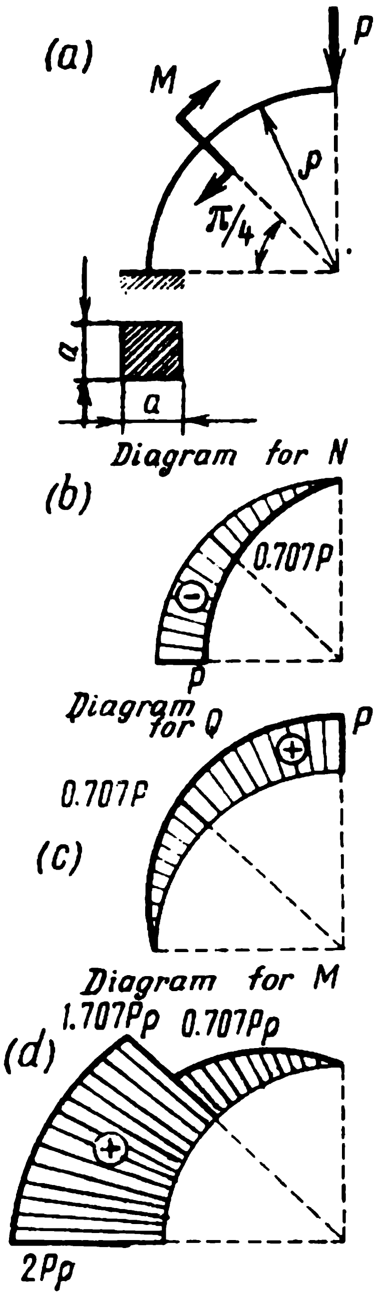


Fig. 167

For the square cross section

$$W_{ext} = W_{int} = W = \frac{a^3}{6} = \frac{10^3}{6} \text{ cm}^3;$$

$$F = a^2 = 10^2 \text{ cm}^2;$$

$$\alpha_{int}^{ext} = \frac{1 \pm \frac{a}{6\rho}}{1 \pm \frac{a}{2\rho}} = \frac{1 \pm \frac{10}{6 \times 50}}{1 \pm \frac{10}{2 \times 50}} \cong \begin{cases} 0.939 \\ 1.074 \end{cases}$$

Therefore

$$\max \sigma_{ext} = \frac{M}{W}\alpha_{ext} + \frac{N}{F} = \frac{10^5 \times 6}{10^3} 0.939 - \frac{10^3}{10^2}$$

$$\cong 563 - 10 \cong 553 \text{ kgf/cm}^2;$$

$$\min \sigma_{int} = \frac{M}{W}\alpha_{int} + \frac{N}{F} =$$

$$= -\frac{10^5 \times 6}{10^3} 1.074 - \frac{10^3}{10^2}$$

$$\cong -644 - 10 = -654 \text{ kgf/cm}^2;$$

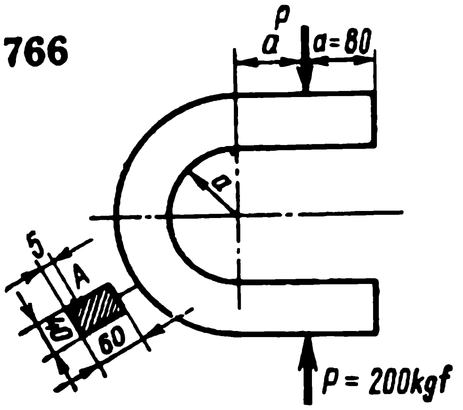
$$\tau_{max} = \frac{3}{2} \times \frac{Q}{F} = \frac{3}{2} \times \frac{10^3}{10^2} = 15 \text{ kgf/cm}^2;$$

$$e = \frac{i^2}{\rho} = \frac{a^2}{12\rho} = \frac{10^2}{12 \times 50} = \frac{1}{6}$$

$$\cong 0.167 \text{ cm} = 1.67 \text{ mm}$$

Problems 766 through 769. Determine the maximum tensile stresses σ_{max}, maximum compressive stresses σ_{min} and normal stresses σ_A at a given point A in the dangerous section of the beam.

766



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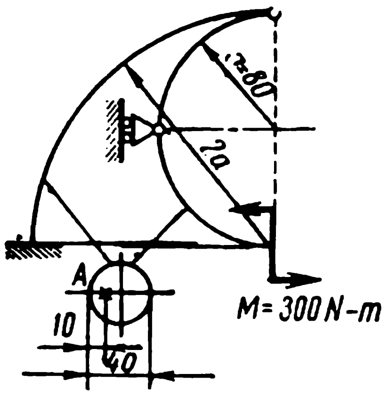

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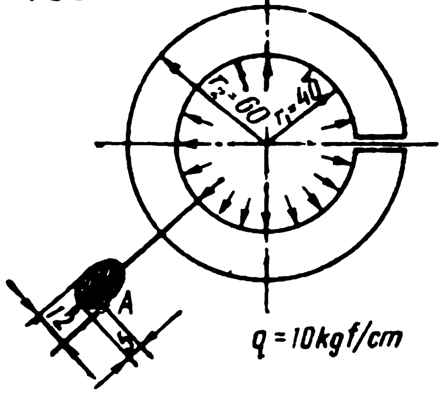

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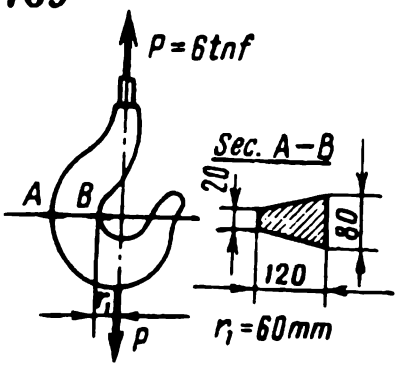


11.3.

Strength Calculations

To determine the safe load on a curved beam and to check the strength of the beam use is made of the following equation:

$$\max|\sigma_{int}| = \left|\frac{M}{W_{int}}\alpha_{int} + \frac{N}{F}\right| \leqslant [\sigma] \tag{181}$$

The required cross-sectional dimensions of the beam are selected by the trial-and-error method. The first trial can be made by considering the beam to be straight in accordance with the inequality

$$\frac{\max|M|}{W_{int}} \leqslant [\sigma] \tag{182}$$

The strength of the cross section should be checked taking into account the curvature of the beam and the axial force, using formula (181). The overstress should not exceed 5%.

If the material of the beam has different tensile and compressive strengths, the strength conditions at the dangerous cross section should comply with the requirements for both the internal and external fibres in accordance with the allowable stress values, $[\sigma_t]$ and $[\sigma_c]$.

Example 90. Given: $[\sigma] = 4000$ kgf/cm^2; $a = 20$ cm; $d = 1$ cm, and the equation for the geometric axis of the spring is $y = -a \sin \frac{\pi}{a} x$ (Fig. 168).

Determine P.

Solution. The dangerous sections are at the apexes of the sine curve

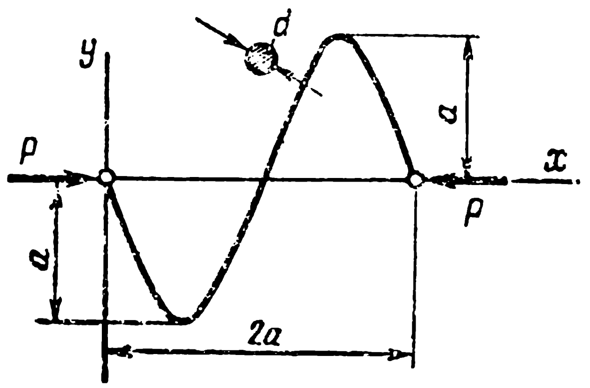


Fig. 168

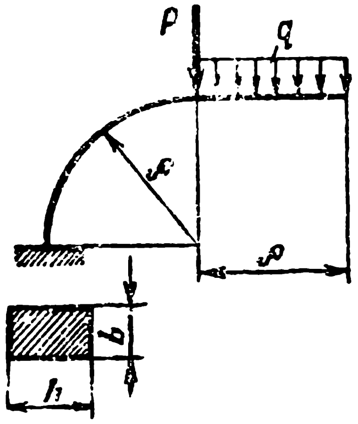


Fig. 169

where $M = Pa = 20P$ kgf-cm; $N = -P$ kgf and the radius of curvature of the geometric axis is

$$\rho_{x=\frac{a}{2}} = \left(\frac{[1+(y')^2]^{3/2}}{y''}\right)_{x=\frac{a}{2}} = \left(\frac{\left[1+\pi^2\cos^2\frac{\pi}{a}x\right]^{3/2}}{\frac{\pi^2}{a}\sin\frac{\pi}{a}x}\right)_{x=\frac{a}{2}}$$

$$= \frac{a}{\pi^2} \cong 0.1a = 2 \text{ cm}$$

For a round cross section of diameter d

$$W_{ext} = W_{int} = W \cong 0.1d^3 \cong 0.1 \text{ cm}^3; \quad F = \frac{\pi d^2}{4} = \frac{\pi}{4} \cong 0.785 \text{ cm}^2;$$

$$\alpha_{int} = \frac{1-\frac{d}{8\rho}}{1-\frac{d}{2\rho}} = \frac{1-\frac{1}{8\times 2}}{1-\frac{1}{2\times 2}} = 1.25$$

In accordance with the strength condition (181)

$$\max|\sigma_{int}| = \frac{M}{W}\alpha_{int} - \frac{N}{F} = \frac{20P}{0.1}1.25 + \frac{P}{0.785} \cong 250P \leqslant 4000$$

The permissible force

$$P \leqslant \frac{4000}{250} = 16 \text{ kgf}$$

Example 91. Let $P = 6$ kN, $q = 12$ kN/m, $\rho = 16 \times 10^{-2}$ m, $b = 3/4h$ and $[\sigma] = 200$ MN/m^2 (Fig. 169).

Determine h and b.

Solution. In the dangerous (fixed) section of the beam

$$M = P\rho + \frac{3}{2} q\rho^2 = 6 \times 10^3 \times 16 \times 10^{-2}$$

$$+ \frac{3}{2} \times 12 \times 10^3 \times 16^2 \times 10^{-4} = 1421 \text{ N·m};$$

$$N = -P - q\rho = -6 \times 10^3 - 12 \times 10^3 \times 16 \times 10^{-2} = 7920 N$$

Tentatively we determine the cross-sectional dimensions of the beam from stress calculations as for a straight beam

$$W = \frac{bh^2}{6} = \frac{h^3}{8} > \frac{M}{[\sigma]} = \frac{1421}{200 \times 10^6} \cong 7.1 \times 10^{-6} \text{ m}^3 = 7.1 \text{ cm}^3$$

whence

$$h > \sqrt[3]{7.1 \times 8} = 3.84 \text{ cm}$$

Taking into account the curvature of the beam and the axial force in the dangerous section, we take $h = 4$ cm and $b = 3$ cm. For these dimensions

$$W = \frac{3 \times 16}{6} = 8 \text{ cm}^3; \quad F = 3 \times 4 = 12 \text{ cm}^2;$$

$$\alpha_{int} = \frac{1 - \frac{h}{6\rho}}{1 - \frac{h}{2\rho}} = \frac{1 - \frac{4}{6 \times 16}}{1 - \frac{4}{2 \times 16}} \cong 1.095$$

In accordance with formula (181) the design stress for the internal fibre of the beam is

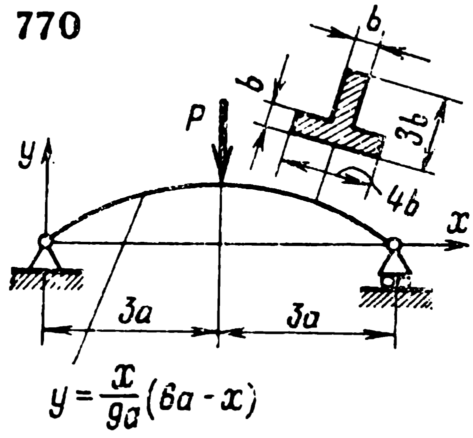
$$\max|\sigma_{int}| = \left|\frac{M}{W}\alpha_{int} + \frac{N}{F}\right| = \frac{1421}{8 \times 10^{-6}} \times 1.095$$

$$+ \frac{7920}{12 \times 10^{-4}} \cong 201.1 \times 10^6 \text{ N/m}^2 = 201.1 \text{ MN/m}^2$$

Since the overstress equals 0.55%, the selected cross section can be considered satisfactory.

Problems 770 and 771. Determine the permissible loads P and M.

770

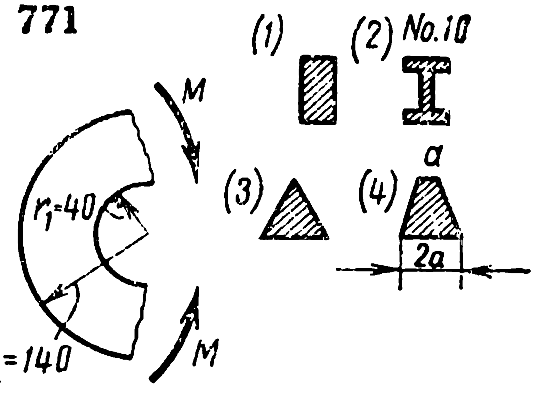


$y = \frac{x}{9a}(6a - x)$

$a = 50$ cm, $b = 20$ cm

$[\sigma]_t = 500$ kgf/cm^2; $[\sigma]_c = 1200$ kgf/cm^2

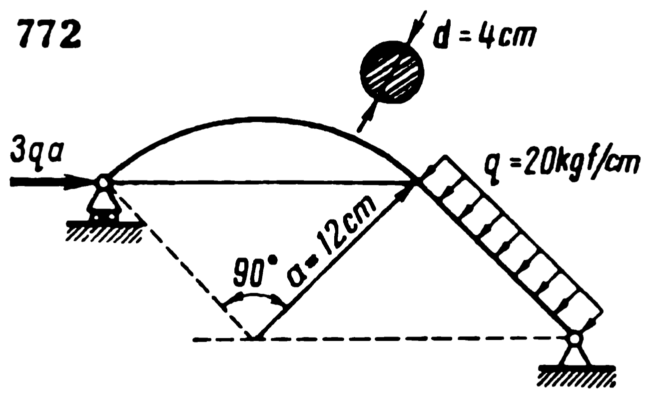
771



$F_1 = F_2 = F_3 = F_4 = 12.0$ cm^2

$[\sigma] = 1000$ kgf/cm^2

772



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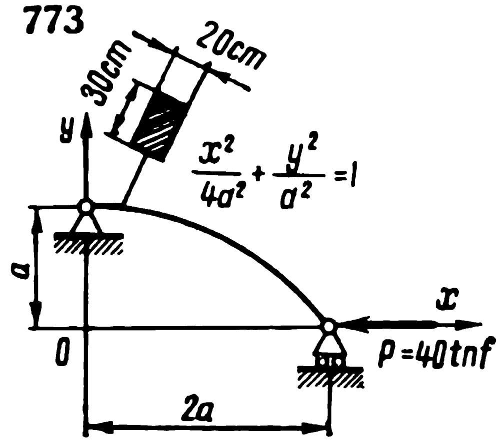


Problems 772 and 773. Check the strength of the curved beams. Consider the permissible normal stress equal to $[\sigma] = 1600$ kgf/cm^2.

CHAPTER 12. STRAIN ENERGY METHOD OF DESIGNING ELASTIC SYSTEMS

12.1.

Determining Elastic Generalized Displacements

METHOD OF THE FICTITIOUS GENERALIZED ZERO FORCE

The expression determining the amount of elastic strain energy U stored in a body or a system due to the static effect of forces can be represented in the form of a homogeneous function of the second order either of the generalized forces P_i or of the generalized displacements δ_i, if they are linearly dependent on one another.

The generalized forces P_i represent any force effects (forces, moments, groups of forces, groups of moments, etc.) which are convenient to select for strain energy calculations.

The generalized displacements δ_i are quantities which determine the displacements over which the generalized forces perform work (for example, linear displacements correspond to concentrated forces; angular displacements, to moments, etc.).

The elastic generalized displacement δ within a body or a system due to generalized forces can be determined by the use of Castigliano's formula

$$\delta = \left(\frac{\partial U_f}{\partial P_f}\right)_{P_f=0} \tag{183}$$

in which P_f is the fictitious generalized force corresponding to the sought-for generalized displacement. The force is applied to the body or system at the point at which the displacement is determined; U_f is the elastic strain energy of the body or system expressed by the homogeneous function of the second order of all the acting generalized forces P_i and of the fictitious generalized force P_f.

If there is a generalized force P, corresponding to the sought-for displacement, at the place where the generalized displacement is being determined, there is no need to apply P_f, and then

$$\delta = \frac{\partial U}{\partial P} \tag{184}$$

If $\left(\frac{\partial U_f}{\partial P_f}\right)_{P_f=0} > 0$ $\left(\text{or } \frac{\partial U}{\partial P} > 0\right)$, then the direction of the generalized displacement δ coincides with the direction of action of P_f (or P).

If $\left(\frac{\partial U_f}{\partial P_f}\right)_{P_f=0} < 0$ $\left(\text{or } \frac{\partial U}{\partial P} < 0\right)$, then the direction of the generalized displacement δ is opposite to the direction of the action of P_f (or P).

The linear displacement found by Castigliano's formula is the projection of the displacement of the point of application of the respective force on the direction of its line of action.

METHOD OF THE FICTITIOUS GENERALIZED UNIT FORCE

In the most general case of the action of forces on an elastic bar system consisting of rectilinear elements, the generalized displacements can be conveniently determined from Maxwell-Mohr's formula

$$\delta=\sum\int\frac{N\overline{N}}{EF}\,dx+\sum\int\frac{M_z\overline{M}_z}{EI_z}\,dx+\sum\int\frac{M_y\overline{M}_y}{EI_y}\,dx$$
$$+\sum\int\frac{M_t\overline{M}_t}{GI_t}\,dx+\sum k_y\int\frac{Q_y\overline{Q}_y}{GF}\,dx+\sum k_z\int\frac{Q_z\overline{Q}_z}{GF}\,dx \quad (185)$$

in which N, M_z, M_y, M_t, Q_y and Q_z = respective internal forces in an arbitrary cross section of each part of the system due to all the generalized forces acting on the system

$\overline{N}$, $\overline{M}_z$, $\overline{M}_y$, $\overline{M}_t$, $\overline{Q}_y$ and $\overline{Q}_z$ = similar forces but are due to the action on the system of only a fictitious generalized force, corresponding to the unknown (sought-for) displacement and equal to dimensionless unity

E and G = Young's and the shear moduli of elasticity of the material at a portion of the element

F = cross-sectional area where the stresses are being determined

I_z and I_y = principal centroidal moments of inertia of area F

I_t = moment of inertia of area F in torsion

k_y and k_z = coefficients depending on the shape of the cross section, which characterize the non-uniformity of the shearing stresses in bending

dx = element of the geometric axis of the portion.

Integration is carried out over the length of each portion, and summation over all the portions.

For plane hinged-bar systems with forces applied at the joints

$$\delta=\sum\frac{N\overline{N}}{EF}\,l \quad (186)$$

in which l is the length of the portions.

For systems in which the portions are subject only to torsion

$$\delta=\sum\int\frac{M_t\overline{M}_t}{GI_t}\,dx \quad (187)$$

For plane girder-frame systems in which the effect of the internal forces N and Q on the deformation is small, the formula can be rewritten as

$$\delta=\sum\int\frac{M\overline{M}}{EI}\,dx \quad (188)$$

For systems with small-curvature elements

$$\delta=\sum\int\frac{M\overline{M}}{EI}\,ds \tag{189}$$

in which ds is an element of the geometric axis of the curved portion.

For more accurate calculations

$$\delta=\sum\int\frac{N\overline{N}}{EF}\,ds+\sum\int\frac{M\overline{M}}{EI}\,ds \tag{190}$$

Example 92. Given: P, a, l, E and I (Fig. 170a). Determine the deflection f.

Solution. First we find the maximum deflection f in the middle of the beam by formula (188) which can be written as follows

$$f=\frac{1}{EI}\sum\int M\overline{M}\,dx$$

The bending moments in arbitrary sections of the beam are: due to the action of the given forces P

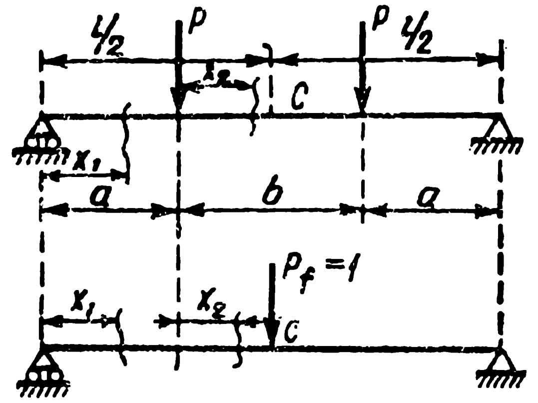


Fig. 170

$$M_a=Px_1;\quad M_b=Pa$$

due to the action of the fictitious unit force $P_f=1$ (Fig. 170b)

$$\overline{M}_a=\frac{x_1}{2};\quad \overline{M}_b=\frac{a+x_2}{2}$$

The sought-for deflection is

$$f=2\frac{P}{2EI}\left[\int_0^a x_1^2\,dx_1+a\int_0^{l/2-a}(a+x_2)\,dx_2\right]=\frac{Pa^3}{24EI}\left(3\frac{l^2}{a^2}-4\right)$$

Then we find the effect of the transverse (shearing) force on the deflection. According to formula (185) the deflection due to the transverse force in transverse bending is

$$f_Q=\frac{k}{GF}\sum\int Q\overline{Q}\,dx$$

Since in the portions of the beam the transverse forces due to the given fictitious load are equal, respectively, to

$$Q_a=P;\quad Q_b=0;\quad \overline{Q}_a=\frac{1}{2};\quad \overline{Q}_b=\frac{1}{2}$$

then

$$f_Q = 2k\frac{Pa}{2GF} = k\frac{Pa}{GF} \quad \text{and} \quad \frac{f_Q}{f} = 24k\frac{EI}{GFa^3\left(3\frac{l^2}{a^2}-4\right)}$$

$$= 24k\frac{E}{G}\frac{i^2}{a^2\left(3\frac{l^2}{a^2}-4\right)}$$

in which $i^2 = \frac{I}{F}$.

Taking into account the fact that for isotropic materials the ratio $\frac{E}{G}$ is in the range $2 \leqslant \frac{E}{G} \leqslant 3$, the ratio $\frac{f_Q}{f}$ can have the following extreme values (limits):

$$\frac{f_Q}{f} = (48 \text{ to } 72)\frac{ki^2}{a^2\left(3\frac{l^2}{a^2}-4\right)}$$

If $a \to 0$, then

$$\frac{f_Q}{f} \to (16 \text{ to } 24)\, k\frac{i^2}{l^2}$$

if $a = \frac{l}{2}$, then $\frac{f_G}{f} = (24 \text{ to } 36)\, k\frac{i^2}{l^2} = 1.5\left(\frac{f_Q}{f}\right)_{a\to 0}$

For the second case $\left(a = \frac{l}{2}\right)$ in which the transverse force has the maximum effect on the deflection, we shall consider beams of rectangular and round cross sections.

Since for a rectangle of the height h and for a circle of the diameter d, $i^2 = \frac{h^2}{12}$ or $i^2 = \frac{d^2}{16}$, $k = \frac{6}{5}$ and $k = \frac{32}{27}$, respectively, then for beams of rectangular cross section

$$\frac{f_Q}{f} = (2.4 \text{ to } 3.6)\frac{h^2}{l^2}$$

and for beams of round cross section

$$\frac{f_Q}{f} = \left(\frac{16}{9} \text{ to } \frac{8}{3}\right)\frac{d^2}{l^2} \cong (1.778 \text{ to } 2.667)\frac{d^2}{l^2}$$

It is evident from the above that f_Q can amount to 5% and more of f when

$$l \leqslant \left(\sqrt{\frac{2.4}{0.05}} \text{ to } \sqrt{\frac{3.6}{0.05}}\right) h \cong (7 \text{ to } 8)\, h$$

and

$$l \leqslant \left(\sqrt{\frac{16}{9\times 0.05}} \text{ to } \sqrt{\frac{8}{3\times 0.05}}\right) d \cong (6 \text{ to } 7)\, d$$

i.e. for not very short beams.

If, for example, we take a No. 20a rolled steel I-beam, then for this beam $\frac{E}{G} = 2.5$; $i = 8.37$ cm; $k = 2.78$ and

$$\frac{f_Q}{f} = 24 \times 2.78 \times 2.5 \frac{8.37^2}{2l^2} = \frac{5840}{l^2}$$

Hence, f_Q will amount to 5% from f beginning with

$$l = \sqrt{\frac{5840}{0.05}} \cong 341 \text{ cm} \cong 17h$$

Example 93. Given: P, a, b, c, F, I_t, E and G (Fig. 171a). Determine δ, the vertical displacement of the point of application of force P, taking into account all the kinds of deformation of the bar.

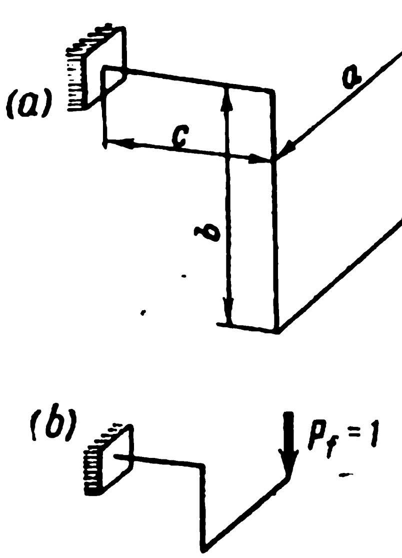


Fig. 171

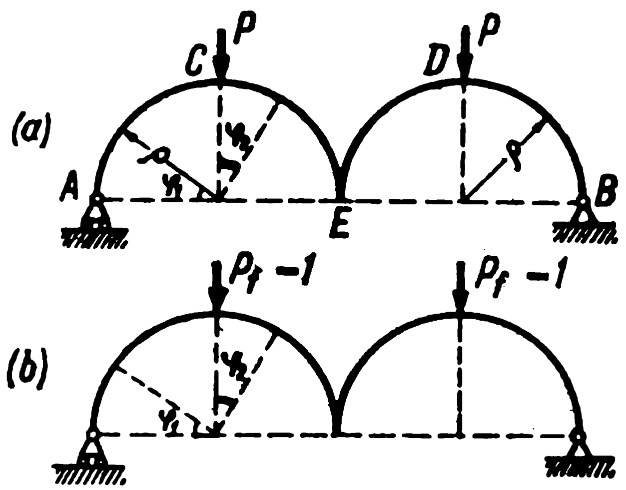


Fig. 172

Solution. The sought-for displacement δ can be found by formula (185).

Only portion b is in tension. The axial internal forces due to forces P and $P_f = 1$ (Fig. 171b) are $N = P$ and $\bar{N} = 1$.

All the portions of the bar are subject to bending. The bending moments and transverse (shearing) forces in arbitrary cross sections of the portions of the bar due to the action of forces P and $P_f = 1$ are equal, respectively, to: $M_a = Px$; $\bar{M}_a = x$; $M_b = Pa$; $\bar{M}_b = a$; $M_c = Px$; $\bar{M}_c = x$; $Q_a = P$; $\bar{Q}_a = 1$; $Q_b = 0$; $\bar{Q}_b = 0$; $Q_c = P$ and $\bar{Q}_c = 1$.

Only portion c is subject to torsion. The torques due to forces P and $P_f = 1$ equal: $M_t = Pa$ and $\bar{M}_t = a$.

Substituting the found internal forces into formula (185) we obtain the following value of the sought-for displacement

$$\delta=\frac{Pb}{EF}+\frac{P}{EI}\left(\int_0^a x^2\,dx+a^2b+\int_0^c x^2\,dx\right)+\frac{kP}{GF}(a+c)$$

$$+\frac{Pa^2c}{GI_t}=P\left\{\frac{1}{E}\left[\frac{b}{F}+\frac{1}{I}\left(\frac{a^3}{3}+a^2b+\frac{c^3}{3}\right)\right]+\frac{1}{G}\left[\frac{k}{F}(a+c)+\frac{a^2c}{I_t}\right]\right\}$$

Example 94. Given: P, ρ, E and I for a curved beam of small curvature (Fig. 172*a*).

Determine δ_C, the vertical displacement of cross section C.

Solution. To avoid violating the symmetry of the system about the vertical axis passing through point E, we shall apply $P_f = 1$ in sections C and D (Fig. 172*b*) and consider only one half of the system.

In arbitrary cross sections specified by angles φ_1 and φ_2 the bending moments:
due to the action of forces P are

$$M_I = -P\rho\,(1-\cos\varphi_1);\quad M_{II} = -P\rho$$

due to forces $P_f = 1$ are

$$\overline{M}_I = -\rho\,(1-\cos\varphi_1);\quad \overline{M}_{II} = -\rho$$

Then we find the required displacement by the use of formula (189)

$$\delta_C=\delta_D=\frac{P}{EI}\left(\int_0^{\pi/2} M_I\overline{M}_I\,d\varphi_1+\int_0^{\pi/2} M_{II}\overline{M}_{II}\,d\varphi_2\right)$$

$$=\frac{P\rho^3}{EI}\left[\int_0^{\pi/2}(1-\cos\varphi_1)^2\,d\varphi_1+\int_0^{\pi/2}d\varphi_2\right]$$

$$=\left(\frac{5}{4}\pi-2\right)\frac{P\rho^3}{EI}\cong 1.927\,\frac{P\rho^3}{EI}$$

Example 95. Given: a load of intensity q, uniformly distributed along the horizontal, ρ, $P = 2q\rho$, E and I (Fig. 173*a*).

Determine displacement δ of the movable support.

Solution. From the conditions of statics it follows for the given system that

$$B_x = 4q\rho;\quad Ay = 4q\rho;\quad B_y = 2q\rho$$

The bending moments in the portions of the system due to the given forces are

$$M_I = Px = 2q\rho x;\quad M_{II} = -P\rho\,(1+\sin\varphi)-A_y\rho\,(1-\cos\varphi)$$

$$+\frac{q\rho^2}{2}(1-\cos\varphi)^2=\frac{q\rho^2}{2}(\cos^2\varphi+6\cos\varphi-4\sin\varphi-11);$$

$$M_{III} = B_x x = 4q\rho x$$

From the conditions of statics for the auxiliary system it follows that (Fig. 173*b*):

$$B'_x = 1; \quad A'_y = \frac{1}{2}; \quad B'_y = \frac{1}{2}$$

The bending moments in the portions of the auxiliary system are

$$\overline{M}_I = x; \quad \overline{M}_{II} = -\rho(1+\sin\varphi) - \frac{1}{2}\rho(1-\cos\varphi) = \frac{\rho}{2}(\cos\varphi - 2\sin\varphi - 3);$$

$$\overline{M}_{III} = x$$

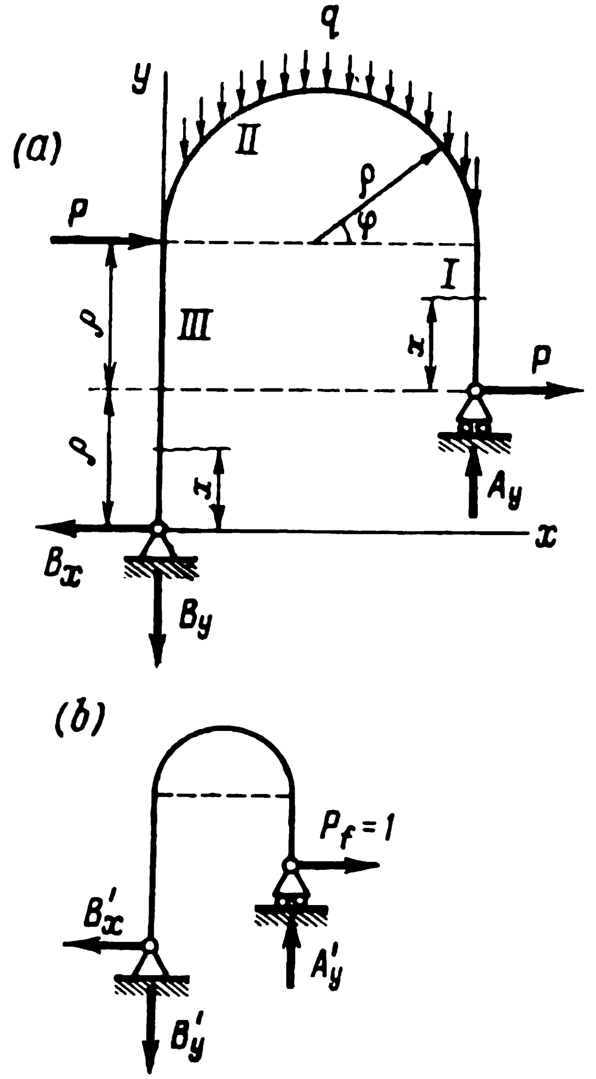


Fig. 173

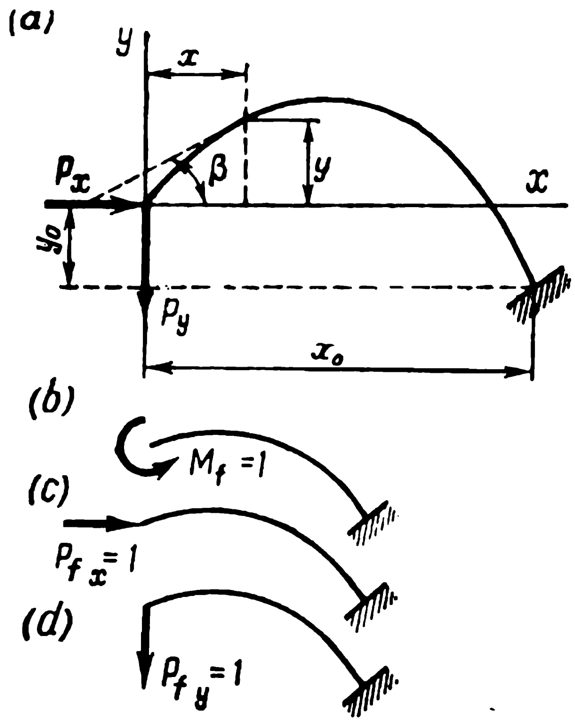


Fig. 174

Using formula (189) we find the sought-for displacement

$$\delta = \frac{1}{EI}\left[2q\rho\int_0^{\rho} x^2dx + \frac{q\rho^4}{4}\int_0^{\pi}(\cos^2\varphi + 6\cos\varphi - 4\sin\varphi - 11)\right.$$

$$\left.\times(\cos\varphi - 2\sin\varphi - 3)\,d\varphi + 4q\rho_0\int_0^{2\rho} x^2dx\right]$$

$$= \frac{q\rho^4}{EI}\left[\frac{2}{3} + \frac{1}{4}\left(\frac{3}{2}\pi + 64 - \frac{4}{3} + 4\pi + 33\pi\right) + \frac{32}{3}\right]$$

$$= \frac{q\rho^4}{EI}\left(27 + \frac{77}{8}\pi\right) \cong 57.24\frac{q\rho^4}{EI}$$

Example 96. Given: P_x; P_y; the length s of the geometric axis of the beam bent into an arbitrary curve, E and I (Fig. 174*a*).

Determine the angle of rotation θ, and the horizontal δ_x and vertical δ_y displacement of the section at which forces P_x and P_y are applied.

Solution. In an arbitrary cross section of the beam, with the coordinates of its centre of gravity being x, y, and having the angle β between the tangent to the axis of the beam at this section and the x-axis, the bending moments and the axial internal forces are:
due to the given forces P_x and P_y (Fig. 174*a*):

$$M = P_x y + P_y x; \quad N = -P_x \cos\beta + P_y \sin\beta$$

due to $M_f = 1$ (Fig. 174*b*): $\overline{M} = 1$; $\overline{N} = 0$
due to $P_{f_x} = 1$ (Fig. 174*c*): $\overline{M} = y$; $\overline{N} = -\cos\beta$
due to $P_{f_y} = 1$ (Fig. 174*d*): $\overline{M} = x$; $\overline{N} = \sin\beta$

Using formula (189) we find

$$\theta = \frac{1}{EI}\left(P_x \int_0^s y\,ds + P_y \int_0^s x\,ds\right) = \frac{1}{EI}(P_x S_x + P_y S_y)$$

in which $S_x = \int_0^s y\,ds$ and $S_y = \int_0^s x\,ds$ are the static moments of the arc s about axes x and y;

$$\delta_x = \frac{1}{EI}\left(P_x \int_0^s y^2\,ds + P_y \int_0^s xy\,ds\right)$$

$$+ \frac{1}{EF}\left(P_x \int_0^s \cos^2\beta\,ds - P_y \int_0^s \sin\beta\cos\beta\,ds\right)$$

$$= \frac{1}{EI}(P_x I_x + P_y I_{xy}) + \frac{1}{EF}\left(P_x \int_0^s \cos^2\beta\,ds - \frac{1}{2}P_y \int_0^s \sin 2\beta\,ds\right)$$

in which $I_x = \int_0^s y^2\,ds$ and $I_{xy} = \int_0^s xy\,ds$ are the axial moment of inertia and the product of inertia of the arc s about axis x and axes xy;

$$\delta_y = \frac{1}{EI}\left(P_x \int_0^s yx\,ds + P_y \int_0^s x^2\,ds\right)$$

$$+ \frac{1}{EF}\left(-P_x \int_0^s \cos\beta\sin\beta\,ds + P_y \int_0^s \sin^2\beta\,ds\right)$$

$$= \frac{1}{EI}(P_x I_{xy} + P_y I_y) + \frac{1}{EF}\left(-\frac{P_x}{2} \int_0^s \sin 2\beta\,ds + P_y \int_0^s \sin^2\beta\,ds\right)$$

in which $I_y = \int\limits_0^s x^2\,ds$ is the axial moment of inertia of the arc s about axis y.

Let us consider a beam of small curvature whose geometric axis is bent to the arc of a parabola: $y = -\frac{x^2}{2a}$; $P_x = 0$ and $P_y = P$ (Fig. 175).

Since

$$\frac{dy}{dx} = -\frac{x}{a} \text{ and } ds = \frac{dx}{a}\sqrt{a^2+x^2}$$

then

$$S_y = \frac{1}{a}\int\limits_0^a x\sqrt{a^2+x^2}\,dx = \frac{1}{a}\left|\frac{1}{3}\sqrt{(a^2+x^2)^3}\right|_0^a = \frac{a^2}{3}(2\sqrt{2}-1);$$

$$I_{xy} = -\frac{1}{2a^2}\int\limits_0^a x^3\sqrt{a^2+x^2}\,dx = -\frac{1}{2a^2}\left|\frac{1}{5}\sqrt{(a^2+x^2)^5}\right.$$

$$\left.-\frac{a^2}{3}\sqrt{(a^2+x^2)^3}\right|_0^a = \frac{a^3}{15}(\sqrt{2}+1);$$

$$I_y = \frac{1}{a}\int\limits_0^a x^2\sqrt{a^2+x^2}\,dx = \frac{1}{a}\left|\frac{x}{4}\sqrt{a^2+x^2} - \frac{a}{8}\left[x\sqrt{a^2+x^2}\right.\right.$$

$$\left.\left.+a^2\ln\left(x+\sqrt{a^2+x^2}\right)\right]\right|_0^a = \frac{a^3}{8}\left(3\sqrt{2}+\ln\frac{1}{\sqrt{2}+1}\right)$$

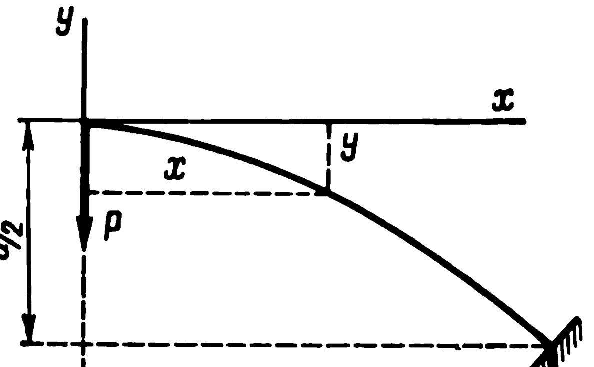


Fig. 175

Substituting the obtained values into the formulas for determining θ, δ_x and δ_y in Example 96 we obtain:

$$\theta = \frac{PS_y}{EI} = \frac{2\sqrt{2}-1}{3}\times\frac{Pa^2}{EI} \cong 0.609\,\frac{Pa^2}{EI};$$

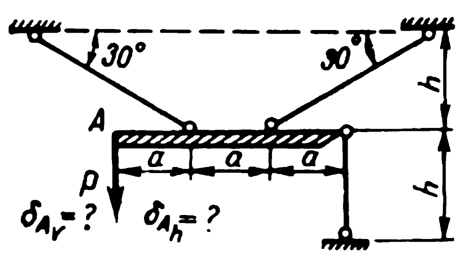
$$\delta_x = \frac{PI_{xy}}{EI} = -\frac{\sqrt{2}+1}{15}\times\frac{Pa^3}{EI} \cong -0.161\frac{Pa^3}{EI};$$

$$\delta_y = \frac{PI_y}{EI} = \frac{1}{8}\left(3\sqrt{2}+\ln\frac{1}{\sqrt{2}+1}\right)\frac{Pa^3}{EI} \cong 0.432\,\frac{Pa^3}{EI}$$

Problems 774 through 787. Determine the generalized displacements shown in the drawings.

Here and subsequently, in all problems on the determination of generalized displacements, assume that the rigidity of the bar cross sections is known. Unless otherwise specified, assume that the moduli of elasticity of the materials and the geometric characteristics of the cross sections of all the elements of the systems and of all the portions of the bars to be the same. In Problems 782 and 783, in determining the deflection, take the transverse (shearing) force into account.

774



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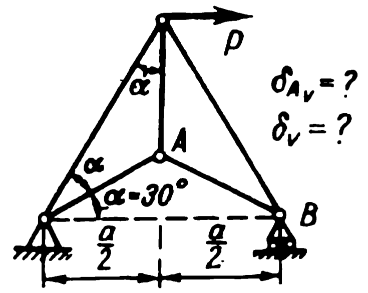

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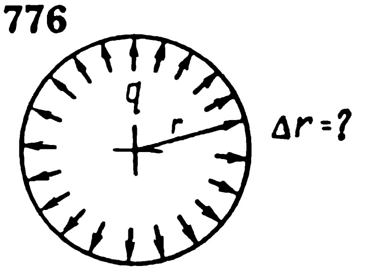

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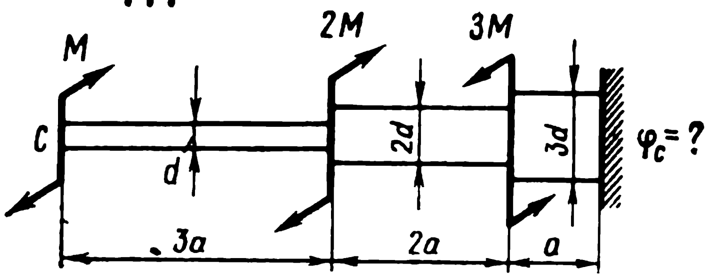

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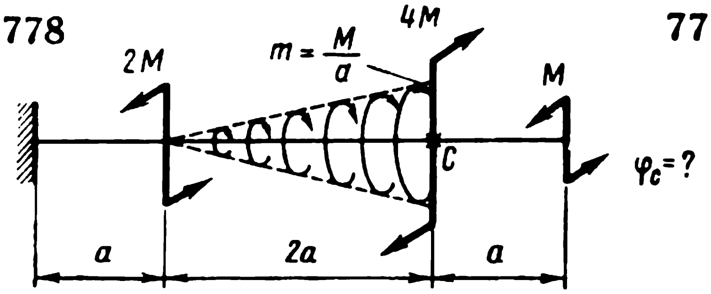

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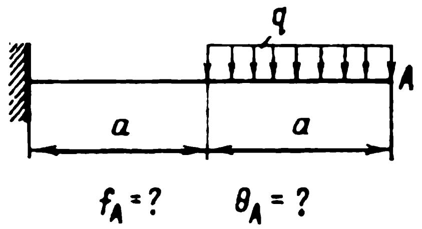

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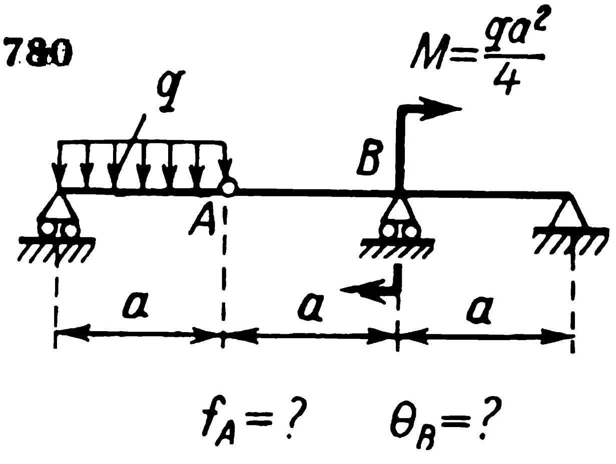

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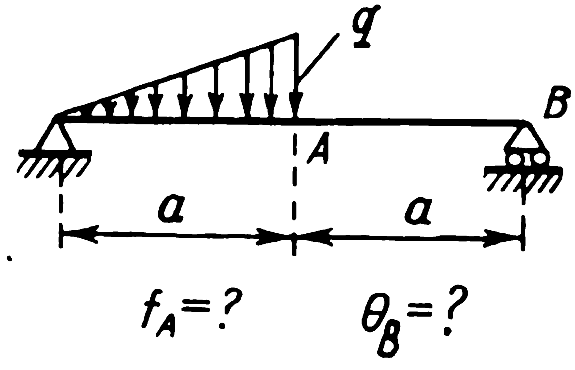

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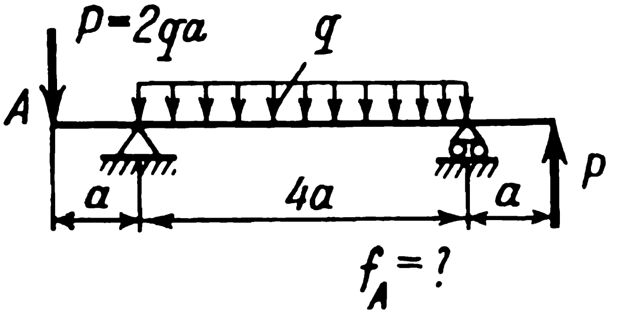

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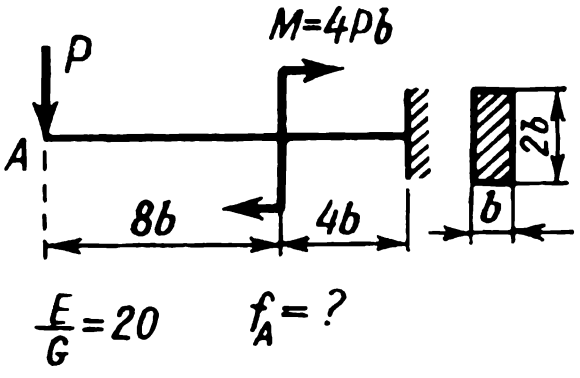

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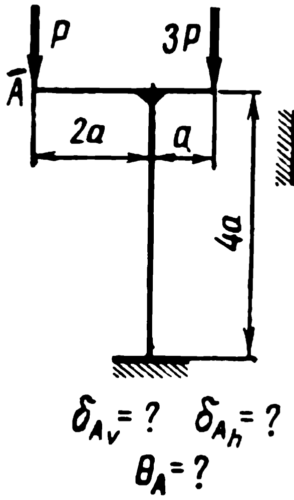

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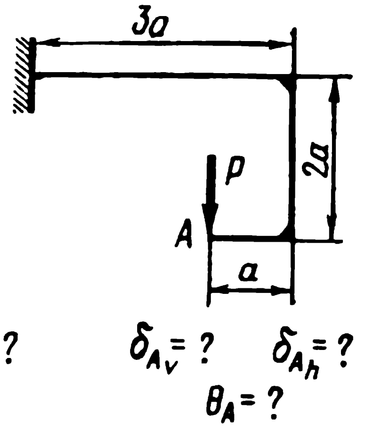

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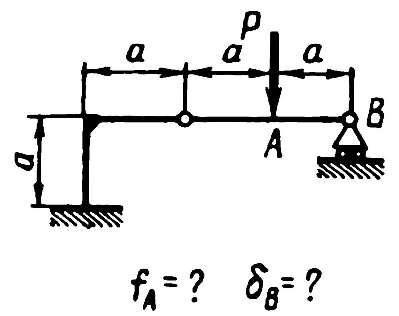

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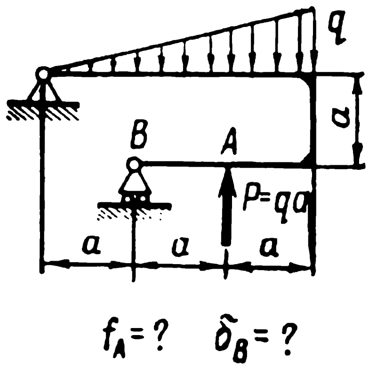
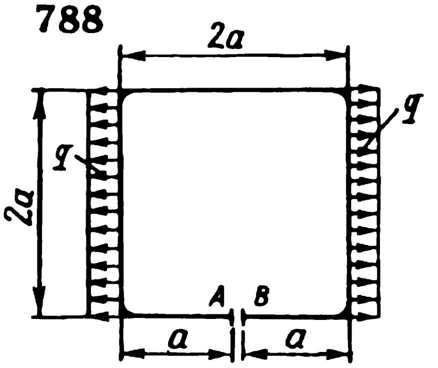


Problems 788, 789 and 790. Determine the change in distance δ_{AB} between cross sections A and B.

788



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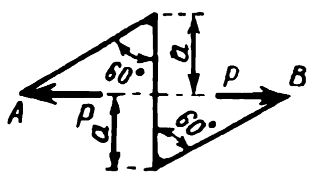

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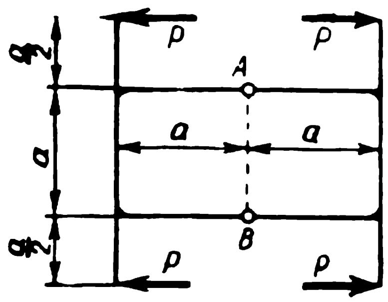
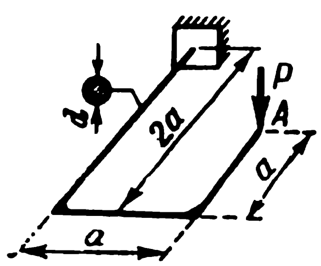


Problems 791, 792 and 793. Determine the vertical displacement δ of cross section A.

791



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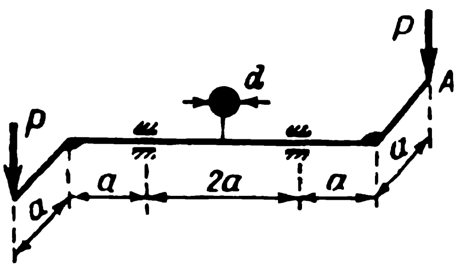

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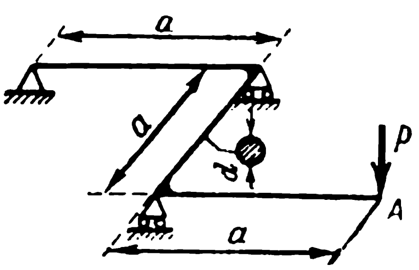
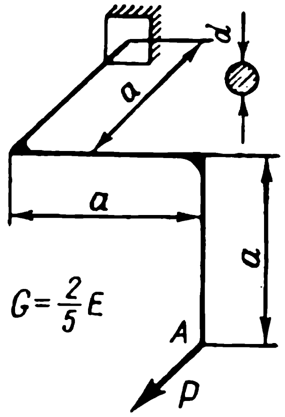


Problems 794, 795 and 796. Determine the linear displacement δ of cross section A in the direction of force P applied at this cross section.

794



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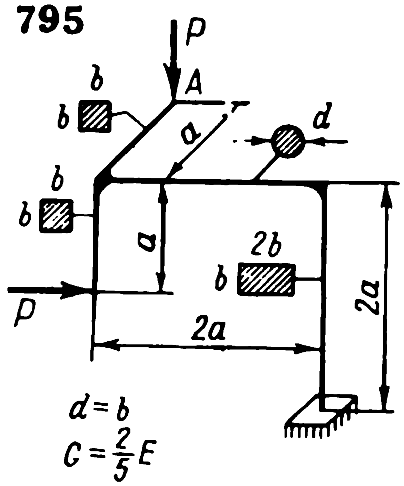

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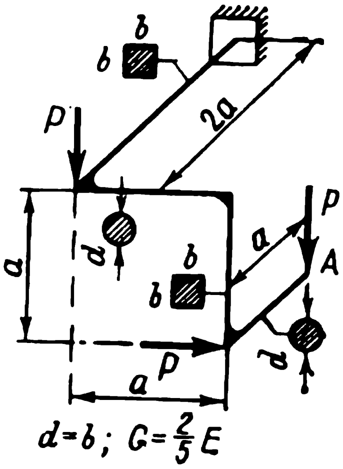
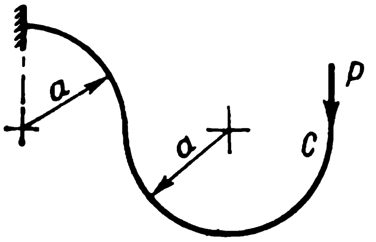


Problems 797 through 805. Determine the linear vertical δ_v, horizontal δ_h and angular θ displacements of section C.

In Problem 801 determine the angle of rotation of the section to the left of the joint.

797



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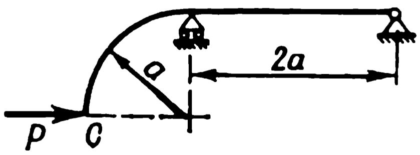

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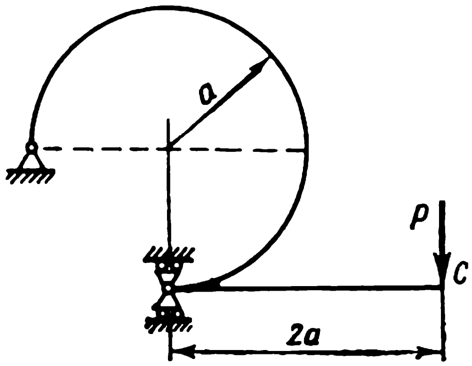

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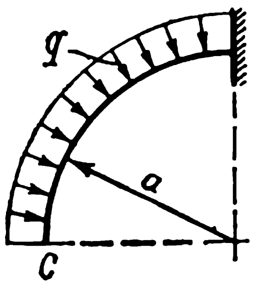

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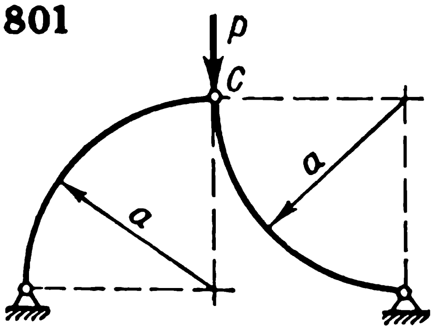

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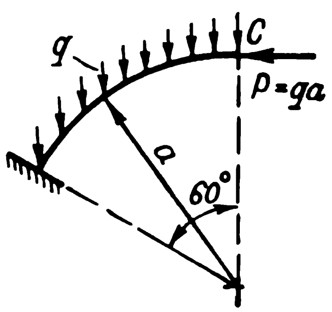

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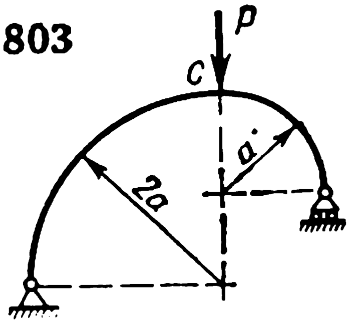

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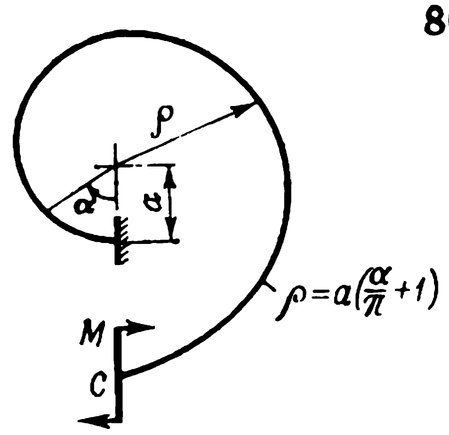

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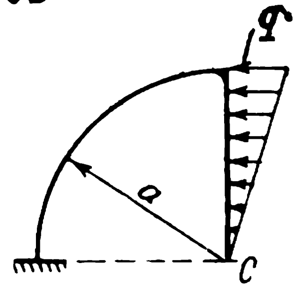
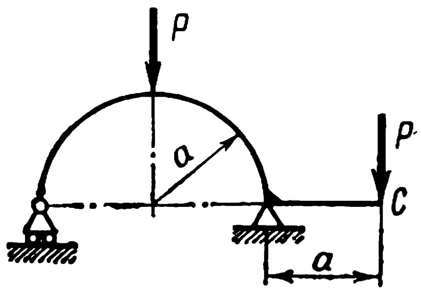


Problems 806 through 811. Determine the displacement δ of the movable support and the vertical displacement δ_v of section C. In Problem 811 determine only δ.

806



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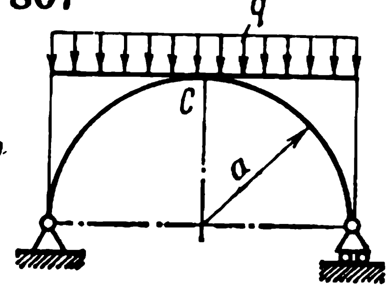

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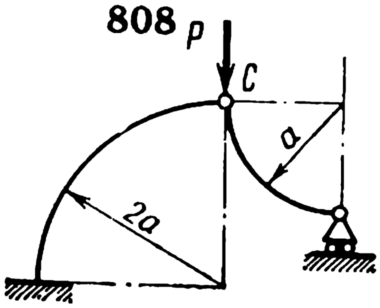

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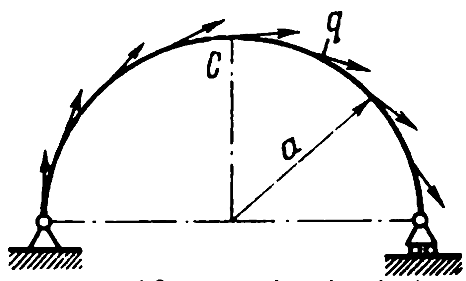
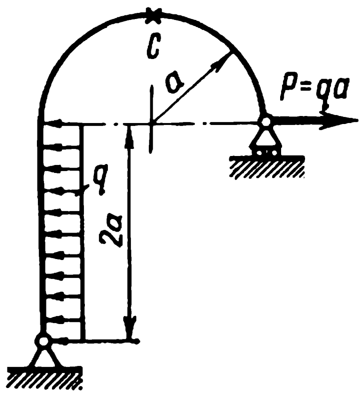
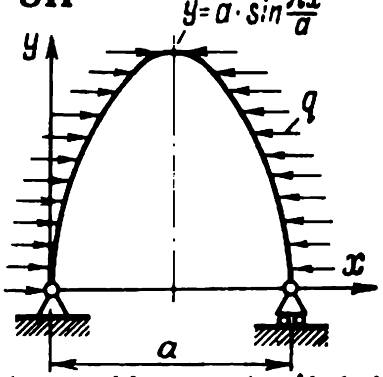


Load uniformly distributed along a tangent to the arc

810



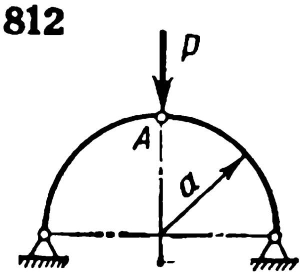
811



Load uniformly distributed along a vertical projection

Problems 812 through 816. Determine the vertical displacement δ of section A.

812



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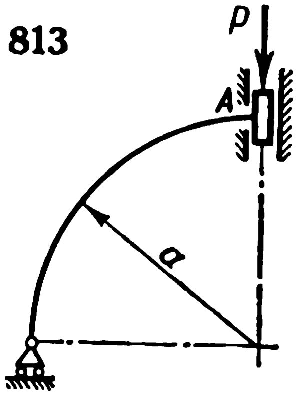

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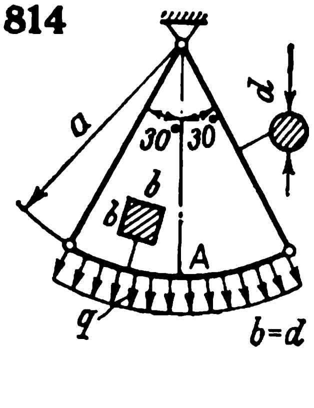

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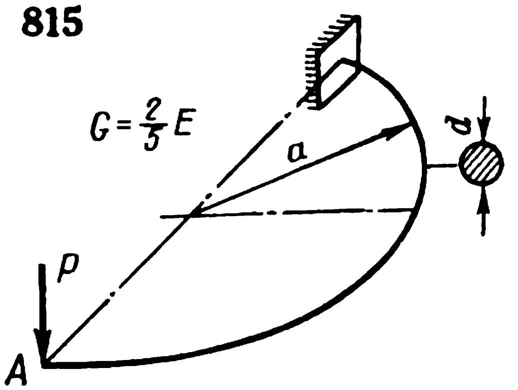

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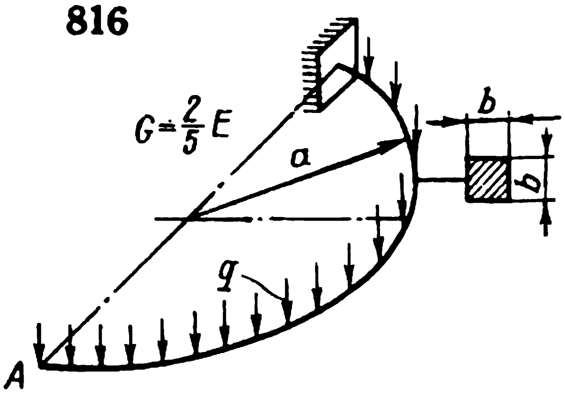
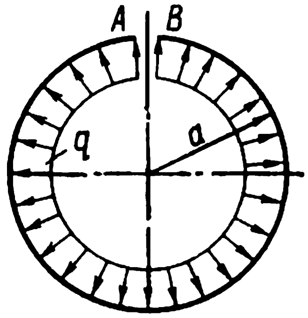


Problems 817, 818 and 819. Determine the change in distance δ_{AB} between cross sections A and B.

817



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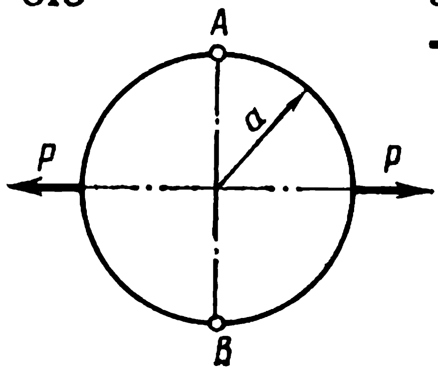

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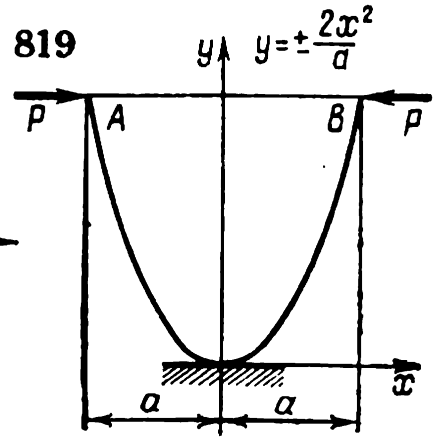
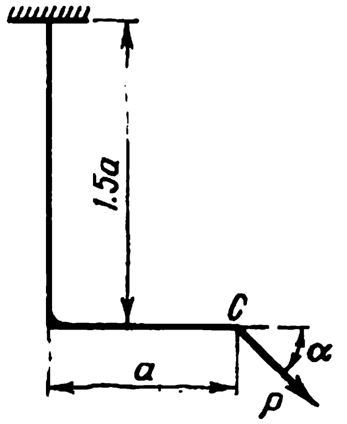
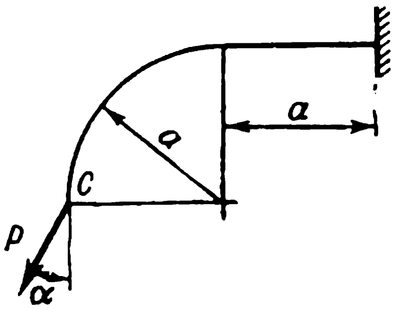


Problems 820 and 821. Determine the magnitude of angle α at which cross section C will have only vertical displacement.

820



821



METHOD OF MULTIPLICATION OF DIAGRAMS—VERESHCHAGIN'S RULE

If the rigidity of a cross section of the bar is constant over a certain portion, each integral of the Maxwell-Mohr formula (185) can be evaluated by finding the product of the area ω of the diagram of internal force due to the given forces (Fig. 176) by the ordinate ξ of the diagram of a similar internal force due to the fictitious unit genera-

lized force (which must be rectilinear). This ordinate (ξ) should be directly below the centroid of the first diagram.

This is called Vereshchagin's rule. In practice it is used for determining linear and angular displacements in girder-frame systems due to bending moments. The formula for the displacement is written in the following form:

$$\delta = \sum \frac{\omega\xi}{EI} \tag{191}$$

in which the summation is carried out over all the portions of the system.

The portions should be differentiated not only by the load, but by the sign of the diagram M or $\overline{M}$ and by the uniformity of the cross section. For diagrams of M and $\overline{M}$ with the same sign the product $\omega\xi > 0$. For diagrams with different signs, $\omega\xi < 0$.

For this reason the position of the diagrams for M and $\overline{M}$ can be arbitrary with respect to the lines of their zero values on the portions provided the sign from one side is the same.

If the diagrams for both M and $\overline{M}$ are rectilinear, it makes no difference for which of them the area ω or ordinate ξ is found.

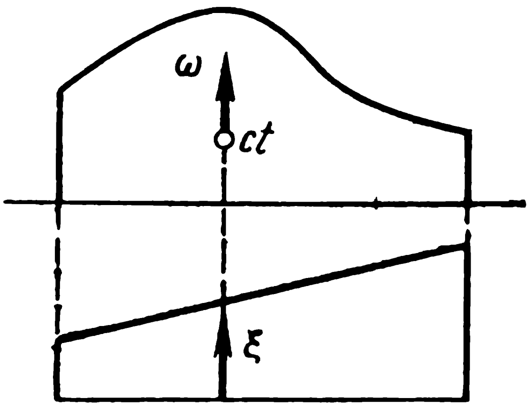


Fig. 176

A diagram of complex outline for M can be broken down into parts for which the areas ω_1, ω_2, ω_3 . . . (Fig. 177) and the centroids are

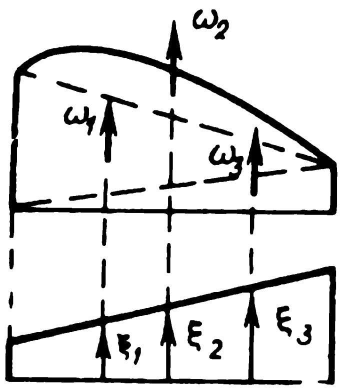


Fig. 177

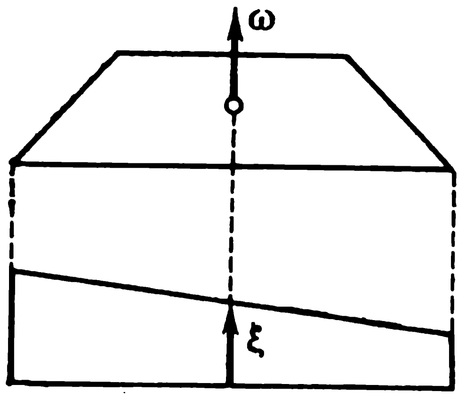


Fig. 178

found separately. Directly under the centroid of each part the ordinates of ξ_1, ξ_2, ξ_3 . . . should be measured, then

$$\omega\xi = \omega_1\xi_1 + \omega_2\xi_2 + \omega_3\xi_3 + \ldots$$

In cases when the diagram of M is represented by one straight line over several portions of the bar (Fig. 178), the area ω of the whole

diagram for M can be multiplied by the corresponding coordinate ξ of the diagram for $\overline{M}$.

If the cross section is variable over a portion of the bar, the area ω of the diagram of the reduced moment M_r is calculated instead of that for the true bending moment M. Thus

$$M_r = M\frac{I_0}{I}$$

in which I_0 is the moment of inertia of a constant cross section to which the portion is conventionally reduced and I is the moment of inertia of the variable cross section.

Example 97. Given: P, a, E and I (Fig. 179*a*).

Determine δ_A, the horizontal displacement of section A.

Solution. Figures 179*c* and *d* are the diagrams of the bending moments. The hatched portions are due to the given forces, the unhatched portions are due to the fictitious force $P_f = 1$ applied at section A (Fig. 179*b*) and directed horizontally to the right.

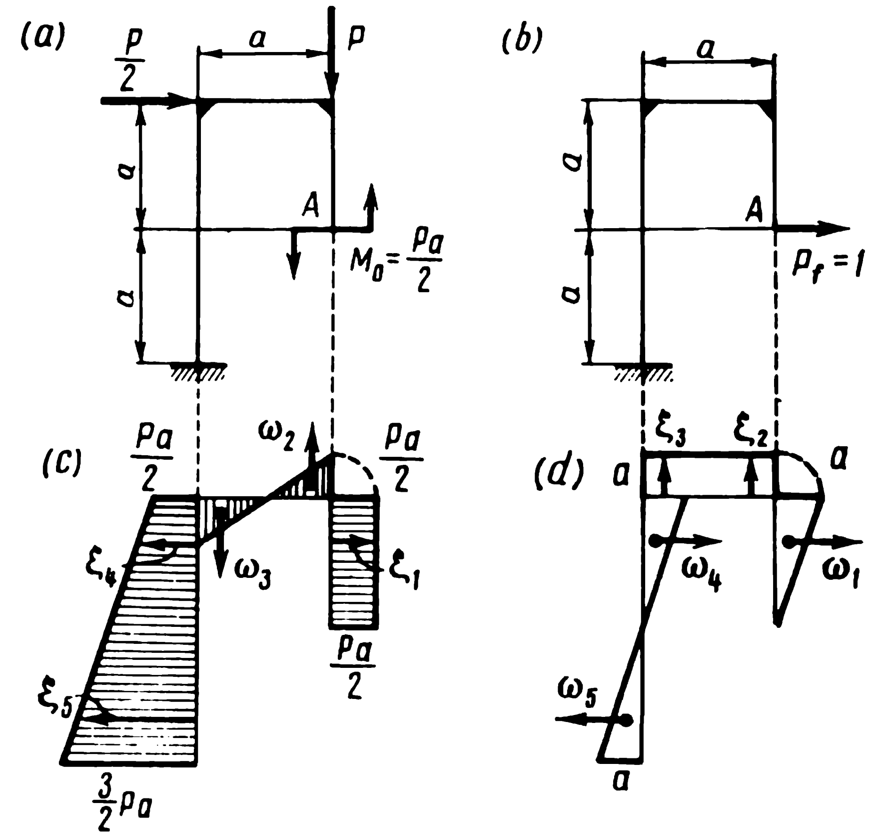


Fig. 179

Since

$$\omega_1 = \frac{a^2}{2}; \quad \xi_1 = \frac{Pa}{2}; \quad \omega_2 = \frac{Pa^2}{8}; \quad \xi_2 = a; \quad \omega_3 = \frac{Pa^2}{8};$$

$$\xi_3 = a; \quad \omega_4 = \frac{a^2}{2}; \quad \xi_4 = \frac{2}{3}Pa; \quad \omega_5 = \frac{a^2}{2}; \quad \xi_5 = \frac{4}{3}Pa$$

then, taking the sign rule into account, the required displacement can be found from formula (191)

$$\delta_A = \frac{1}{EI}(\omega_1\xi_1 + \omega_2\xi_2 - \omega_3\xi_3 - \omega_4\xi_4 + \omega_5\xi_5) = \frac{7}{12} \times \frac{Pa^3}{EI}$$

Example 98. Given: P, a, E and I (Fig. 180a). Determine θ_A and f_C.

Solution. Figure 180b is the diagram for M due to the given force P.

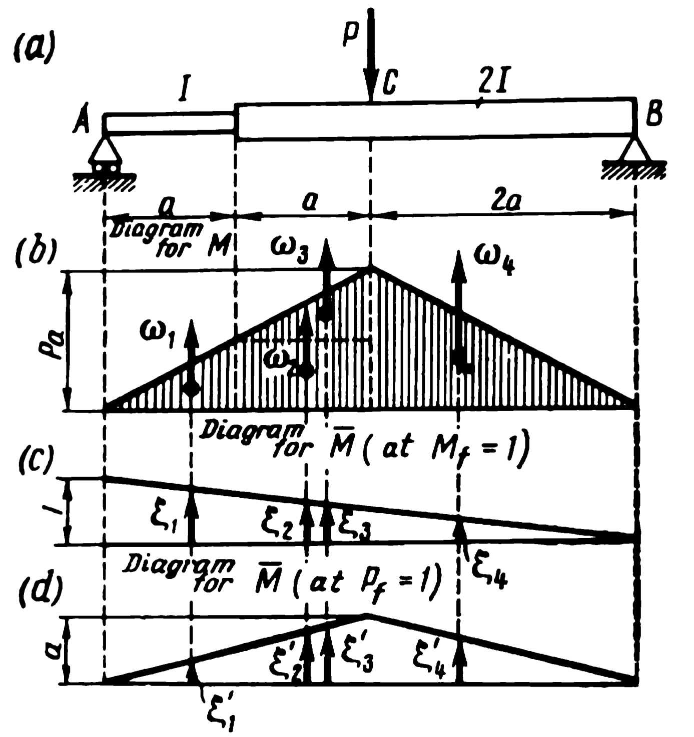


Fig. 180

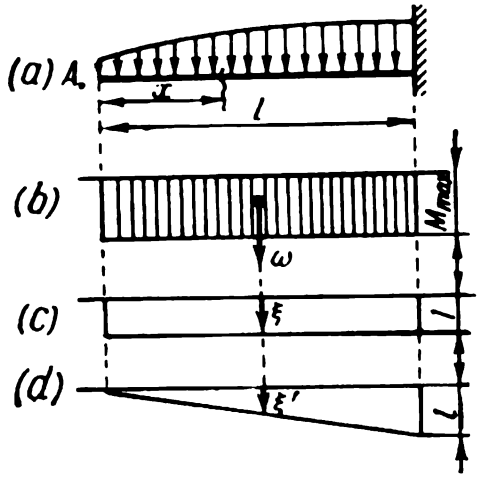


Fig. 181

In Fig. 180c the diagram for $\bar{M}$ is due to the fictitious moment $M_f = 1$ applied at section A and directed clockwise, and in Fig. 180d, the diagram for $\bar{M}$ is due to the fictitious force $P_f = 1$ applied at section C and directed vertically downward.

Since

$$\omega_1 = \frac{Pa^2}{4}; \quad \xi_1 = \frac{5}{6}; \quad \xi_1' = \frac{a}{3}; \quad \omega_2 = \frac{Pa^2}{2}; \quad \xi_2 = \frac{5}{8};$$

$$\xi_2' = \frac{3}{4}a; \quad \omega_3 = \frac{Pa^2}{4}; \quad \xi_3 = \frac{7}{12}; \quad \xi_3' = \frac{5}{6}a; \quad \omega_4 = Pa^2;$$

$$\xi_4 = \frac{1}{3}; \quad \xi_4' = \frac{2}{3}a$$

then, taking into account the difference in the moments of inertia of the cross sections in the two portions, the required displacements, using formula (191), are

$$\theta_A = \frac{1}{EI}\left[\omega_1\xi_1 + \frac{1}{2}(\omega_2\xi_2 + \omega_3\xi_3 + \omega_4\xi_4)\right] = \frac{29}{48}\frac{Pa^2}{EI};$$

$$f_C = \frac{1}{EI}\left[\omega_1\xi_1' + \frac{1}{2}(\omega_2\xi_2' + \omega_3\xi_3' + \omega_4\xi_4')\right] = \frac{17}{24}\frac{Pa^3}{EI}$$

Example 99. Given: l, E and $[\sigma]$ for a beam of constant strength with a constant height h (Fig. 181a).

Determine θ_A and f_A.

Solution. In accordance with the condition for designing a beam of constant strength it follows that

$$W_x = \frac{I_x}{\frac{h}{2}} = \frac{M_x}{[\sigma]}$$

In the dangerous section

$$W = \frac{2I}{h} = \frac{M_{max}}{[\sigma]}$$

If the beam is conventially reduced to a constant dangerous cross section, then $I_0 = I$ and the reduced bending moment is

$$M_r = M_x \frac{I_0}{I_x} = M_{max} = \text{const}$$

The diagram for the reduced moment (Fig. 181b) is a rectangle of the height

$$M_{max} = \frac{2I[\sigma]}{h}$$

Figures 181c and d are diagrams for M due to $M_f = 1$ and $P_f = 1$ applied at section A and directed counterclockwise for $M_f = 1$ and vertically downward for $P_f = 1$.

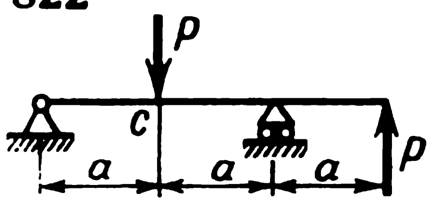
Since $\omega = M_{max} l = \frac{2I[\sigma]}{h} l$; $\xi = 1$ and $\xi' = \frac{l}{2}$,

then the required displacements are

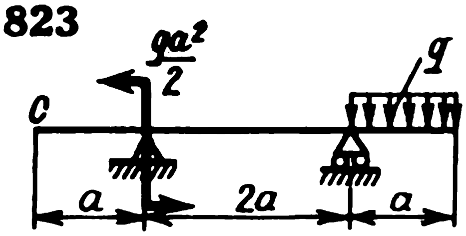
$$\theta_A = \frac{\omega\xi}{EI} = \frac{M_{max} l}{EI} = \frac{2[\sigma]l}{Eh}; \quad f_A = \frac{\omega\xi'}{EI} = \frac{M_{max} l^2}{2EI} = \frac{[\sigma]l^2}{Eh}$$

Problems 822 through 826. Determine the deflection f and angle of rotation θ of section C.

822



823



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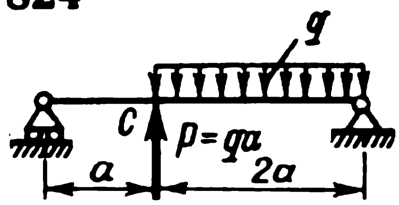
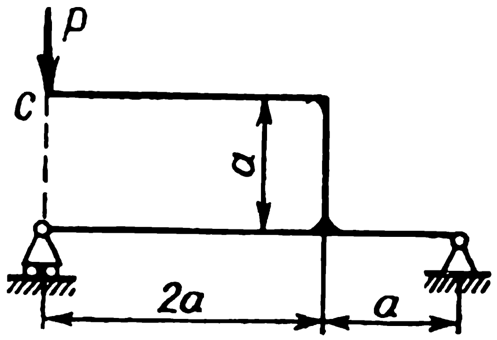


Problems 827 through 830. Determine the vertical δ_v and horizontal δ_h displacements of the section at which force P is applied.

825



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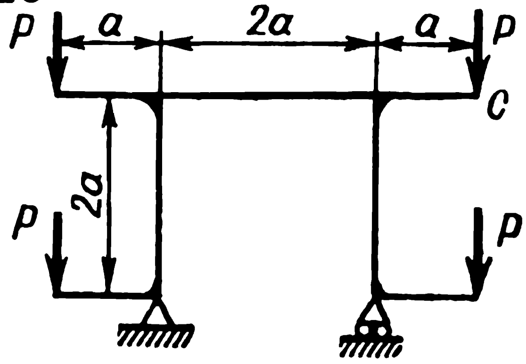

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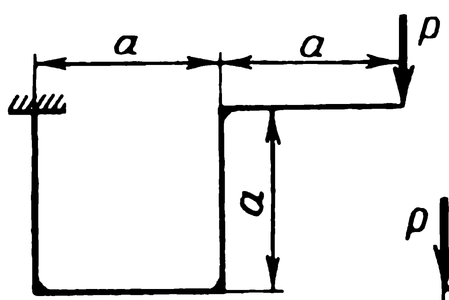

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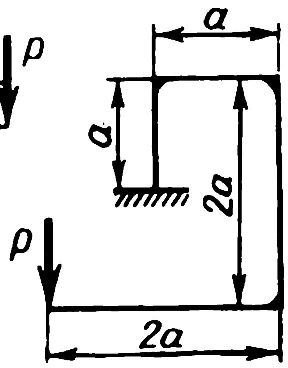

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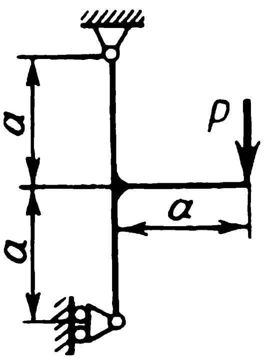

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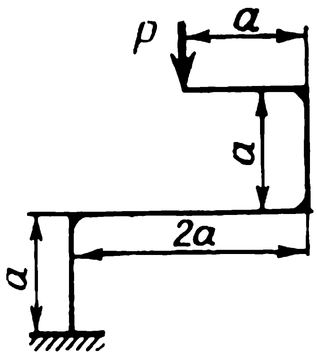
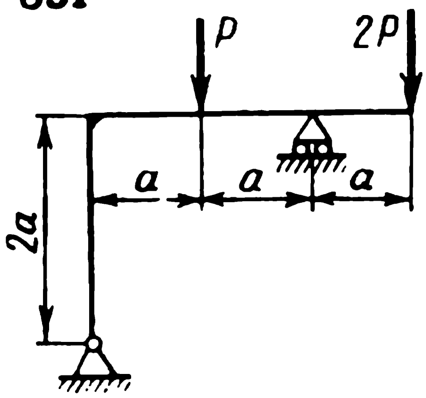


Problems 831, 832 and 833. Determine the displacement δ of the movable hinged support.

831



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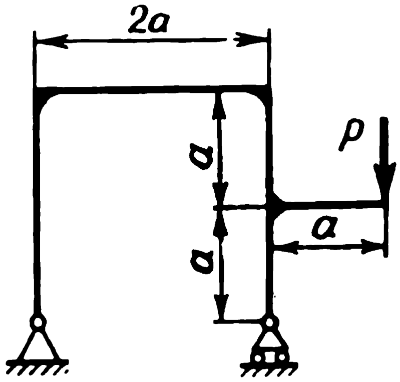

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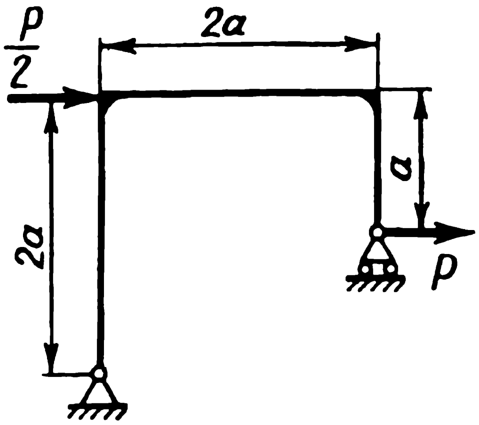


12.2.

Analysis of Statically Indeterminate Systems

PRINCIPLE OF LEAST WORK

Statically indeterminate elastic systems can be analysed by applying the principle of least work. In accordance with this principle the redundant generalized unknown forces are of values such that the generalized forces acting on the system accomplish the least amount of work.

The problems are solved according to the following general procedure.

The statically indeterminate system is relieved of redundant constraints until it becomes statically determinate, but geometrically unchanged. This is called the *basic system.*

To make the given system equivalent to the basic one, the latter is loaded with all the acting P_i and all the redundant unknown X_i generalized forces.

Then we determine the elastic strain energy of the basic system as a second-order function of P_i and X_i.

Since the generalized displacements corresponding to the redundant unknown generalized forces equal zero, equations of the following form can be written

$$\frac{\partial U}{\partial X_i}=0\,(i=1,\ 2,\ 3\ \ldots) \tag{192}$$

All the redundant generalized forces X_i are determined from these equations.

Equations (192) are the conditions for minimum elastic strain energy of the system as a function of the redundant unknown generalized forces.

For bar systems, the equations of the principle of least work can be expressed by the Maxwell-Mohr formula.

If the system consists of straight elements in tension, compression, transverse bending and torsion, each of equations (192) can be written in the following form:

$$\sum\int\frac{N\overline{N}}{EF}dx+\sum\int\frac{M\overline{M}}{EI}dx+\sum k\int\frac{Q\overline{Q}}{GF}dx + \sum\int\frac{M_t\overline{M}_t}{GI_t}dx=0 \tag{193}$$

in which N, M, Q and M_t = respective internal forces and torque in an arbitrary cross section of each portion of the basic equivalent system due to all the given P_i and all the redundant unknown X_i generalized forces

$\overline{N}$, $\overline{M}$, $\overline{Q}$ and $\overline{M}_t$ = similar internal forces and torque in the basic system but due to only one of the redundant unknown generalized forces $X_i=1$.

Thus, in order to solve an n-fold statically indeterminate problem, the system should be dealt with in the $n+1$ state: the basic equivalent state due to the action of all the forces P_i and X_i and n auxiliary forces, each of which is under the action of only one force of $X_i=1$.

For planar hinged-bar systems with forces applied at the joints, equations (193) can be simplified to

$$\sum\int\frac{N\overline{N}}{EF}dx=0 \tag{194}$$

For planar girder-frame systems in which the values of the axial internal forces N and transverse (shearing) forces Q are small, the following simplified equations can be used:

$$\sum \int \frac{M\overline{M}}{EI} dx = 0 \tag{195}$$

For systems subject only to torsion

$$\sum \int \frac{M_t \overline{M}_t}{GI_t} dx = 0 \tag{196}$$

For planar statically indeterminate beams of small curvature:

$$\sum \int \frac{M\overline{M}}{EI} ds = 0 \tag{197}$$

If more accuracy is required in the calculations, the equations take the axial internal forces into account:

$$\sum \int \frac{N\overline{N}}{EF} ds + \sum \int \frac{M\overline{M}}{EI} ds = 0 \tag{198}$$

Statically indeterminate systems should be relieved of redundant constraints so that the basic system obtained is the simplest and most convenient for calculations.

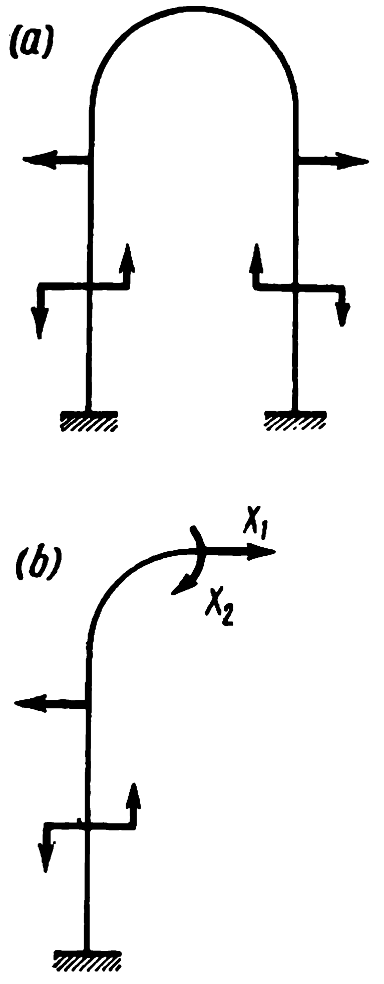


Fig. 182

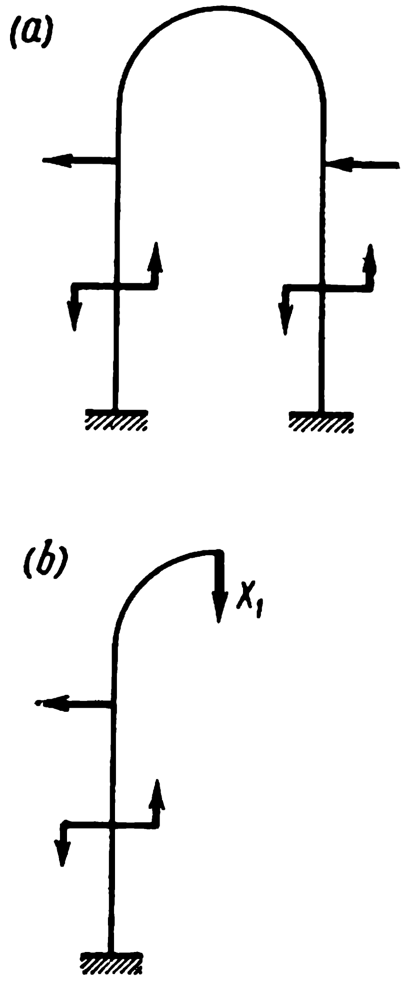


Fig 183

Geometrically symmetrical systems with loads symmetrical about a line (Fig. 182*a*) and with skew-symmetric loads or loads symmetrical about a point (Fig. 183*a*) can be more expediently relieved of redundant constraints by cutting them along the plane of symmetry. This reduces the number of redundant unknown generalized forces and enables only one cut-off portion of the system to be dealt with (Figs. 182*b* and 183*b*).

In the section coinciding with the plane of symmetry, the skew-symmetrical forces Q and M_t become zero in case of a load symmetrical about a line, and forces N and M, symmetrical about a line, become zero in case of a skew-symmetrical load (Fig. 184).

For the straight elements of the system, the integrals in equation (193) can be solved by the method of multiplying the diagrams.

After analysing a statically indeterminate system, the generalized displacement of some section can be determined in dealing with either the given or any other feasible basic equivalent system. It proves expedient to select a system for which the internal forces due to a fictitious unit generalized force can be determined in the simplest way.

Example 100. Let $P = 8$ tnf; $a = 1$ m; $\beta_1 = 30°$; $\beta_2 = 60°$ and $E_I = E_{II} = E_{III} = E = 2 \times 10^6$ kgf/cm^2 (Fig. 185*a*).

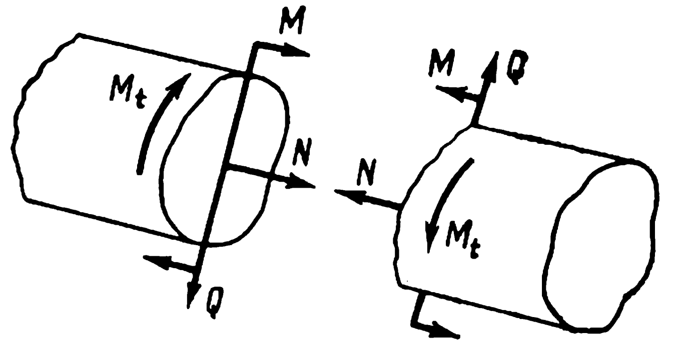


Fig. 184

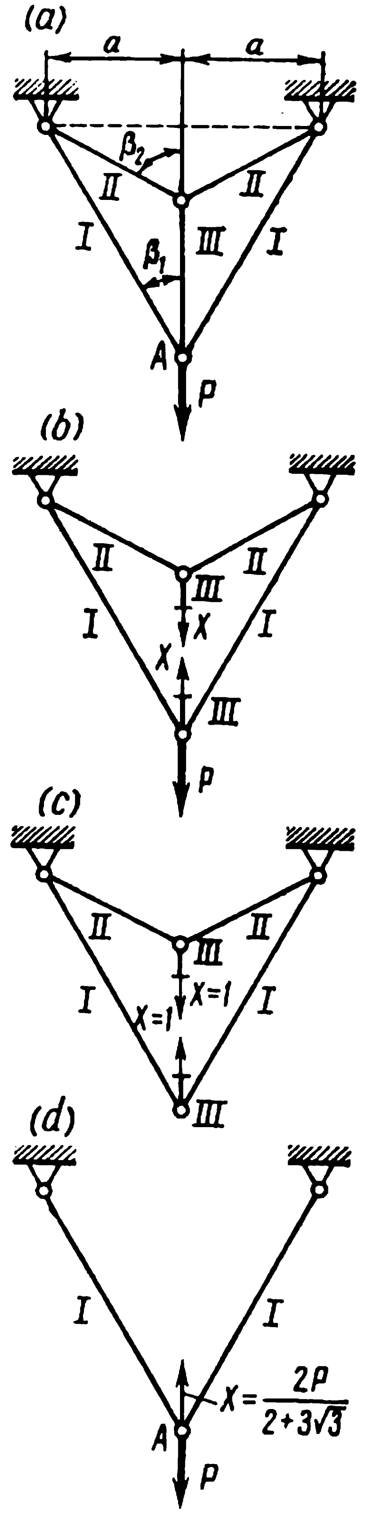


Fig. 185

Determine $\sigma_{I,\,II,\,III}$ and δ_A.

Solution. Since the elements of the given system have cross sections of equal rigidity and are subject only to tensile forces constant throughout the length of the elements, the statically indeterminate system

can be analysed by the use of the simplified equation (194)

$$\sum N\bar{N}l = 0 \tag{a}$$

We shall take the system illustrated in Fig. 185*b* as the equivalent basic system. From the conditions of statics we find for this system that

$$N_I = \frac{P-X}{2\cos\beta_1} = \frac{P-X}{\sqrt{3}}; \quad N_{II} = \frac{X}{2\cos\beta_2} = X; \quad N_{III} = X$$

From the conditions of statics for the auxiliary system (Fig. 185*c*) we obtain

$$\bar{N}_I = -\frac{1}{2\cos\beta_1} = -\frac{1}{\sqrt{3}}; \quad \bar{N}_{II} = \frac{1}{2\cos\beta_2} = 1; \quad \bar{N}_{III} = 1$$

Taking into account that $l_I = \frac{a}{\sin\beta_1} = 2$ m;

$$l_{II} = \frac{a}{\sin\beta_2} = \frac{2}{\sqrt{3}}\ \text{m}; \quad l_{III} = a(\cot\beta_1 - \cot\beta_2)$$

$$= \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\ \text{m}$$

to determine the redundant unknown internal force we shall transform equation (a) as follows

$$2N_I\bar{N}_I l_I + 2N_{II}\bar{N}_{II}l_{II} + N_{III}\bar{N}_{III}l_{III} = -2\frac{P-X}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \times 2$$

$$+2X\frac{2}{\sqrt{3}} + X\frac{2}{\sqrt{3}} = \frac{2}{3}[-2P + (2+3\sqrt{3})X] = 0$$

from which

$$X = N_{III} = N_{II} = \frac{2P}{2+3\sqrt{3}} \cong 0.278P$$

and

$$N_I = \frac{P - \dfrac{2P}{2+3\sqrt{3}}}{\sqrt{3}} = \frac{3P}{2+3\sqrt{3}} \cong 0.417P$$

The normal stresses in the cross sections of the elements of the system are

$$\sigma_{II} = \sigma_{III} = \frac{X}{F} \cong \frac{0.278 \times 8 \times 10^3}{2} = 1112\ \text{kgf/cm}^2;$$

$$\sigma_I = \frac{N_I}{F} \cong \frac{0.417 \times 8 \times 10}{2} = 1668\ \text{kgf/cm}^2$$

To determine the vertical displacement δ_A of joint A, we shall make use of the statically determinate system in Fig. 185*d*.

Since in this system $N_I = \frac{3P}{2+3\sqrt{3}}$, and, due to the fictitious unit force $P_f = 1$ applied at the joint A vertically downward, $N_I = \frac{1}{2\cos\beta_1} = \frac{1}{\sqrt{3}}$, the sought-for displacement is

$$\delta_A = \frac{2}{EF} N_I \bar{N}_I l_I = \frac{2}{EF} \times \frac{3P}{2+3\sqrt{3}} \times \frac{2a}{\sqrt{3}}$$
$$= \frac{2\times 0.417 \times 8 \times 10^3 \times 2 \times 10^2}{2\times 10^6 \times 2 \times 1.732} \cong 0.19 \text{ cm}$$

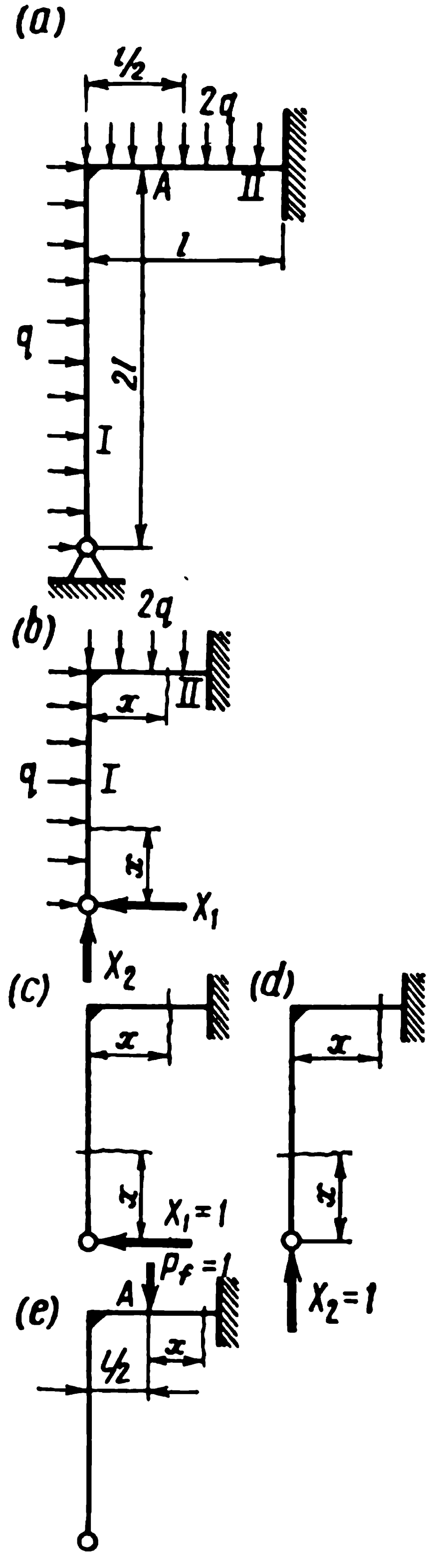


Fig. 186

Example 101. Given: q, l, E and I (Fig. 186*a*). Find f_A.

Solution. Since the rigidities of the cross sections of the vertical and horizontal portions of the half-frame are the same, the simplified equation (195) is used to analyse the statically indeterminate system. Thus

$$\sum \int M\bar{M}\, dx = 0 \qquad \text{(b)}$$

The bending moments M_I and M_{II} in the portions of the equivalent basic system (Fig. 186*b*) and $\bar{M}_I$ and $\bar{M}_{II}$ in the portions of the auxiliary systems (Fig. 186*c* and *d*) are respectively equal to

$$M_I = X_1 x - \frac{qx^2}{2}; \quad M_{II} = X_1 2l + X_2 x - 2ql^2 - qx^2;$$
$$\bar{M}_I = x; \qquad \bar{M}_{II} = 2l;$$
$$\bar{M}_I = 0; \qquad \bar{M}_{II} = x;$$

Next we set up two equations (b):

$$\int_0^{2l} \left(X_1 x - \frac{qx^2}{2}\right) x\, dx + 2l \int_0^{l} (X_1 2l + X_2 x - 2ql^2 - qx^2)\, dx = 0$$

and

$$\int_0^{l} (X_1 2l + X_2 x - 2ql^2 - qx^2)\, x\, dx = 0$$

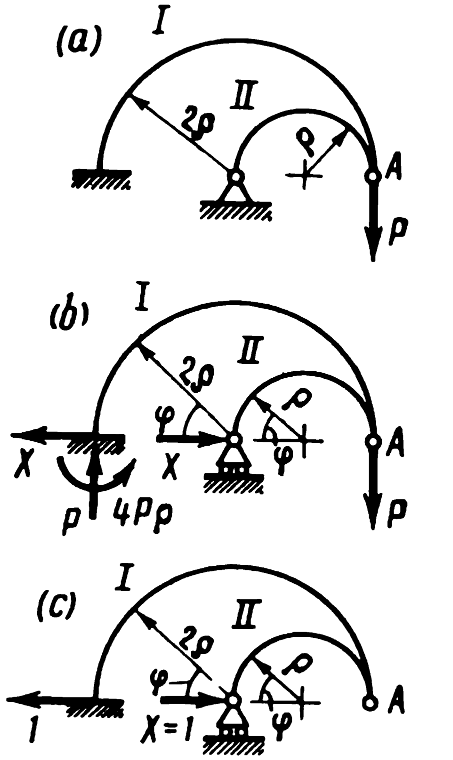
After solving the integrals, the equations can be written as

$$\left.\begin{aligned} 20X_1 + 3X_2 - 20ql &= 0; \\ 12X_1 + 4X_2 - 15ql &= 0 \end{aligned}\right\}$$

from which

$$X_1=\frac{35}{44}ql; \quad X_2=\frac{15}{11}ql$$

To determine the deflection f_A of section A, we find M for the equivalent basic system and $\overline{M}$ for the auxiliary system (Fig. 186e) in the section to the right of section A at the distance x:



$$M=X_1 2l+X_2\left(\frac{l}{2}+x\right)$$

$$-2ql^2-q\left(\frac{l}{2}+x\right)^2$$

$$=-\frac{q}{44}(44x^2-16lx-l^2);$$

$$\overline{M}=-x$$

From formula (188) the sought-for deflection f is equal to

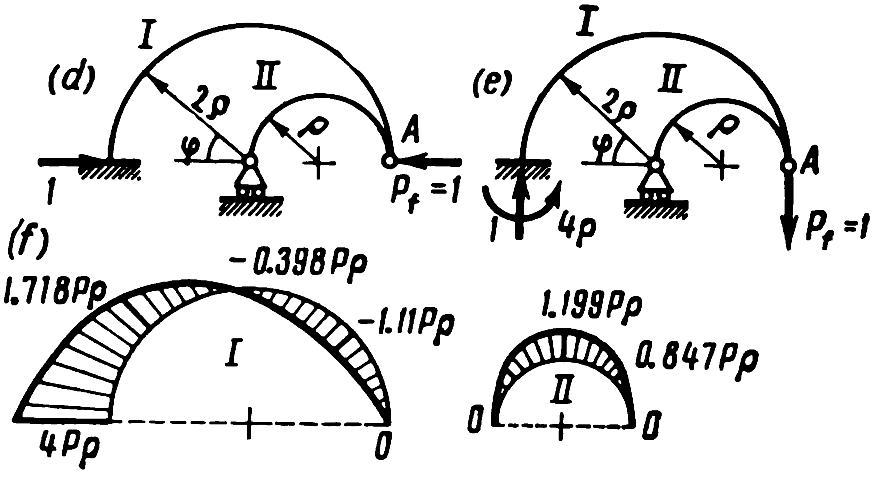


Fig. 187

$$f_A=-\frac{q}{44EI}\int_0^{l/2}(44x^2-16lx-l^2)\,x\,dx=-\frac{5ql^4}{2112EI}$$

Example 102. Given: P, ρ, E and $I_{II}=2I_I$ for beams of small curvature (Fig. 187a).

Determine the horizontal δ_x and vertical δ_y displacements of joint A and plot the diagrams for the bending moments.

Solution. Since the beams are of small curvature and Young's modulus of their materials is the same, a simplified formula (197) can be used to analyse the statically indeterminate system

$$\sum \frac{1}{I} \int M\overline{M}\,ds = 0 \tag{c}$$

We assume the horizontal reaction at the hinged support to be the redundant unknown X.

Making use of equations of statics and the fact that we have an unsupported hinge A, we can express the reactions at the supports of the equivalent basic system (Fig. 187*b*) in terms of the given force P and redundant unknown X. In the auxiliary system (Fig. 187*c*), loaded with force $X = 1$, only a horizontal reaction is developed in the fixed section.

The bending moments in portions I and II of the equivalent basic and auxiliary systems equal, respectively,

$$M_I = 4P\rho - P2\rho\,(1 - \cos\varphi) - X2\rho\sin\varphi$$
$$= 2P\rho\,(1 + \cos\varphi) - 2X\rho\sin\varphi;$$

$$M_{II} = X\rho\sin\varphi; \qquad \overline{M}_I = -\rho\sin\varphi \text{ and } \overline{M}_{II} = \rho\sin\varphi$$

Since the elements of the geometric axes of the beams are

$$ds_I = 2\rho\,d\varphi; \quad ds_{II} = \rho\,d\varphi; \quad I_{II} = 2I_I$$

equation (c) can be written

$$8\int_0^{\pi} [X\sin\varphi - P(1+\cos\varphi)]\sin\varphi\,d\varphi + \frac{1}{2}X\int_0^{\pi}\sin^2\varphi\,d\varphi = 0$$

After solving the integrals we obtain

$$\frac{17}{4}\pi X - 16P = 0$$

from which

$$X = \frac{64}{17\pi}P \cong 1.198P$$

The horizontal δ_x and vertical δ_y displacements of joint A are determined by the use of the auxiliary systems shown in Fig. 187*d* and *e*, respectively.

Since no internal forces are developed in portion II of these systems, we shall consider only portion I. For this portion the bending moments are:

due to the given forces:

$$M_I = 2P\rho(1+\cos\varphi) - 2X\rho\sin\varphi = 2P\rho\left(1 + \cos\varphi - \frac{64}{17\pi}\sin\varphi\right)$$

due to the horizontally applied force $P_f = 1$:

$$\overline{M}_I = 2\rho \sin\varphi$$

and due to the vertical force $P_f = 1$:

$$\overline{M}_I = 4\rho - 2\rho(1-\cos\varphi) = 2\rho(1+\cos\varphi)$$

Hence the sought-for displacements are

$$\delta_x = \frac{2\rho}{EI_I}\int_0^\pi M_I\overline{M}_I\,d\varphi = \frac{8P\rho^3}{EI_I}\int_0^\pi \left(1+\cos\varphi - \frac{64}{17\pi}\sin\varphi\right)$$

$$\times \sin\varphi\,d\varphi = \frac{16}{17}\frac{P\rho^3}{EI_I} \cong 0.941\frac{P\rho^3}{EI_I}$$

$$\delta_y = \frac{2\rho}{EI_I}\int_0^\pi M_I\overline{M}_I\,d\varphi = \frac{8P\rho^3}{EI_I}\int_0^\pi \left(1+\cos\varphi - \frac{64}{17\pi}\sin\varphi\right)$$

$$\times (1+\cos\varphi)\,d\varphi = 8\left(\frac{3}{2}\pi - \frac{128}{17\pi}\right)\frac{P\rho^3}{EI_I} \cong 18.5\frac{P\rho^3}{EI_I}$$

The diagrams for the bending moments M_I and M_{II} are shown in Fig. 187*f*.

THE FORCE METHOD

In applying the force method for the analysis of statically indeterminate elastic systems canonical equations of the following type are set up and solved:

$$\left.\begin{array}{l}
\delta_{11}X_1 + \delta_{12}X_2 + \delta_{13}X_3 + \ldots + \delta_{1n}X_n + \delta_{1p} = 0; \\
\delta_{21}X_1 + \delta_{22}X_2 + \delta_{23}X_3 + \ldots + \delta_{2n}X_n + \delta_{2p} = 0; \\
\delta_{31}X_1 + \delta_{32}X_2 + \delta_{33}X_3 + \ldots + \delta_{3n}X_n + \delta_{3p} = 0; \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\delta_{n1}X_1 + \delta_{n2}X_2 + \delta_{n3}X_3 + \ldots + \delta_{nn}X_n + \delta_{np} = 0
\end{array}\right\} \quad (199)$$

Each of these equations expresses the condition (192) that a generalized displacement in a statically indeterminate system corresponding to each redundant unknown generalized force X_1, X_2, X_3, ... X_n is equal to zero.

The absolute terms δ_{ip} of the equations and all the coefficients δ_{ii} and δ_{ik} are the generalized displacements in the basic system in the direction of the i-th (shown by the first subscript) redundant unknown generalized force X_i: δ_{ip} are due to the action of all the given generalized forces P, and δ_{ii} and δ_{ik} are due to each unit redundant unknown generalized force $X_i = 1$ or $X_k = 1$ (shown by the second subscript).

All these generalized displacements can be determined by any pertinent method or taken from tables whenever possible.

The displacements δ_{ip} can be greater or less than zero or equal to zero; they depend on the given forces, the geometry of the system and the choice of the basic system.

The displacements δ_{ii} and δ_{ik} do not depend on the given forces; they are fully determined by the geometry of the system and the selection of the redundant unknowns. The principal coefficients δ_{ii} are intrinsically positive nonzero magnitudes; the auxiliary coefficients $\delta_{ik} = \delta_{ki}$ can be greater than zero, less than or equal to zero.

The basic system should be chosen so that as many auxiliary coefficients as possible become zero. Symmetrical systems should be relieved of constraints as indicated on page 320.

An n-fold statically indeterminate system should be considered in $n + 1$ states: the basic state being due to all the given generalized forces and n auxiliary states due to each redundant unknown generalized force equal to unity.

If a statically indeterminate system is subject only to temperature changes, the absolute terms of the canonical equations are δ_{it} which represent the generalized displacements corresponding to the i-th redundant unknown generalized force in the basic system due to the change of temperature. In case of the simultaneous action of a load and changes in temperature on a system, the absolute terms in the canonical equations are represented by the sum $\delta_{ip} + \delta_{it}$.

Inaccuracies in manufacturing the members of a system are accounted for by the introduction of the quantity $\delta_{i\Delta}$ into the absolute terms of the canonical equations; this quantity denotes the generalized displacements corresponding to the i-th redundant unknown generalized force in the basic system due to the inaccuracy Δ in manufacturing the members.

The values δ_{it} and $\delta_{i\Delta}$ are taken with either the positive or negative sign depending on whether the directions of these displacements coincide with or oppose the accepted direction of X_i.

For systems which are singly statically indeterminate, the canonical equation of the method of forces is of the form

$$\delta_{11}X_1 + \delta_{1p} = 0$$

and hence the redundant unknown generalized force has the value

$$X_1 = -\frac{\delta_{1p}}{\delta_{11}} \tag{200}$$

In analysing singly statically indeterminate planar girder-frame systems or systems with curvilinear elements of small curvature for which the role of the axial internal force and of the transverse force is insignificant, we can write

$$\delta_{1p} = \sum \int \frac{M\overline{M}}{EI}\,ds; \quad \delta_{11} = \sum \int \frac{\overline{M}^2}{EI}\,ds$$

and

$$X_1 = -\frac{\sum \int \frac{M\overline{M}}{EI} ds}{\sum \int \frac{\overline{M}^2}{EI} ds} \tag{201}$$

in which ds is an element of length of the geometric axis of a portion.

Example 103. Given: P, q and a (Fig. 188*a*).

Analyse the statically indeterminate system, considering the deformation of the portions to be due only to the bending moment.

Solution. Take the reaction at the movable support as the redundant unknown X_1. Since $M\overline{M} \neq 0$ and $\overline{M}^2 \neq 0$ only on the portion of length $2a$, and on this portion for the basic (Fig. 188*b*) and auxiliary (Fig. 188*c*) systems $M = -P(a+x) - \frac{qx^2}{2}$ and $\overline{M} = x$, then, according to formula (201)

$$X_1 = \frac{\int_0^{2a} M\overline{M}\,dx}{\int_0^{2a} \overline{M}^2\,dx} = \frac{\int_0^{2a} \left[P(a+x) + \frac{qx^2}{2}\right] x\,dx}{\int_0^{2a} x^2\,dx} = \frac{\frac{14}{3} Pa^3 + 2qa^4}{\frac{8}{3} a^3} = \frac{1}{4}(7P + 3qa)$$

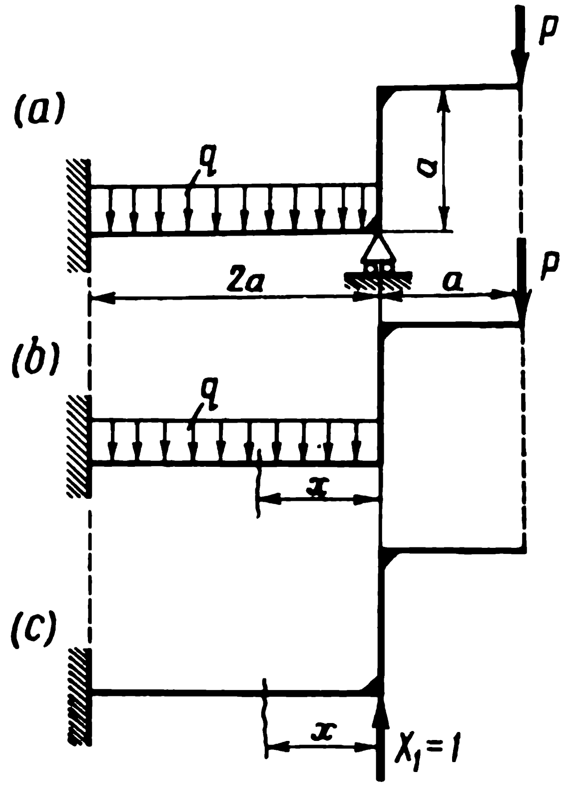


Fig. 188

Example 104. Given: q, a, E and I (Fig. 189*a*).

Plot the diagram for the bending moment.

Solution. To analyse the statically indeterminate system we relieve the frame by cutting the cross-bar in the middle (Fig. 189*b*).

The basic system is shown in Fig. 189*c* and the auxiliary systems with the diagrams representing the bending moments due to $X_1 = 1$, $X_2 = 1$ and $X_3 = 1$ are shown in Figs. 189*d*, *e* and *f*.

Since the diagrams for $\overline{M}$ due to $X_1 = 1$ and $X_3 = 1$ are symmetrical about a line and the diagram for $\overline{M}$ due to $X_2 = 1$ is symmetrical only about a point, the auxiliary coefficients $\delta_{12} = \delta_{21} = 0$, $\delta_{23} =$

$= \delta_{32} = 0$ and the canonical equations of the force method become

$$\left.\begin{aligned} &\delta_{11}X_1 + \delta_{13}X_3 + \delta_{1p} = 0; \\ &\delta_{22}X_2 + \delta_{2p} = 0; \\ &\delta_{31}X_1 + \delta_{33}X_3 + \delta_{3p} = 0 \end{aligned}\right\}$$

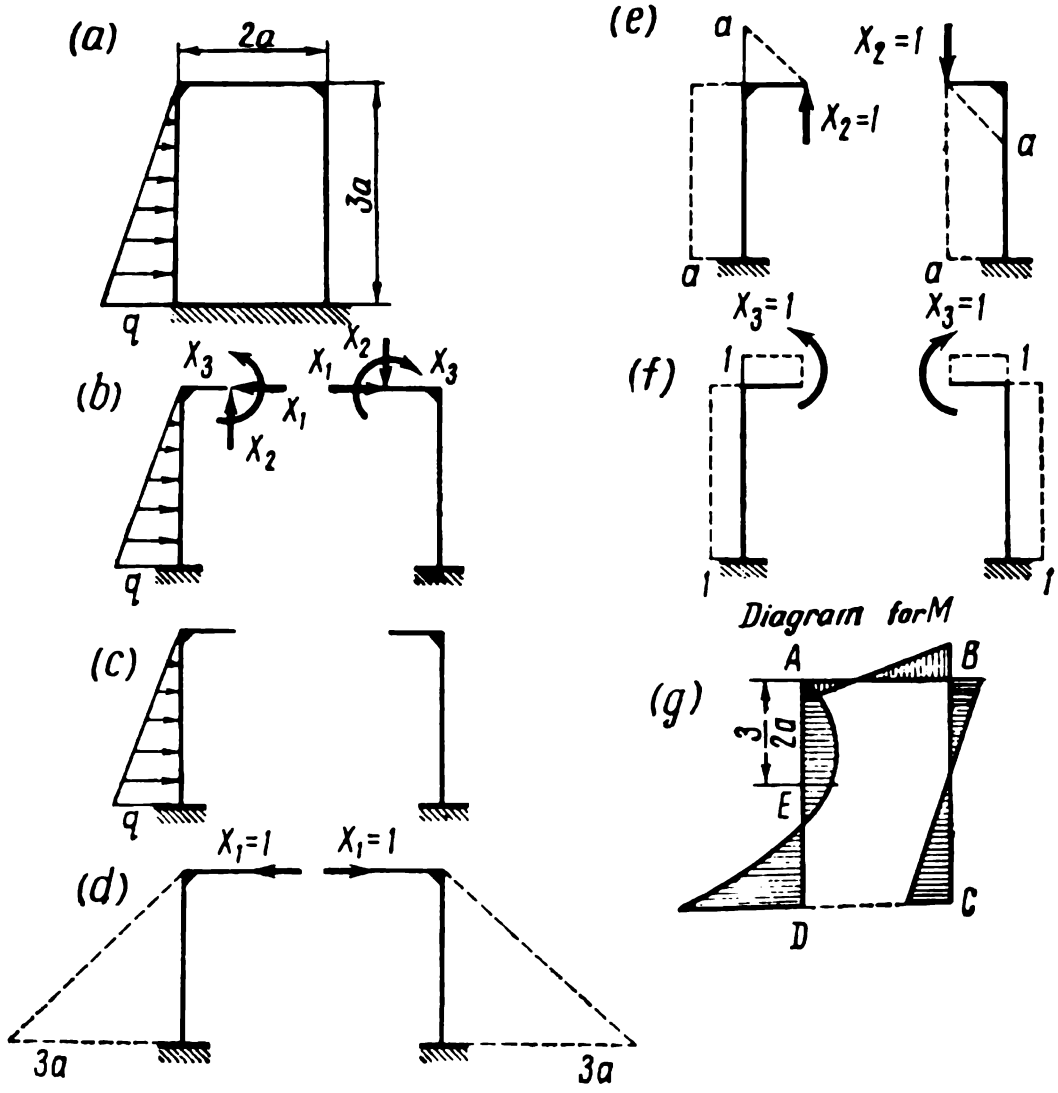


Fig. 189

The coefficients δ_{11}, δ_{22}, δ_{33} and $\delta_{13} = \delta_{31}$ can be found by multiplying the diagrams. Thus

$$EI\delta_{11} = 2 \times 3 \times a\frac{3}{2}a\frac{2}{3}3a = 18a^3;$$

$$EI\delta_{22} = 2\left(a\frac{a}{2} \times \frac{2}{3}a + a \times 3a \times a\right) = \frac{20}{3}a^3;$$

$$EI\delta_{33} = 2(1 \times a \times 1 + 1 \times 3a \times 1) = 8a;$$

$$EI\delta_{13} = EI\delta_{31} = 2 \times 1 \times 3a \times \frac{1}{2} \times 3a = 9a^2$$

The absolute terms are found by Mohr's integrals

$$EI\delta_{1p}=-\int_0^{3a}\frac{qx^3}{18a}x\,dx=-\frac{27}{10}qa^4;\quad EI\delta_{2p}=-\int_0^{3a}\frac{qx^3}{18a}a\,dx=-\frac{9}{8}qa^4;$$

$$EI\delta_{3p}=-\int_0^{3a}\frac{qx^3}{18a}dx=-\frac{9}{8}qa^3$$

After substituting the obtained values into the canonical equations and performing certain transformations we obtain

$$\left.\begin{aligned}&20aX_1+10X_3=3qa^2;\\&160X_2=27qa;\\&72aX_1+64X_3=9qa^2\end{aligned}\right\}$$

from which

$$X_1=\frac{\begin{vmatrix}3\,qa^2 & 10\\ 9\,qa^2 & 64\end{vmatrix}}{\begin{vmatrix}20a & 10\\ 72a & 64\end{vmatrix}}=\frac{192-90}{1280-720}qa\cong 0.1821qa;$$

$$X_2=\frac{27}{160}qa\cong 0.1688qa;$$

$$X_3=\frac{\begin{vmatrix}20a & 3\,qa^2\\ 72a & 9\,qa^2\end{vmatrix}}{560a}=\frac{180-216}{560}qa^2\cong-0.0643qa^2$$

Using the superposition method the diagrams for the bending moment M in all the portions of the frame (Fig. 189g) can be obtained:

in section A: $M = 0.1688\ qa \times a - 0.0643\ qa^2 \times 1 = 0.1045\ qa^2$;

in section B: $M = -0.1688\ qa \times a - 0.0643\ qa^2 \times 1 = -0.2331qa^2$;

in section C: $M = 0.1821\ qa \times 3a - 0.1688\ qa \times a - 0.0643qa^2 = 0.3132\ qa^2$;

in section D: $M = 0.1821\ qa \times 3a + 0.1688\ qa \times a - 0.0643\ qa^2 - \frac{18a}{q}\ 27a^3 = -0.8492\ qa^2$;

in section E: $M = 0.1821\ qa \times 1.5a + 0.1688\ qa \times a - 0.0643qa^2 - \frac{q}{18a} \times \frac{27}{8}\ a^3 = 0.1902\ qa^2$

Example 105. Given: a, b, c, d, E, G $[\sigma]$, α, and $\Delta t^\circ > 0$ (Fig. 190*a*).

Formulate the strength condition for the system on portion c.

Solution. The basic system with changes in its geometry due to the temperature increase of Δt degrees is shown in Fig. 190*b* and the auxiliary system, in Fig. 190*c*.

Since

$$\delta_{1t} = -\alpha b\Delta t \quad \text{and} \quad \delta_{11} = \frac{b}{EF} + \frac{1}{EI}\left(\int_0^a x^2\,dx + a^2 b + \int_0^c x^2\,dx\right)$$

$$+\frac{a^2c}{GI_p} = \frac{b}{EF} + \frac{1}{EI}\left(\frac{a^3}{3} + a^2b + \frac{c^3}{3}\right) + \frac{a^2c}{GI_p}, \text{ then}$$

$$X_1 = -\frac{\delta_{1t}}{\delta_{11}}$$

$$= \frac{ab\Delta t}{\frac{b}{EF} + \frac{1}{EI}\left(\frac{a^3}{3} + a^2b + \frac{c^3}{3}\right) + \frac{a^2c}{GI_p}}$$

where

$$F = \frac{\pi d^2}{4}; \quad I = \frac{\pi d^4}{64} \text{ and } I_p = \frac{\pi d^4}{32}.$$

In the dangerous fixed section of the portion c, the bending moment is $M = X_1 c$ and the torque is $M_t = X_1 a$.

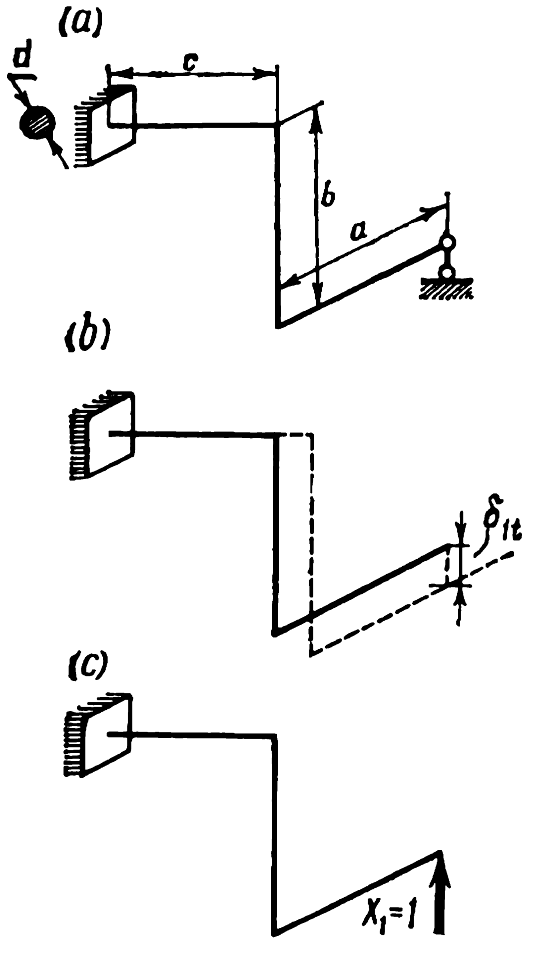


Fig. 190

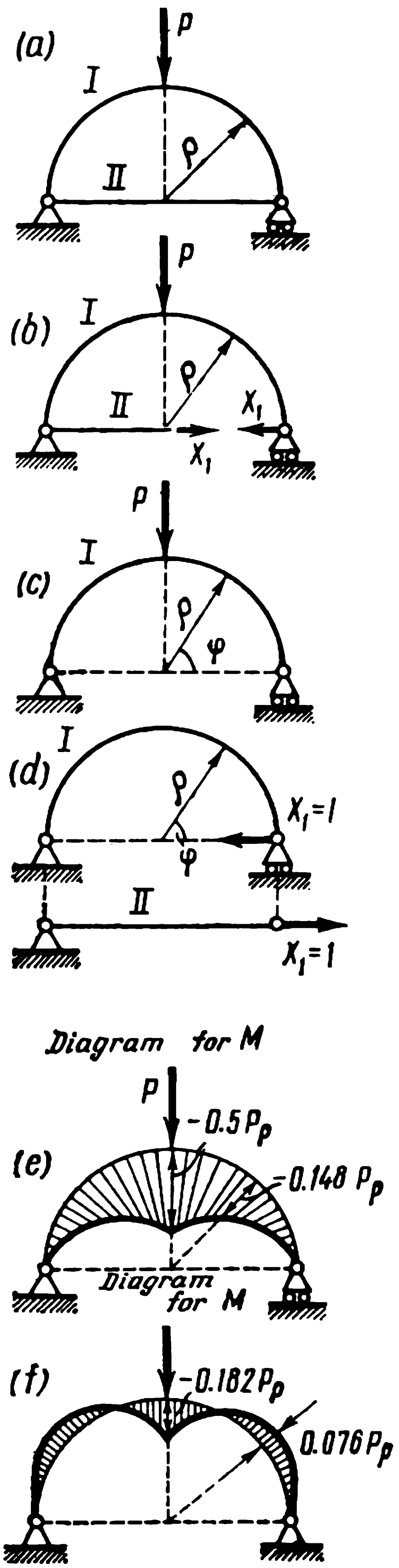


Fig. 191

The strength condition becomes

$$\frac{\sqrt{M^2+(\eta M_t)^2}}{W}=\frac{X_1\sqrt{c^2+(\eta a)^2}}{W}\leqslant[\sigma]$$

in which $W=\frac{\pi d^3}{32}$ and $\eta=1$ according to the third or $\eta=\sqrt{0.75}$ according to the fourth strength theory.

Example 106. Given: P, ρ, E_I, I_I, E_{II} and F_{II}, with the curved beam being one of small curvature (Fig. 191*a*).

Determine the displacement δ of the movable support.

Solution. First we relieve the constraints of the system by disjoining elements I and II at the movable joint (Fig. 191*b*). The basic and auxiliary systems are shown in Figs. 191*c* and *d*.

Since

$$\delta_{1p}=-\frac{2}{E_I I_I}\int_0^{\pi/2}\frac{P}{2}\rho(1-\cos\varphi)\rho\times\sin\varphi\times\rho\,d\varphi=-\frac{P\rho^3}{2E_I I_I}$$

and

$$\delta_{11}=\frac{2}{E_I I_I}\int_0^{\pi/2}\rho^2\sin^2\varphi\times\rho\,d\varphi+\frac{2\rho}{E_{II}F_{II}}=\frac{\pi\rho^3}{2E_I I_I}+\frac{2\rho}{E_{II}F_{II}}$$

then

$$X_1=-\frac{\delta_{1p}}{\delta_{11}}=\frac{P\rho^3}{2E_I I_I}\times\frac{1}{\frac{\pi\rho^3}{2E_I I_I}+\frac{2\rho}{E_{II}F_{II}}}=\frac{P}{\pi}\times\frac{1}{1+\frac{4}{\pi}\times\frac{E_I I_I}{\rho^2 E_{II}F_{II}}}$$

The sought-for displacement δ is determined as the total elongation of tie rod II, i.e.

$$\delta=\frac{X_1 2\rho}{E_{II}F_{II}}=\frac{2P\rho}{\pi E_{II}F_{II}}\times\frac{1}{1+\frac{4}{\pi}\times\frac{E_I I_I}{\rho^2 E_{II}F_{II}}}$$

If there is no tie rod II (i.e. $E_{II}F_{II}=0$), then

$$X_1=0 \text{ and } \delta=\frac{P\rho^3}{2E_I I_I}$$

If both supports are stationary (i.e. $E_{II}F_{II}=\infty$), then

$$X_1=\frac{P}{\pi} \text{ and } \delta=0$$

For the first case ($E_{II}F_{II}=0$) the bending moment in an arbitrary section is

$$M=-\frac{P}{2}\rho(1-\cos\varphi)$$

The diagram for M is as shown in Fig. 191*e*. For the second case ($E_{II}F_{II}=\infty$) the bending moment in an arbitrary section is

$$M=P\rho\left(\frac{\sin\varphi}{\pi}-\frac{1}{2}+\frac{\mathrm{cor}\,\varphi}{2}\right)$$

The diagram for M is of the form shown in Fig. 191*f*.

12.3.

Plane Thin-Walled Rings

A plane thin-walled ring is any plane rigidly closed elastic bar system in which the lengths of the portions greatly exceed their cross-sectional dimensions. Such a system is three-fold statically indeterminate. The redundant unknowns are the bending moment X_1, the axial internal force X_2 and the transverse (shearing) force X_3, i.e. the internal forces in a cross section cut through to relieve redundant constraints of the ring (Fig. 192). Therefore rigidly closed systems belong to internally statically indeterminate systems.

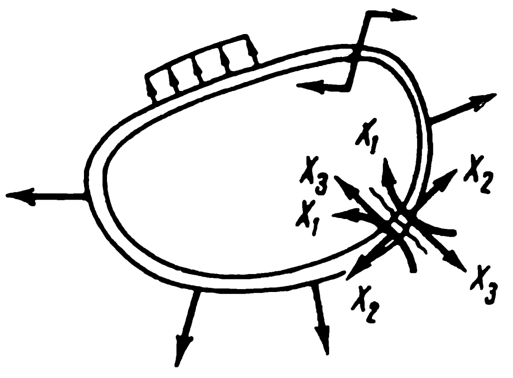


Fig. 192

The statically indeterminate rings can be analysed by the use of either the principle of least work or (which is more convenient), of the canonical equations of the force method. Since the rings are thin-walled, only the deformation due to the bending moment need be considered in formulating the equations for their analysis.

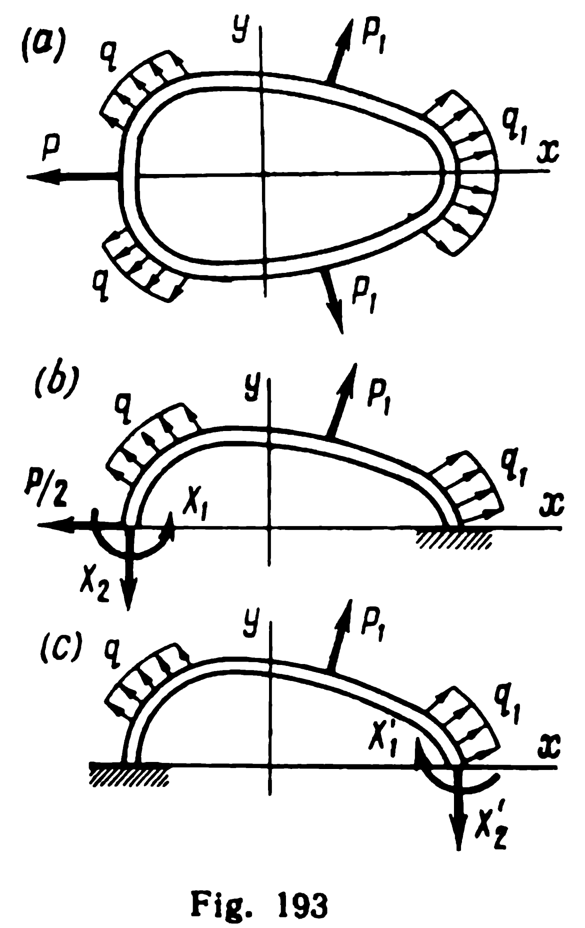


Fig. 193

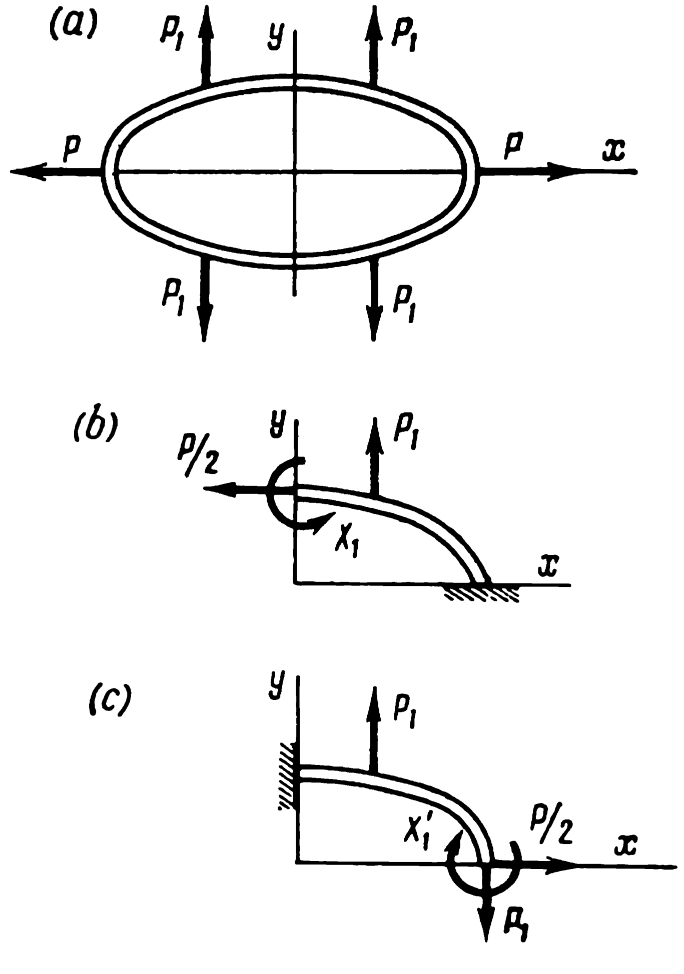


Fig. 194

If in its geometry and load the ring is symmetrical with respect to one axis (Fig. 193*a*), then the transverse (shearing) forces in the cross sections coinciding with the axis of symmetry equal zero. Hence, the bending moment (X_1 or X_1') and the axial force (X_2 or X_2') will be the redundant unknowns in these sections. Only one of its symmetrical halves need be considered instead of the whole ring (Fig. 193*a* and *b*).

If in its geometry and loading the ring is symmetrical with respect to two axes (Fig. 194*a*), in the cross sections passing through the axes of symmetry, the transverse (shearing) forces equal zero and the axial forces can be determined from the conditions of statics as the sums of the projections of the external and internal forces applied to the half ring, on the respective axis of symmetry. In this case only the bending moment (X_1 or X_1') is the redundant unknown. Instead of the whole ring, only a quarter of it, included between the axes of symmetry (Fig. 194*b* or *c*) need be considered.

If the ring has more than two axes of symmetry, only one part of it, included between the cross sections coinciding with the adjacent axes of equal symmetry, need be considered.

In these sections the transverse (shearing) forces equal zero, the axial internal forces are found from the conditions of statics and the bending moment is the redundant unknown.

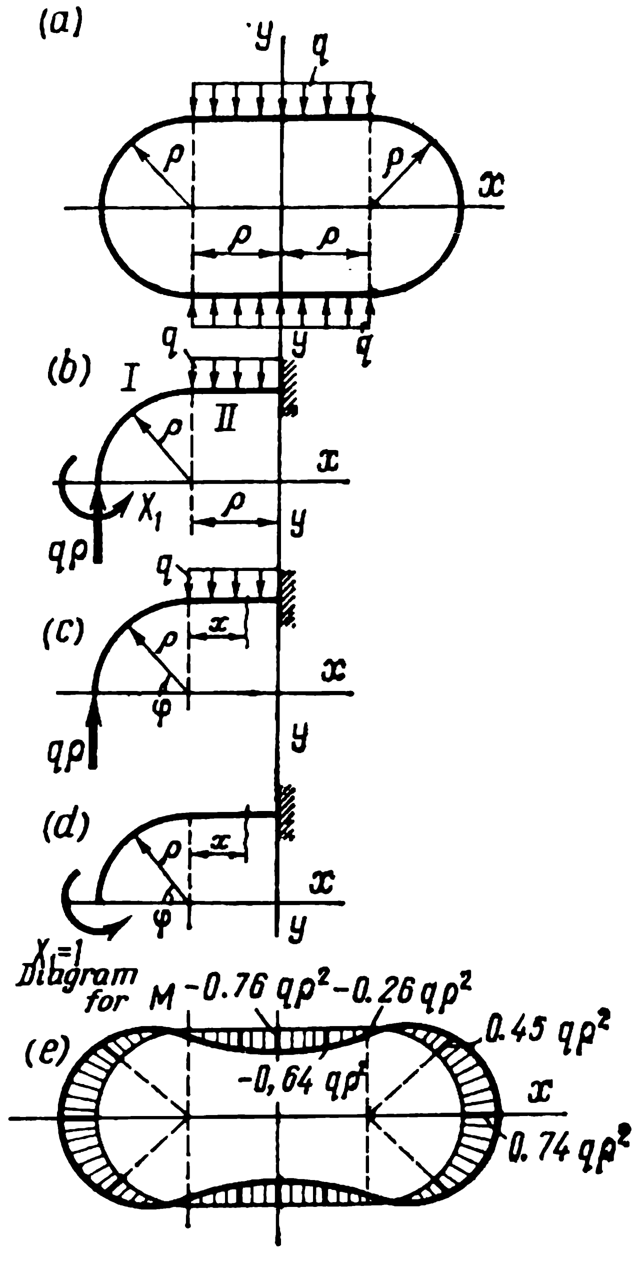


Fig. 195

Example 107. Given: q, ρ, E and I for a thin-walled ring symmetrical about the x- and y-axes (Fig. 195*a*).

Determine by what amount δ the straight portions of the ring are brought closer together.

Solution. We consider one quarter of the ring (Fig. 195*b*). In the sections coinciding with the x-axis the transverse (shearing) force equals zero, the axial internal force is equal to $q\rho$ and the bending moment X_1 is the redundant unknown generalized force.

The bending moments in arbitrary sections of portions I and II of the basic (Fig. 195c) and auxiliary (Fig. 195d) systems are given by

$$M_I = -q\rho^2(1-\cos\varphi);$$

$$M_{II} = -q\rho(\rho+x)+q\frac{x^2}{2}; \quad \overline{M}_I = 1; \quad \overline{M}_{II} = 1$$

Since

$$EI\delta_{1p} = \int_0^s M_I\overline{M}_I\,ds + \int_0^\rho M_{II}\overline{M}_{II}\,dx = -q\rho^3\int_0^{\pi/2}(1-\cos\varphi)\,d\varphi$$

$$-q\int_0^\rho\left[\rho(\rho+x)-\frac{x^2}{2}\right]dx = -q\rho^3\left(\frac{\pi}{2}+\frac{1}{3}\right)$$

and

$$EI\delta_{11} = \int_0^s \overline{M}_I^2\,ds + \int_0^\rho \overline{M}_{II}^2\,dx = \rho\int_0^{\pi/2}d\varphi + \int_0^\rho dx = \rho\left(\frac{\pi}{2}+1\right)$$

the bending moment X_1 will equal

$$X_1 = -\frac{\delta_{1p}}{\delta_{11}} = q\rho^2\frac{3\pi+2}{3(\pi+2)} \cong 0.74q\rho^2$$

After relieving the redundant constraints we obtain

$$M_I = -q\rho^2\left[1-\frac{3\pi+2}{3(\pi+2)}-\cos\varphi\right] \cong -q\rho^2(0.26-\cos\varphi);$$

$$M_{II} = -q\left[\rho^2+\rho x-\frac{x^2}{2}-\frac{3\pi+2}{3(\pi+2)}\rho^2\right]$$

$$\cong -q(0.26\rho^2+\rho x-0.5x^2)$$

therefore

$$M_{I_{\varphi=0}} \cong 0.74q\rho^2; \quad M_{I_{\varphi=\frac{\pi}{4}}} \cong 0.45q\rho^2; \quad M_{I_{\varphi=\frac{\pi}{2}}} \cong -0.26q\rho^2;$$

$$M_{II_{x=0}} \cong -0.26q\rho^2; \quad M_{II_{x=\frac{\rho}{2}}} \cong -0.64q\rho^2; \quad M_{II_{x=\rho}} = -0.76q\rho^2$$

The bending moment diagram is shown in Fig. 195e.

To determine the amount the middle of the straight portions of the ring are brought together we apply the fictitious force $P_f = 1$, directed vertically upward in the section with $q\rho$. The bending moments in arbitrary sections of portions I and II of the quarter of the ring due to this force are

$$\overline{M}_I = -\rho(1-\cos\varphi) \text{ and } \overline{M}_{II} = -(\rho+x)$$

Hence the required displacement is

$$\delta = \frac{2}{EI}\left(\int_0^s M_I \overline{M}_I\, ds + \int_0^\rho M_{II} \overline{M}_{II}\, dx\right) = \frac{2q\rho^4}{EI}\int_0^{\pi/2} (0.26 - \cos\varphi)$$

$$\times (1 - \cos\varphi)\, d\varphi + \frac{2q}{EI}\int_0^\rho (0.26\rho^2 + \rho x - 0.5x^2)$$

$$\times (\rho + x)\, dx \cong 1.72\frac{q\rho^4}{EI}$$

Example 108. Given: P, α $\left(\text{in which } 2\alpha = \frac{2\pi}{n}\right)$, ρ, E, I and F (Fig. 196*a*).

Determine $\Delta\rho_P$ and $\Delta\rho_0$, the changes in the radii of the ring along the line of action of the forces and in the middle between the forces.

Solution. We consider a portion of the ring wall separated out by cross sections passing through the middle of the arcs between the forces (Fig. 196*b*).

The transverse (shearing) forces in the cross sections equal zero and the axial internal forces N_0 are found from the sum of projections of the external and internal forces on the vertical: $N_0 = \frac{P}{2\sin\alpha}$.

The bending moments and axial internal forces in an arbitrary section of the basic (Fig. 196*c*) and auxiliary (Fig. 196*d*) systems are respectively equal to

$$M = \frac{P\rho}{2\sin\alpha}(1 - \cos\varphi); \quad N = \frac{P\cos\varphi}{2\sin\alpha}; \quad \overline{M} = 1; \quad \overline{N} = 0$$

Since

$$\delta_{1p} = \frac{\rho}{EI}\int_0^\alpha M\overline{M}\, d\varphi = -\frac{P\rho^2}{2EI\sin\alpha}\int_0^\alpha (1 - \cos\varphi)\, d\varphi$$

$$= -\frac{P\rho^2}{2EI}\left(\frac{\alpha}{\sin\alpha} - 1\right)$$

and

$$\delta_{11} = \frac{\rho}{EI}\int_0^\alpha \overline{M}^2\, d\varphi = \rho\frac{\alpha}{EI}$$

the moment

$$X_1 = -\frac{\delta_{1p}}{\delta_{11}} = \frac{P\rho}{2}\left(\frac{1}{\sin\alpha} - \frac{1}{\alpha}\right)$$

The bending moment in an arbitrary section of the ring is

$$M = \frac{P\rho}{2}\left(\frac{1}{\sin\alpha} - \frac{\cos\varphi}{\sin\alpha} - \frac{1}{\sin\alpha} + \frac{1}{\alpha}\right) = \frac{P\rho}{2}\left(\frac{1}{\alpha} - \frac{\cos\varphi}{\sin\alpha}\right)$$

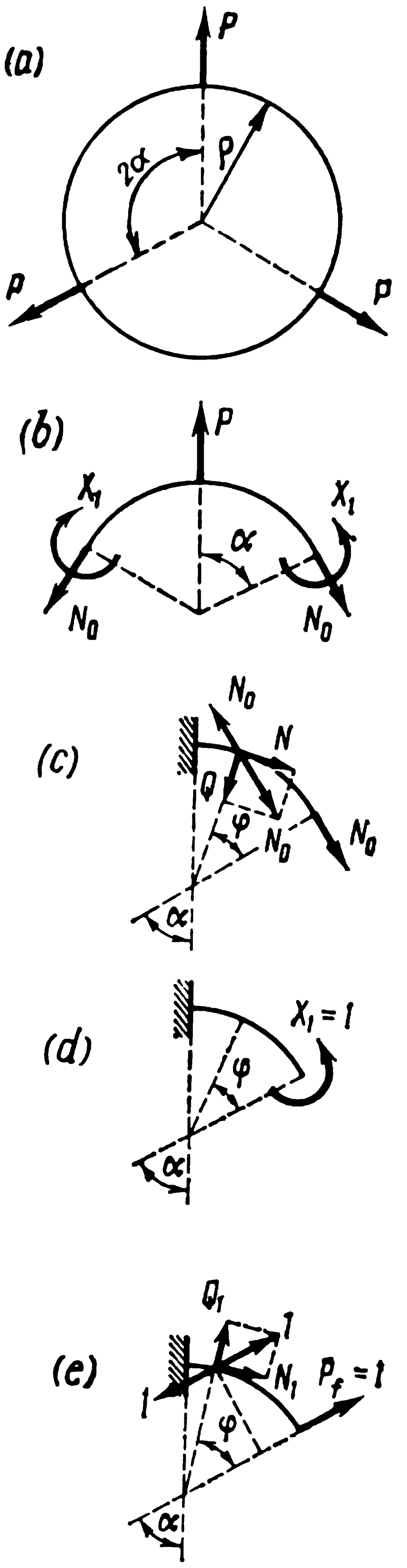


Fig. 196

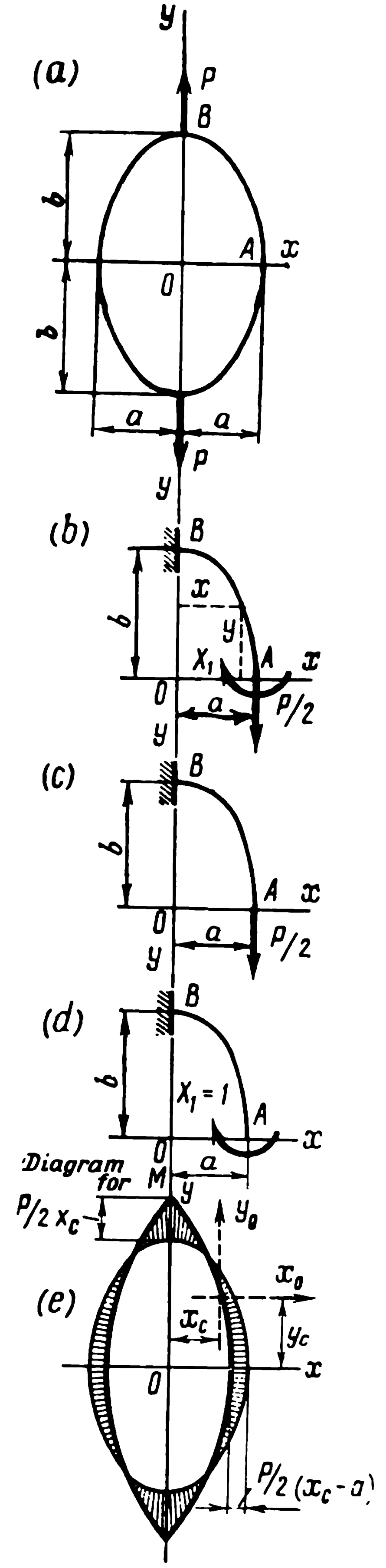


Fig. 197

To determine $\Delta\rho_p$, we apply force $P_f = 1$ instead of force P. Then

$$\overline{M} = \frac{\rho}{2}\left(\frac{1}{\alpha} - \frac{\cos\varphi}{\sin\alpha}\right); \quad \overline{N} = \frac{\cos\varphi}{2\sin\alpha}$$

and

$$\Delta\rho_p = \frac{2\rho}{EI}\int_0^{\alpha} M\overline{M}\,d\varphi + \frac{2\rho}{EF}\int_0^{\alpha} N\overline{N}\,d\varphi$$

$$= \frac{P\rho^3}{2EI}\int_0^{\alpha}\left(\frac{1}{\alpha} - \frac{\cos\varphi}{\sin\alpha}\right)^2 d\varphi + \frac{P\rho}{2EF\sin^2\alpha}\int_0^{\alpha}\cos^2\varphi\,d\varphi$$

And finally

$$\Delta\rho_p = \frac{P\rho^3}{2EI}\left(\frac{\cot\alpha}{2} + \frac{\alpha}{2\sin^2\alpha} - \frac{1}{\alpha}\right) + \frac{P\rho}{4EF}\left(\cot\alpha + \frac{\alpha}{\sin^2\alpha}\right)$$

To determine $\Delta\rho_0$, we apply a radial force $P_f = 1$ at the section at an angle α to the vertical (Fig. 196*e*). Then

$$\overline{M} = -\rho\sin\varphi; \quad \overline{N} = \sin\varphi$$

and

$$\Delta\rho_0 = \frac{P\rho^3}{2EI}\int_0^{\alpha}\left(\frac{\cos\varphi}{\sin\alpha} - \frac{1}{\alpha}\right)\sin\varphi\,d\varphi + \frac{P\rho}{2EF\sin\alpha}\int_0^{\alpha}\cos\varphi\sin\varphi\,d\varphi$$

and finally

$$\Delta\rho_0 = \frac{P\rho^3}{2EI}\left[\frac{\sin\alpha}{2} + \frac{1}{\alpha}(\cos\alpha - 1)\right] + \frac{P\rho}{4EF}\sin\alpha$$

Example 109. Given: P, a, b, E and I for a thin-walled ring symmetrical about axes y and x (Fig. 197*a*).

Determine Δa and Δb.

Solution. We shall consider one quarter of the ring (Fig. 197*b*). In the cross sections coinciding with the x-axis the transverse (shearing) force equals zero and the axial internal force equals $\frac{P}{2}$.

The bending moments in an arbitrary cross section (with a centre of gravity having the coordinates x and y) of the basic (Fig. 197*c*) and auxiliary (Fig. 197*d*) systems are equal, respectively, to

$$M = \frac{P}{2}(a - x) \quad \text{and} \quad \overline{M} = 1$$

Since

$$EI\delta_{1p} = \int_0^s M\overline{M}\,ds = \frac{P}{2}\int_0^s (a - x)\,ds = \frac{P}{2}\left(a\int_0^s ds - \int_0^s x\,ds\right)$$

$$= \frac{P}{2}(as - S_y) = \frac{P}{2}s\left(a - \frac{S_y}{s}\right) = \frac{P}{2}s(a - x_c)$$

in which s = length of the arc of the geometric axis of the wall of the quarter of the ring

$S_y = \int_0^s xds$ = static moment of arc s with respect to the y-axis

$x_c = \frac{S_y}{s}$ = abscissa of the centre of gravity of arc s

$$EI\delta_{11} = \int_0^s \overline{M}^2\,ds = \int_0^s ds = s.$$

Then the bending moment in section A is

$$X_1 = -\frac{\delta_{1p}}{\delta_{11}} = \frac{P}{2}(x_c - a)$$

The bending moment in an arbitrary section of the ring is

$$M = \frac{P}{2}(a - x) + \frac{P}{2}(x_c - a) = \frac{P}{2}(x_c - x); \quad M_{x=a} = \frac{P}{2}(x_c - a)$$

$$\text{if } a > x_c \text{ then } M_{x=a} < 0; \quad M_{x=x_c} = 0; \quad M_{x=0} = \frac{P}{2}x_c > 0$$

Figure 197*e* shows the diagram for the bending moment for the case when $a - x_c < x_c$.

In order to determine the change in dimension a, a horizontal force $P_f = 1$, directed towards the centre O, is applied at section A of the quarter of the ring.

Due to this force $\overline{M} = y$, hence

$$\Delta a = \frac{P}{2EI}\int_0^s (x_c - x)\,y\,ds = \frac{P}{2EI}\left(x_c\int_0^s y\,ds - \int_0^s xy\,ds\right)$$

$$= \frac{P}{2EI}(x_cS_x - I_{xy}) = -\frac{P}{2EI}I_{x_0y_0}$$

in which $S_x = \int_0^s yds$ = static moment of the arc s with respect to the y-axis

$I_{xy} = \int_0^s xy\,ds$ = product of inertia of arc s with respect to the x- and y-axes.

$I_{x_0y_0} = I_{xy} - x_cS_x = I_{xy} - x_cy_cs$ = product of inertia of arc s with respect to axes x_0y_0 passing through the centre of gravity of the quarter of the ring and parallel to axes xy (Fig. 197*e*).

To determine the change of size b in section A of the quarter of the ring we apply a vertical force $P_f = 1$ directed downward.

Due to this force $\bar{M} = a - x$, hence

$$\Delta b = \frac{P}{2EI}\int_0^s (x_c - x)(a - x)\,ds = \frac{P}{2EI}\left(ax_c\int_0^s ds\right.$$

$$\left. - a\int_0^s x\,ds - x_c\int_0^s x\,ds + \int_0^s x^2\,ds\right) = \frac{P}{2EI}(ax_cs - aS_y - x_cS_y + I_y)$$

$$= \frac{P}{2EI}(I_y - x_cS_y) = \frac{P}{2EI}I_{y_0}$$

in which $I_y = \int_0^s x^2\,ds$ = linear moment of inertia of arc s with respect to axis y

$I_{y_0} = I_y - x_cS_y = I_y - x_c^2s$ = linear moment of inertia of arc s with respect to axis y_0.

Particular cases: A. A square frame is stretched along a diagonal (Fig. 198*a*). Since $x_c = y_c = \frac{a}{2}$ (Fig. 198*b*), then

$$M = \frac{P}{2}\left(\frac{a}{2} - x\right); \quad M_{x=c} = \frac{Pa}{4}; \quad M_{x=a} = -\frac{Pa}{4}$$

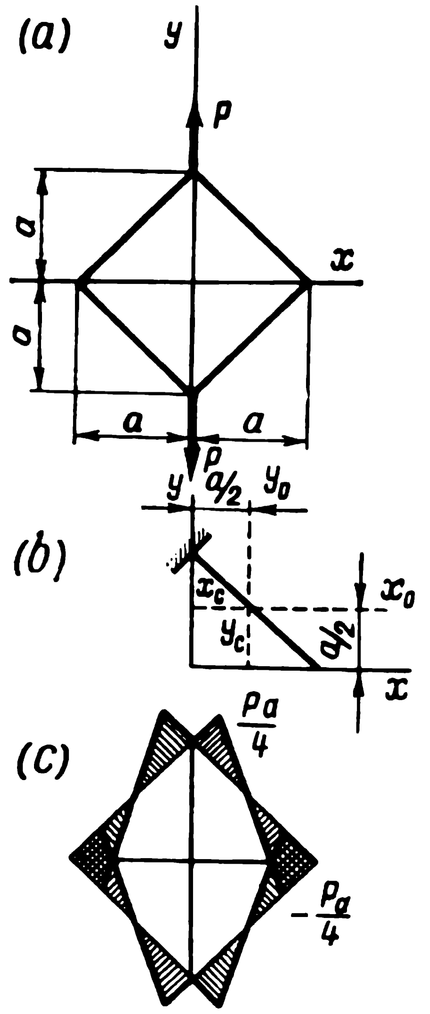


Fig. 198

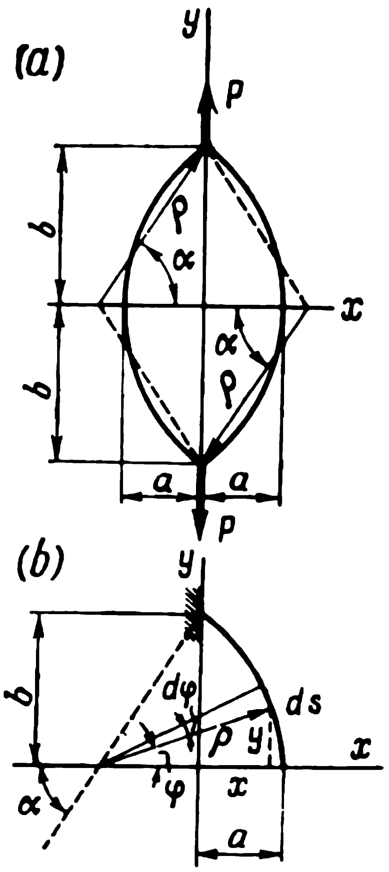


Fig. 199

The bending moment diagram is shown in Fig. 198c. On the centroidal axes x_0y_0:

$$x_0 = -y_0; \quad ds = dx_0 \sqrt{2} = dy_0 \sqrt{2}$$

therefore

$$I_{x_0} = I_{y_0} = \sqrt{2} \int_{-a/2}^{a/2} x_0^2 \, dx_0 = \frac{\sqrt{2}}{12} a^3 \text{ and } I_{x_0y_0}$$

$$= -\sqrt{2} \int_{-a/2}^{a/2} x_0^2 \, dx_0 = -\frac{\sqrt{2}}{12} a^3$$

According to the formulas of Example 109, it follows that the shortening of the horizontal and lengthening of the vertical semidiagonals of the frame are the same in magnitude and equal

$$\Delta a = \frac{P}{2EI} \times \frac{\sqrt{2}}{12} a^3 = \frac{\sqrt{2}}{24} \times \frac{Pa^3}{EI}$$

B. A ring made up of two circular acrs of radius ρ and central angle 2α (Fig. 199a) is stretched vertically.

Since (Fig. 199b)

$$ds = \rho \, d\varphi; \quad s = \rho\alpha; \quad a = \rho(1 - \cos\alpha); \quad b = \rho \sin\alpha;$$

$x = \rho(\cos\varphi - \cos\alpha)$ and $y = \rho \sin\alpha$, then

$$S_y = \int_0^s x \, ds = \rho^2 \int_0^\alpha (\cos\varphi - \cos\alpha) \, d\varphi = \rho^2 (\sin\alpha - \alpha \cos\alpha);$$

$$x_c = \frac{S_y}{s} = \frac{\rho}{\alpha} (\sin\alpha - \alpha\cos\alpha);$$

$$S_x = \int_0^s y \, ds = \rho^2 \int_0^\alpha \sin\varphi \, d\varphi = \rho^2 (1 - \cos\alpha);$$

$$I_{xy} = \int_0^s xy \, ds = \rho^3 \int_0^\alpha (\cos\varphi - \cos\alpha) \sin\varphi \, d\varphi$$

$$= \frac{\rho^3}{2} (1 - 2\cos\alpha + \cos^2\alpha);$$

$$I_y = \int_0^s x^2 \, ds = \rho^3 \int_0^\alpha (\cos\varphi - \cos\alpha)^2 \, d\varphi$$

$$= \frac{\rho^3}{4} (2\alpha + 4\alpha\cos^2\alpha - 3\sin 2\alpha)$$

According to the formulas of Example 109

$$X_1 = \frac{P}{2}(x_c - a) = \frac{P\rho}{2\alpha}(\sin\alpha - \alpha);$$

$$M = \frac{P}{2}(x_c - x) = \frac{P\rho}{2}\left(\frac{\sin\alpha}{\alpha} - \cos\varphi\right);$$

$$\Delta a = \frac{P}{2EI}(x_c S_x - I_{xy}) = \frac{P\rho^3}{2EI}\left(\frac{\sin\alpha}{\alpha} - \frac{\sin 2\alpha}{2\alpha} + \frac{\cos^2\alpha}{2} - \frac{1}{2}\right);$$

$$\Delta b = \frac{P}{2EI}(I_y - x_c S_y) = \frac{P\rho^3}{2EI}\left(\frac{\alpha}{2} - \frac{\sin^2\alpha}{\alpha} + \frac{\sin 2\alpha}{4}\right)$$

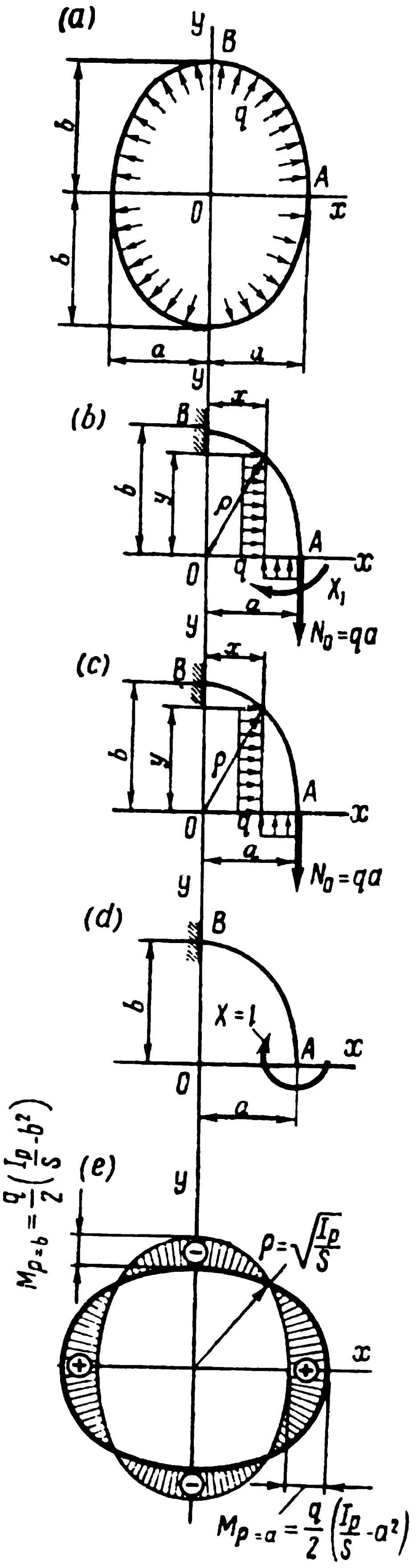


Fig. 200

Example 110. Given: a, b, E, I and q is the internal pressure on the walls of a ring symmetrical about axes x and y (Fig. 200a). Determine Δa and Δb.

Solution. We shall consider one quarter of the ring (Fig. 200b). In the sections coinciding with the x-axis the transverse (shearing) force equals zero, and the axial internal force equals qa.

The bending moments in an arbitrary cross section (with a centre of gravity having the coordinates x and y) of the basic (Fig. 200c) and auxiliary (Fig. 200d) systems are equal, respectively, to

$$M = qa(a-x) - \frac{q}{2}(a-x)^2 - \frac{q}{2}y^2 = \frac{q}{2}[a^2 - (x^2 + y^2)] = \frac{q}{2}(a^2 - \rho^2)$$

in which $\rho = \sqrt{x^2 + y^2}$ is the distance of the section being considered from the origin of coordinates and $\overline{M} = 1$.

Since

$$EI\delta_{1p} = \int_0^s M\overline{M}\,ds = \frac{q}{2}\int_0^s (a^2 - \rho^2)\,ds$$

$$= \frac{q}{2}\left(a\int_0^s ds - \int_0^s \rho^2\,ds\right) = \frac{q}{2}(as - I_p)$$

in which s = length of the arc of the geometric axis of one quarter of the ring

$I_p = \int_0^s \rho^2\, ds$ = polar moment of inertia of arc s with respect to the origin of coordinates,

and $EI\delta_{11}\int_0^s \overline{M}^2\, ds = \int_0^s ds = s$, then the bending moment in section A is

$$X_1 = -\frac{\delta_{1p}}{\delta_{11}} = \frac{q}{2}\left(\frac{I_p}{s} - a^2\right)$$

The bending moment in an arbitrary section of the ring is

$$M = \frac{q}{2}(a^2-\rho^2) + \frac{q}{2}\left(\frac{I_p}{s} - a^2\right) = \frac{q}{2}\left(\frac{I_p}{s} - \rho^2\right);$$

$$M_{\rho=a} = \frac{q}{2}\left(\frac{I_p}{s} - a^2\right); \quad M_{\rho=b} = \frac{q}{2}\left(\frac{I_p}{s} - b^2\right)$$

If ρ increases continuously from value a to $b > a$, then $M_{\rho=a} > 0$; $M_{\rho=b} < 0$ and $M_{\rho=\sqrt{\frac{I_p}{s}}} = 0$.

The bending moment diagram for the case when $b > a$ is shown in Fig. 200*e*.

To determine the change in dimension a in section A of the quarter of the ring we apply a horizontal force $P_f = 1$ directed towards the centre O.

Due to this force $\overline{M} = y$, hence

$$\Delta a = \frac{q}{2EI}\int_0^s \left(\frac{I_p}{s} - \rho^2\right) y\, ds = \frac{q}{2EI}\left(\frac{I_p}{s}\int_0^s y\, ds - \int_0^s \rho^2 y\, ds\right)$$

$$= \frac{q}{2EI}\left(\frac{S_x}{s} I_p - \int_0^s \rho^2 y\, ds\right) = \frac{q}{2EI}(y_c I_p - I_x)$$

in which $S_x = \int_0^s y\, ds$ = static moment of arc s about the x-axis

$y_c = \frac{S_x}{s}$ = ordinate of the centre of gravity of arc s

$$I_x = \int_0^s \rho^2 y\, ds$$

To determine the change in dimension b in section A of the quarter of the ring we apply a vertical downward force $P_f = 1$.

Due to this force $\overline{M} = a - x$, hence

$$\Delta b = \frac{q}{2EI}\int_0^s \left(\frac{I_p}{s} - \rho^2\right)(a-x)\,ds = \frac{q}{2EI}\left(\frac{I_p}{s} a \int_0^s ds\right.$$
$$- a\int_0^s \rho^2\,ds - \frac{I_p}{s}\int_0^s x\,ds + \int_0^s \rho^2 x\,ds\Big) = \frac{q}{2EI}\Big(\frac{I_p}{s} as$$
$$\left. - aI_p - \frac{I_p}{s} S_y + I_y\right) = \frac{q}{2EI}(I_y - x_c I_p)$$

in which $I_y = \int_0^s \rho^2 x\,ds$

$S_y = \int_0^s x\,ds$ = static moment of arc s about axis y

$x_c = \frac{S_y}{s}$ = abscissa of the centre of gravity of arc s.

Particular case. A ring is made up of two semicircles of radius a and of two straight lines of length $2a$ (Fig. 201a).

Since (Fig. 201b) $s = \frac{2+\pi}{2}a$; $x_1 = a$; $y_1 = y$; $ds_1 = dy$;

$x_2 = a\cos\varphi$; $y_2 = a(1+\sin\varphi)$; $ds_2 = a\,d\varphi$;

$\rho_1^2 = a^2 + y^2$; $\rho_2^2 = 2a^2(1+\sin\varphi)$

then

$$S_x = \int_0^a y_1\,ds_1 + \int_0^{\frac{\pi}{2}a} y_2\,ds_2 = \int_0^a y\,dy + a^2\int_0^{\frac{\pi}{2}}(1+\sin\varphi)\,d\varphi = \frac{3+\pi}{2}a^2;$$

$$S_y = \int_0^a x_1\,ds_1 + \int_0^{\frac{\pi}{2}a} x_2\,ds_2 = a\int_0^a dy + a^2\int_0^{\frac{\pi}{2}}\cos\varphi\,d\varphi = 2a^2;$$

$$x_c = \frac{S_y}{s} = \frac{4a}{2+\pi}; \quad y_c = \frac{S_x}{s} = \frac{3+\pi}{2+\pi}a;$$

$$I_x = \int_0^a y_1^2\,ds_1 + \int_0^{\frac{\pi}{2}a} y_2^2\,ds_2 = \int_0^a y^2\,dy + a^3\int_0^{\frac{\pi}{2}}(1+\sin\varphi)^2\,d\varphi = \frac{28+9\pi}{12}a^3;$$

$$I_y = \int_0^a x_1^2\,ds_1 + \int_0^{\frac{\pi}{2}a} x_2^2\,ds_2 = a^2\int_0^a dy + a^3\int_0^{\frac{\pi}{2}}\cos^2\varphi\,d\varphi = \frac{4+\pi}{4}a^3;$$

$$I_p = I_x + I_y = \frac{10+3\pi}{3}a^3;$$

$$I_x = \int_0^a \rho_1^2 y_1\,ds_1 + \int_0^{\frac{\pi}{2}a} \rho_2^2 y_2\,ds_2 = \int_0^a (a^2+y^2)y\,dy$$

$$+2a^4\int_0^{\frac{\pi}{2}} (1+\sin\varphi)^2\,d\varphi = \frac{19+6\pi}{4}a^4$$

$$I_y = \int_0^a \rho_1^2 x_1\,ds_1 + \int_0^{\frac{\pi}{2}a}$$

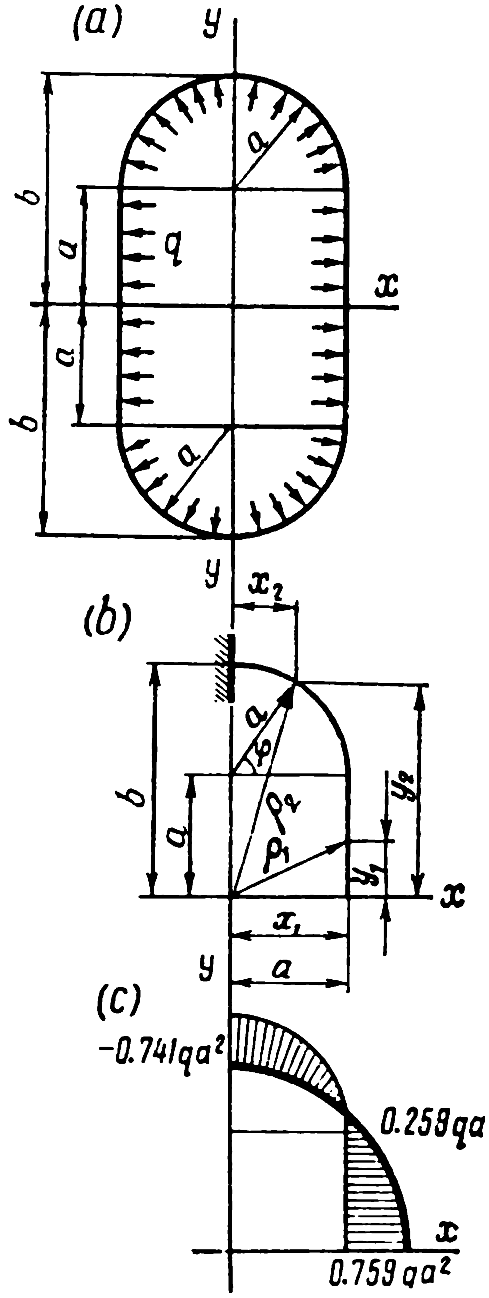


Fig. 201

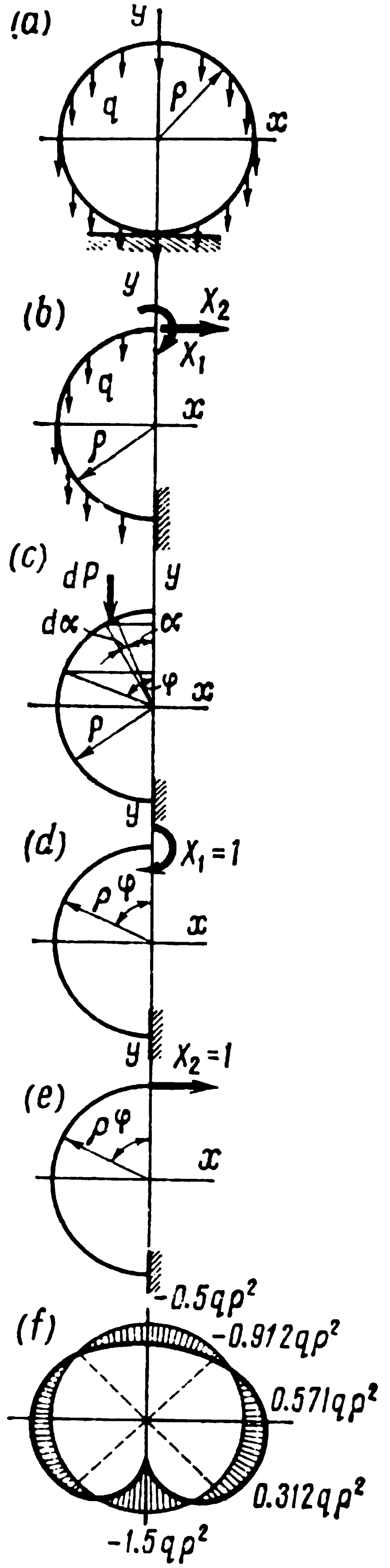


Fig. 202

$$\rho_2^2 x_2\,ds_2 = a\int_0^a (a^2+y^2)\,dy + 2a^4\int_0^{\frac{\pi}{2}} (1+\sin\varphi)\cos\varphi\,d\varphi = \frac{13}{3}a^4$$

For the straight portion:

$$M_1 = \frac{q}{2}\left(\frac{I_P}{s} - \rho_1^2\right) = \frac{qa^2}{2}\left(\frac{14+3\pi}{6+3\pi} - \frac{y^2}{a^2}\right) \cong \frac{qa^2}{2}\left(1.519 - \frac{y^2}{a^2}\right);$$

$$M_{1_{y=0}} = \frac{14+3\pi}{6(2+\pi)}qa^2 \cong 0.759qa^2;$$

$$M_{1_{y=a}} = \frac{4}{3(2+\pi)}qa^2 \cong 0.259qa^2$$

For the curvilinear portion:

$$M_2 = \frac{q}{2}\left(\frac{I_P}{s} - \rho^2\right) = qa^2\left(\frac{4}{6+3\pi} - \sin\varphi\right) \cong qa^2(0.259 - \sin\varphi);$$

$$M_{2_{\varphi=0}} = \frac{4}{3(2+\pi)}qa^2 \cong 0.259qa^2;$$

$$M_{2_{\varphi=\frac{\pi}{4}}} = -\frac{2+3\pi}{3(2+\pi)}qa^2 \cong -0.741qa^2$$

The bending moment diagram for the quarter of the ring is shown in Fig. 201*c*.

The change in dimension a is

$$\Delta a = \frac{q}{2EI}(y_c I_P - I_x) = \frac{6-17\pi-6\pi^2}{24(2+\pi)} \times \frac{qa^4}{EI} \cong -0.864\frac{qa^4}{EI}$$

The change in dimension b is

$$\Delta b = \frac{q}{2EI}(I_y - x_c I_P) = \frac{\pi-14}{12(2+\pi)} \times \frac{qa^4}{EI} \cong -0.160\frac{qa^4}{EI}$$

Example 111. Given: q, ρ, E and I for a thin-walled ring loaded symmetrically about the y-axis (Fig. 202*a*).

Determine δ, the change in the length of the vertical diameter of the ring.

Solution. We shall consider one half of the ring (Fig. 202*b*). In the section coinciding with axis y the transverse (shearing) force equals zero, while the bending moment X_1 and the axial internal force X_2 are the redundant generalized unknown forces.

The canonical equations of the force method are of the form

$$\left.\begin{aligned} \delta_{11}X_1 + \delta_{12}X_2 + \delta_{1P} &= 0; \\ \delta_{21}X_1 + \delta_{22}X_2 + \delta_{2P} &= 0 \end{aligned}\right\}$$

Next we find the bending moment in an arbitrary cross section specified by angle φ. This moment is due to the given load q (Fig. 202*c*).

Since the element of force acting on the element ds of the arc of the geometric axis of the ring wall is $dp = q\,ds = q\rho\,d\alpha$, the resulting element of bending moment in the section at the angle φ is

$$dM = dp\rho\,(\sin\varphi - \sin\alpha) = q\rho^2\,(\sin\varphi - \sin\alpha)\,d\alpha$$

The bending moment in the same section due to all the forces acting on the arc with the central angle φ equals

$$M = q\rho^2 \int_0^{\varphi} (\sin\varphi - \sin\alpha)\,d\alpha = q\rho^2\,(\varphi\sin\varphi + \cos\varphi - 1)$$

For the auxiliary systems (Fig.202 d and e) the bending moments in the same section equal $\overline{M}_1 = 1$, $M_2 = \rho\,(1 - \cos\varphi)$.
Since

$$EI\delta_{1P} = \int_0^{s} M\overline{M}_1\,ds = q\rho^3 \int_0^{\pi} (\varphi\sin\varphi + \cos\varphi - 1)\,d\varphi = 0;$$

$$EI\delta_{2p} = \int_0^{s} M\overline{M}_2\,ds = q\rho^4 \int_0^{\pi} (\varphi\sin\varphi + \cos\varphi - 1)(1 - \cos\varphi)\,d\varphi = -\frac{\pi}{4}\,q\rho^4;$$

$$EI\delta_{11} = \int_0^{s} \overline{M}_1^2\,ds = \rho \int_0^{\pi} d\varphi = \pi\rho;$$

$$EI\delta_{22} = \int_0^{s} \overline{M}_2^2\,ds = \rho^3 \int_0^{\pi} (1 - \cos\varphi)^2\,d\varphi = \frac{3\pi}{2}\,\rho^3 \text{ and}$$

$$EI\delta_{12} = EI\delta_{21} = \int_0^{s} \overline{M}_1\overline{M}_2\,ds = \rho^2 \int_0^{\pi} (1 - \cos\varphi)\,d\varphi = \pi\rho^2$$

the canonical equations of the force method can be rewritten in the following form:

$$\left.\begin{aligned} X_1 + \rho X_2 &= 0; \\ X_1 + \frac{3}{2}\rho X_2 &= \frac{q\rho^2}{4} \end{aligned}\right\}$$

from which

$$X_1 = -\frac{q\rho^2}{2} \text{ and } X_2 = \frac{q\rho}{2}$$

The bending moment in an arbitrary section of the ring wall is

$$M = q\rho^2\,(\varphi\sin\varphi + \cos\varphi - 1) - \frac{q\rho^2}{2} + \frac{q\rho^2}{2}\,(1 - \cos\varphi)$$
$$= q\rho^2\left(\varphi\sin\varphi + \frac{1}{2}\cos\varphi - 1\right)$$

Therefore

$$M_{\varphi=0}=-0.5q\rho^2;\quad M_{\varphi=\frac{\pi}{4}}=q\rho^2\left(\frac{\pi}{4}\times\frac{\sqrt{2}}{2}+\frac{1}{2}\times\frac{\sqrt{2}}{2}-1\right)\cong-0.0912q\rho^2;$$

$$M_{\varphi=\frac{\pi}{2}}=q\rho^2\left(\frac{\pi}{2}-1\right)\cong 0.571q\rho^2;$$

$$M_{\varphi=\frac{3}{4}\pi}=q\rho^2\left(\frac{3\pi}{4}\times\frac{\sqrt{2}}{2}-\frac{1}{2}\times\frac{\sqrt{2}}{2}-1\right)\cong 0.312q\rho^2$$

$$M_{\varphi=\pi}=-1.5q\rho^2$$

The bending moment diagram is shown in Fig. 202*f*.

To determine the change in the length of the vertical diameter of the ring in the section coinciding with axis y of the basic system (Fig. 202*b*) we apply a downward vertical force $P_f=1$. Then we find the bending moment in an arbitrary section due to this force. Thus

$$\overline{M}=\rho\sin\varphi$$

Hence, the required displacement is

$$\delta=\frac{1}{EI}\int_0^s M\overline{M}\,ds=\frac{q\rho^4}{EI}\int_0^\pi\left(\varphi\sin\varphi+\frac{1}{2}\cos\varphi-1\right)\sin\varphi\,d\varphi$$

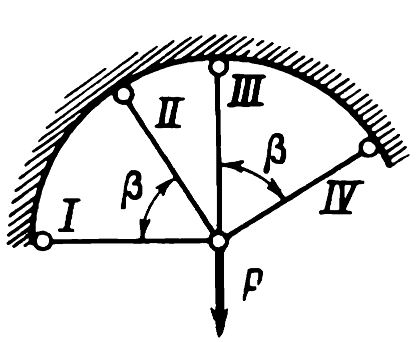
$$=\left(\frac{\pi^2}{4}-2\right)\frac{q\rho^4}{EI}\cong 0.467\frac{q\rho^4}{EI}$$

Problems 834 and 835. Determine the axial internal force N in the enumerated bars of the systems.

The rigidity EF is the same for all the bars.

In these and the following statically indeterminate problems use either the principle of least work or the force method (as you like).

834



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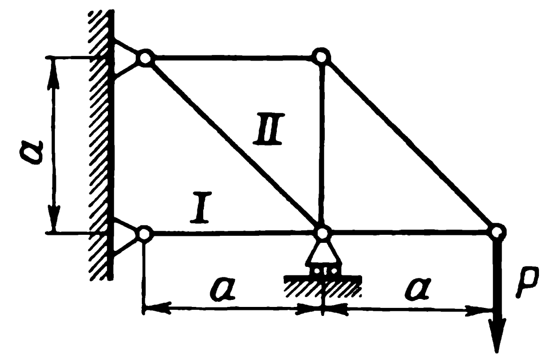
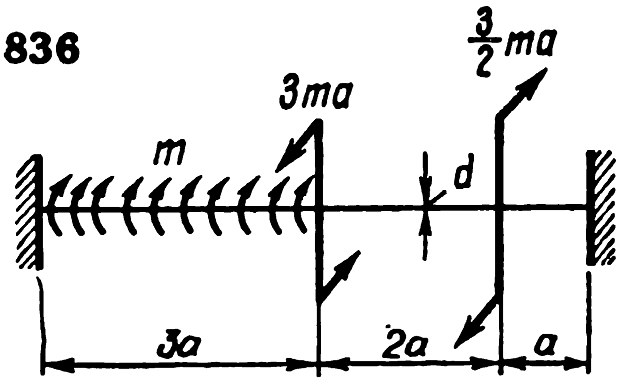


Problems 836 and 837. Determine the moments at the support in the left-hand fixed end of the bars.

836



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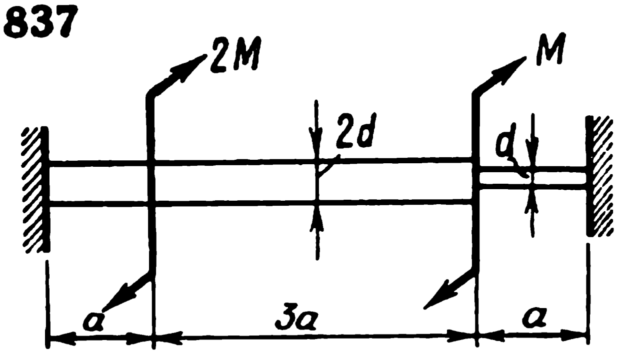
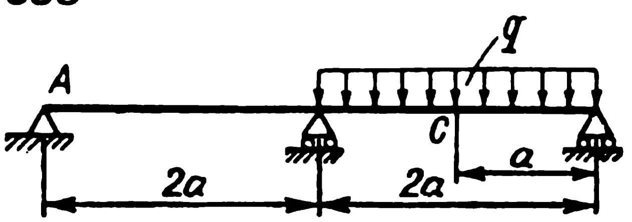


Problems 838 through 841. Determine the reactions in support sections A and vertical displacements δ of section C of the beams. In Problem 841 also determine the reaction moment M_A in section A. The rigidities EI are assumed to be known.

838



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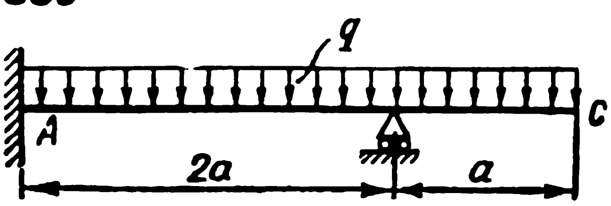

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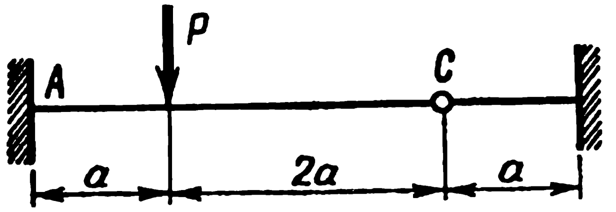

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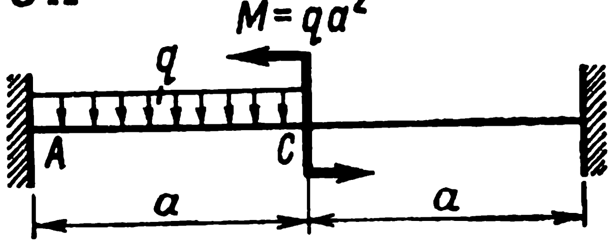

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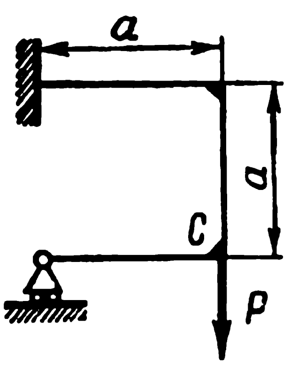

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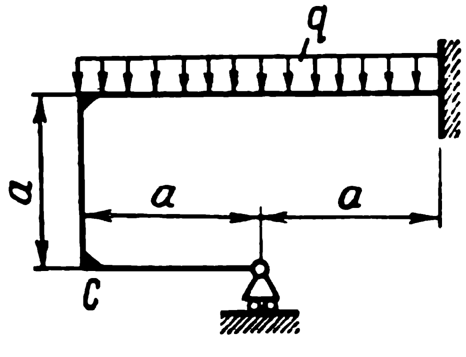
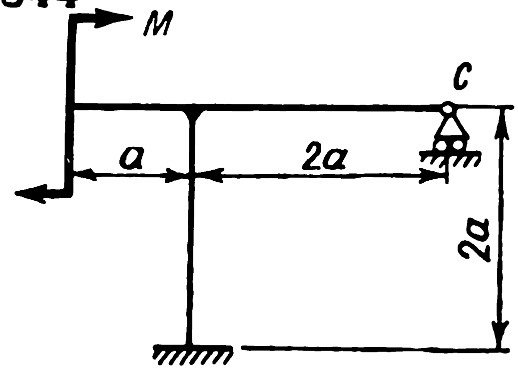


Problems 842 through 845. Determine the reactions of the hinged support of the systems and the linear displacements of section C. In Problems 842 and 843 δ_v are vertical and in Problems 844 and 845 δ_h are horizontal displacements*.

* In Problems 842 through 866 do not take into account the longitudinal strain of the elements of the system and assume the rigidities EI of the elements subject to bending to be equal.

844



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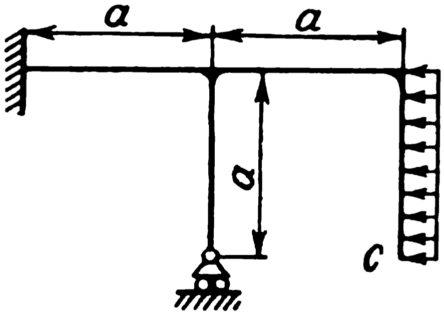
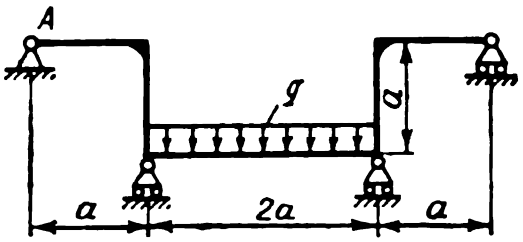


Problems 846 and 847. Determine the reactions of support A and the vertical displacements δ of the section at the axis of symmetry of the systems.

846



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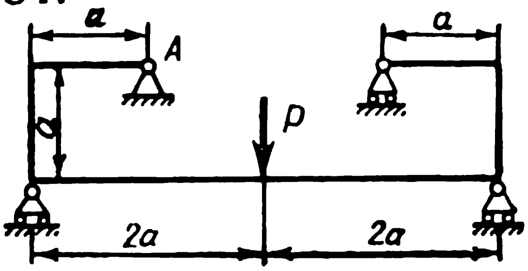
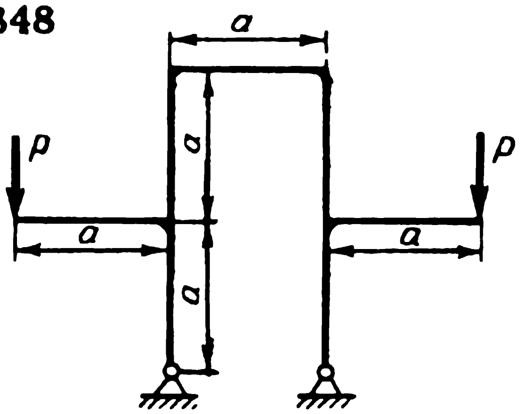


Problems 848 through 851. Determine the horizontal reactions of the supports in the systems.

848



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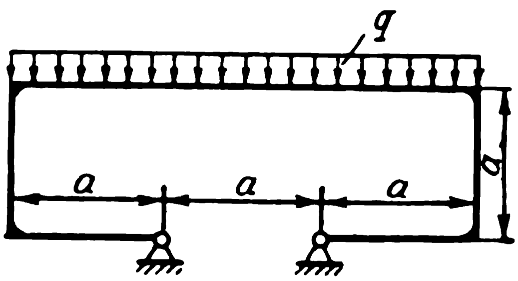

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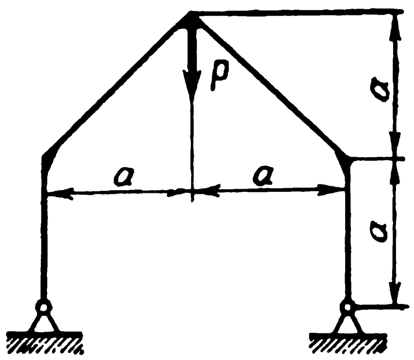

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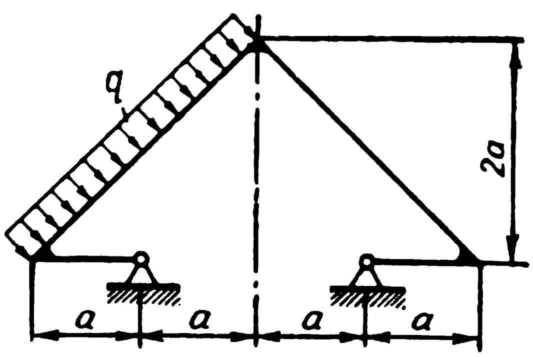
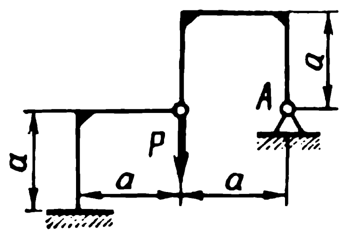


Problems 852 through 857. Determine the component reactions of support A of the systems.

852



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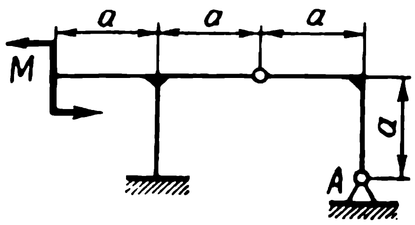

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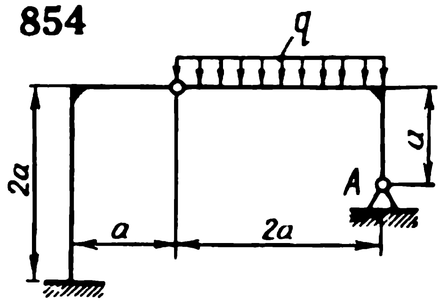

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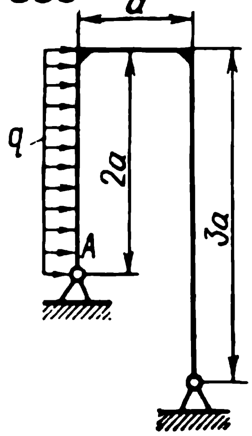

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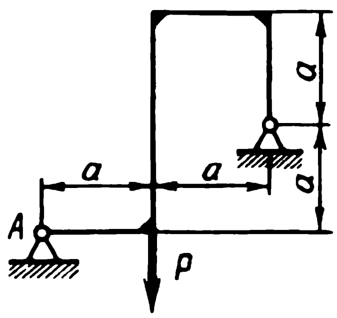

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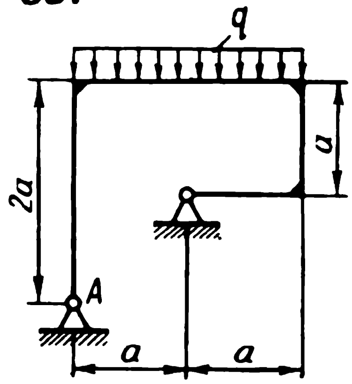
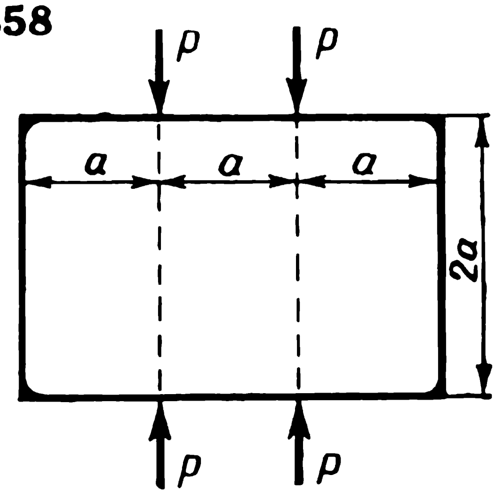


Problems 858 through 861. Determine the maximum bending moment (in absolute value) in the frame systems.

858



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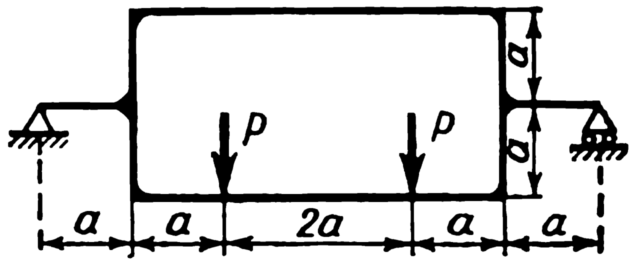

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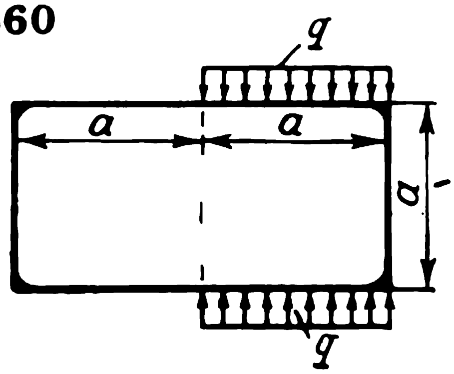

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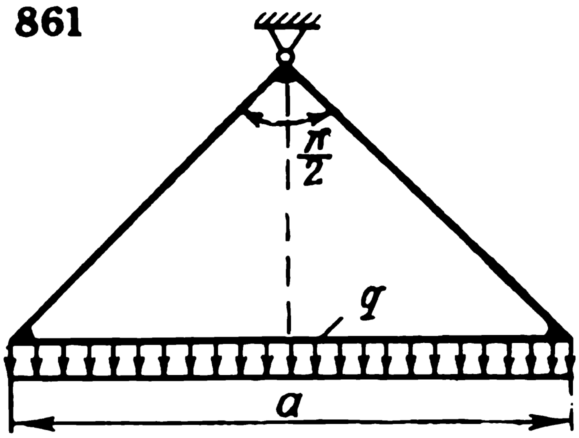
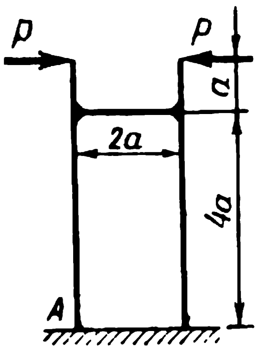


Problems 862 through 870. Determine the reaction components in the support section A of the systems.

Additionally, in Problems 868, 869 and 870 determine the vertical displacements δ of section C, assuming that the sections are round in all the portions of the systems and the material is the same; $G = 0.4\,E$.

862



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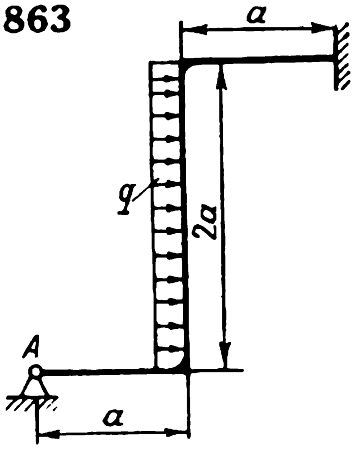

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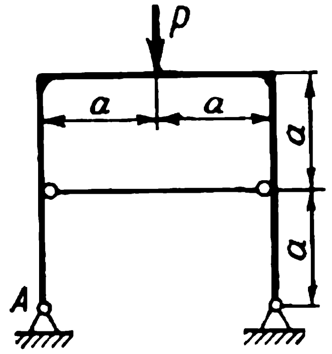

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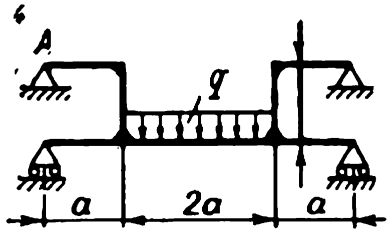

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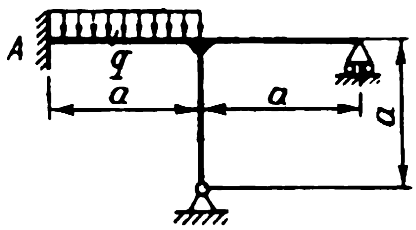

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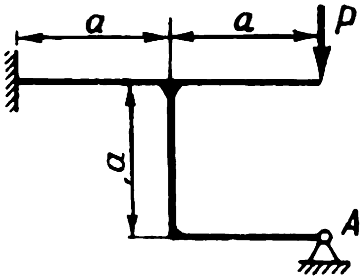

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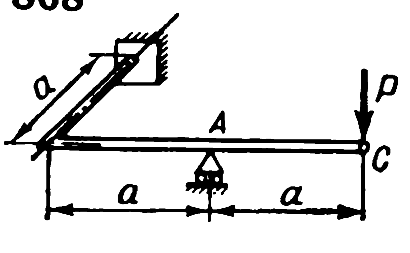

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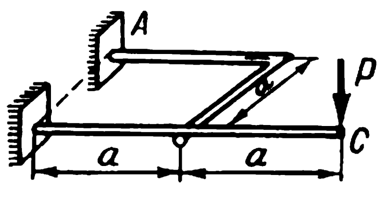

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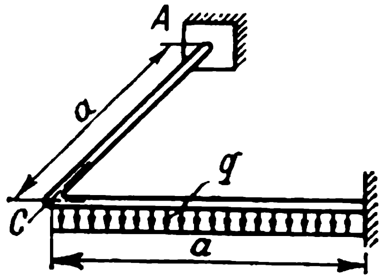
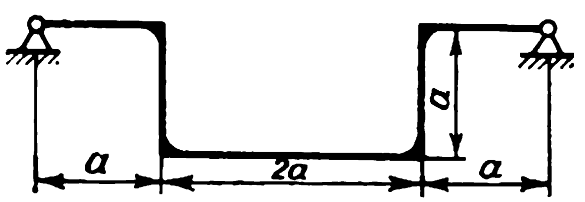


Problems 871 through 875. Determine the maximum normal stresses induced in the elements of the systems due to a rise of temperature by Δt°C.

871



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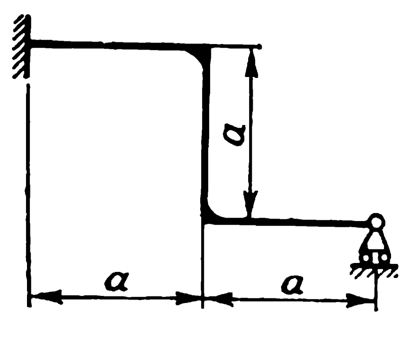
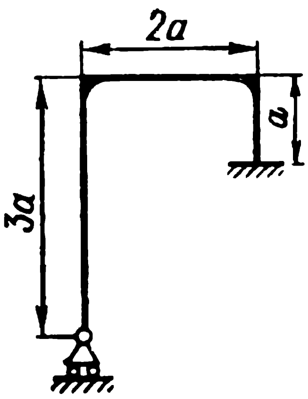


Take into account only the bending strains. All the elements of each system have the same symmetrical section of height h and are of the same material with given E and α values.

873



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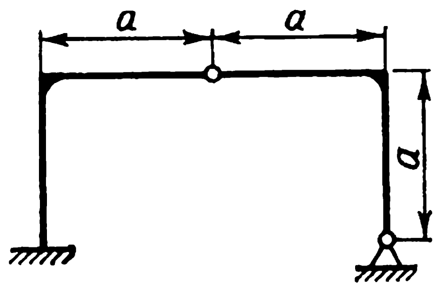

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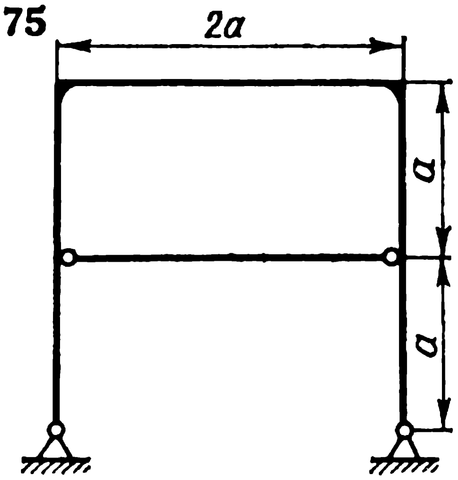
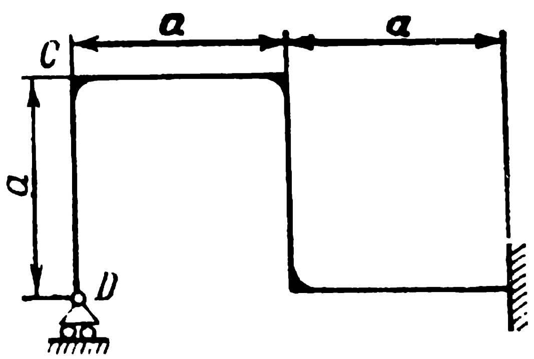


Problems 876 through 879. Determine the maximum assembly stresses in the systems, if the length of element CD exceeds the design value by Δ.

Take into account only the bending strains. All the elements of each system have the same symmetrical section of height h and are of the same material with a given E value.

876



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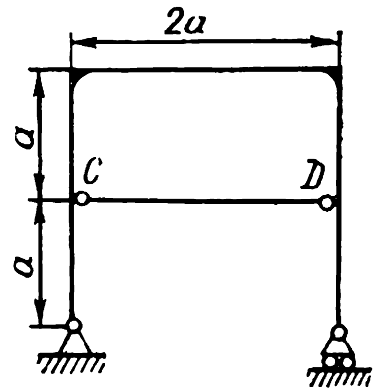

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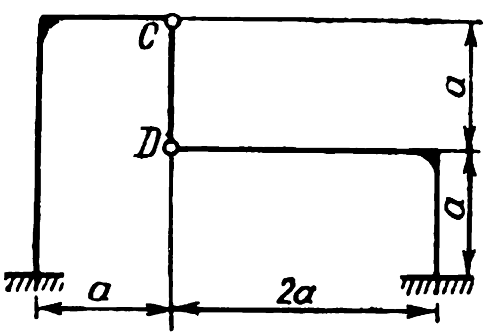

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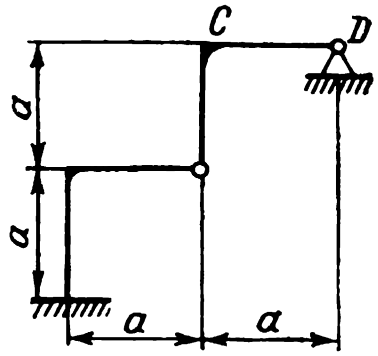
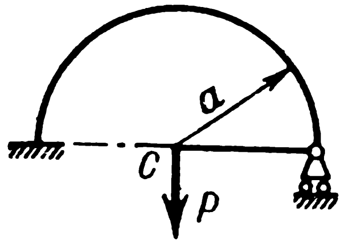


Problems 880 through 884. Determine the reaction of the right-hand hinged support and the vertical displacement δ of section C in the systems*.

In Problem 882 the load q is uniformly distributed along the horizontal.

880



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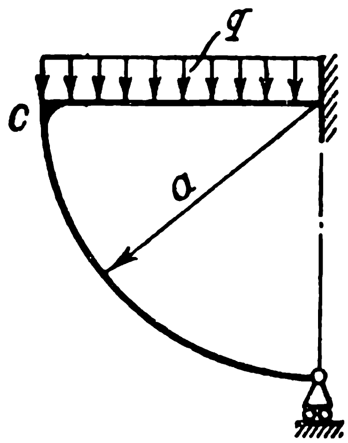

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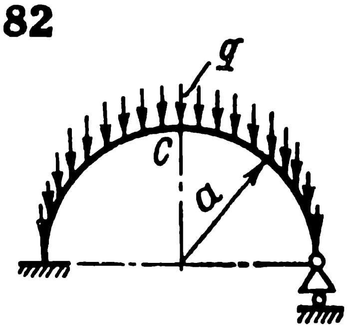

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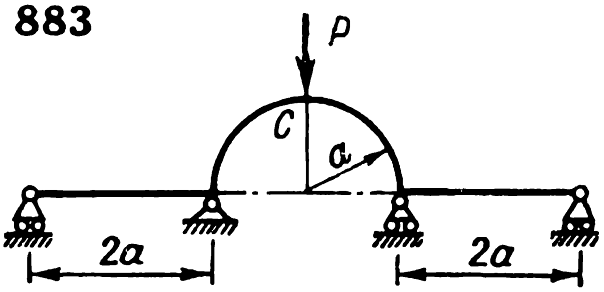

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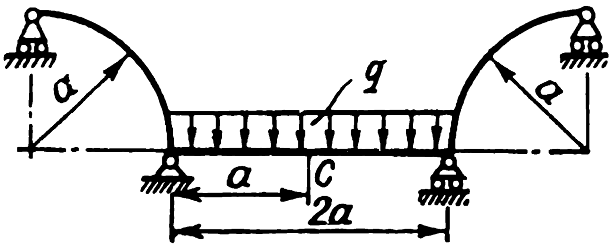

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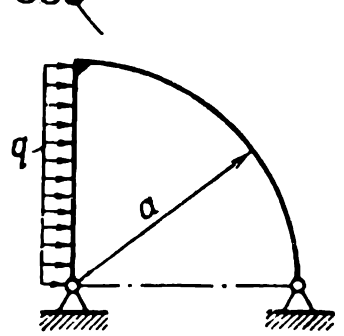

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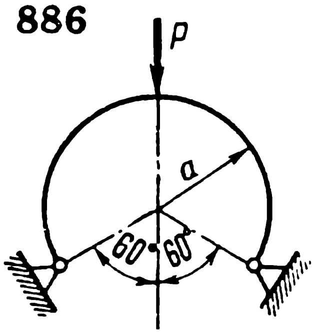

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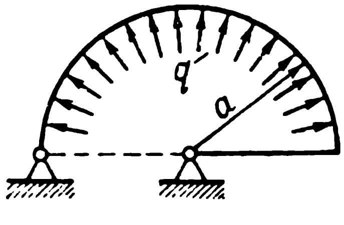

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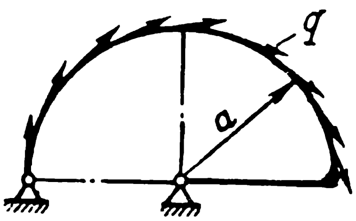

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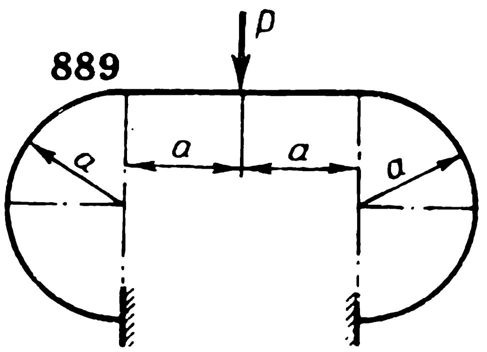

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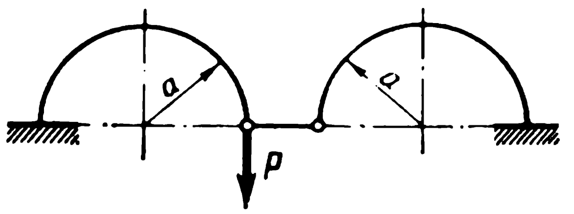
* In Problems 880 through 915 take into account only the strains due to the bending moment and assume the rigidities EI to be equal in all the elements of the systems.

Problems 885 through 889. Determine the reaction components in the support at the left end of the systems.

890



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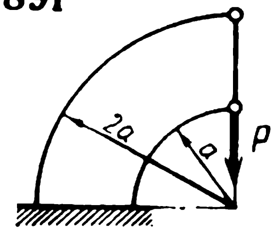

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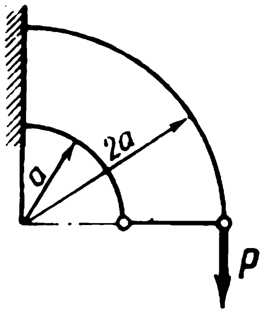

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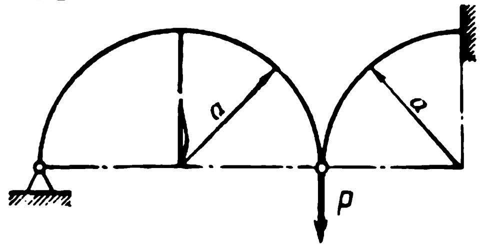

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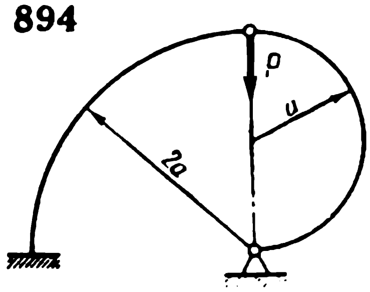

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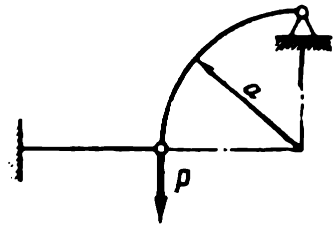
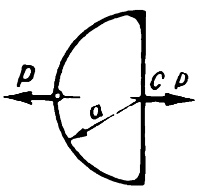


Problems 890 through 895. Determine the vertical displacement of the section at which force P is applied to the systems.

Problems 896 through 899. Determine the bending moments in section C of the systems.

896



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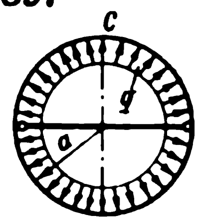

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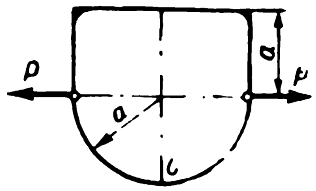

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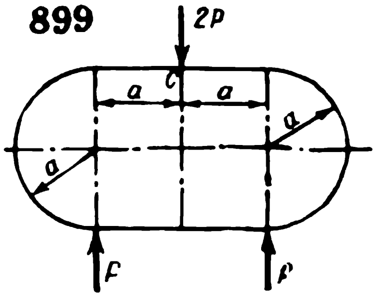
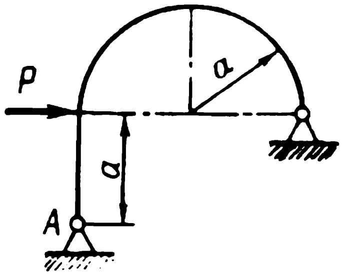


Problems 900 through 906. Determine the reaction components at the support section A of the systems.

In Problem 902 the load q is uniformly distributed along the horizontal.

900



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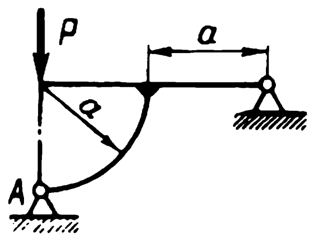

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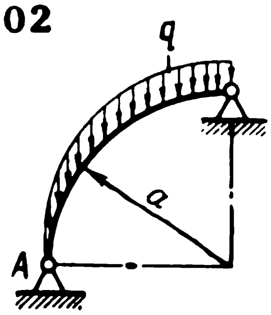

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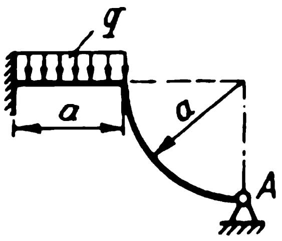

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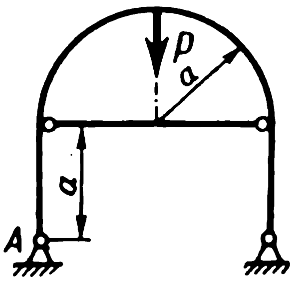

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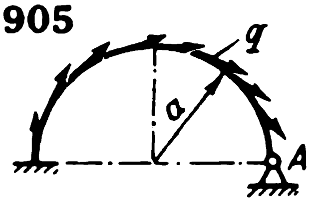

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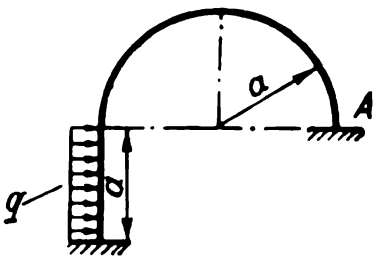
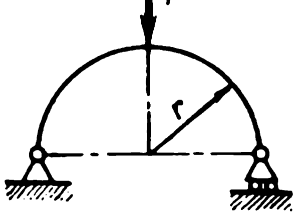
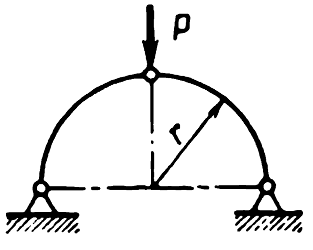


Problems 907 through 910. Determine by how many times the bending moment (maximum value) and the vertical displacement of the middle section in the statically determinate curved beam of the figure for Problem 907*a* exceed those of the indicated systems.

907 *a*





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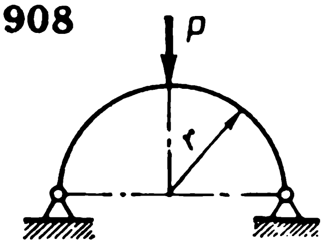

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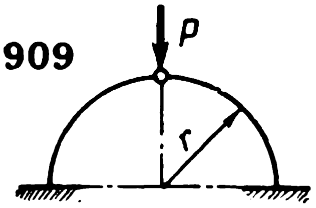

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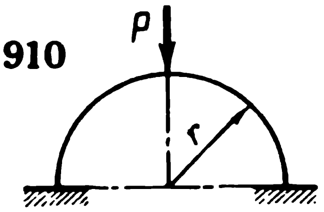
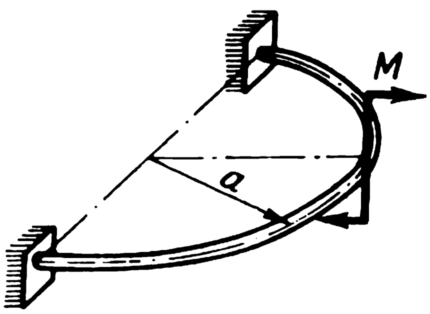


Problems 911, 912 and 913. Determine the reaction moments in the fixed ends and the vertical displacements of the middle section of the curved beams.

The beams are of round cross section; $G = 0.4E$.

911



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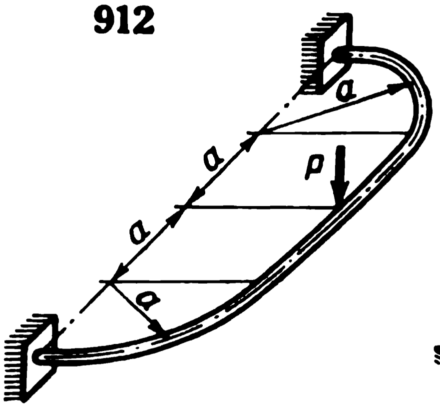

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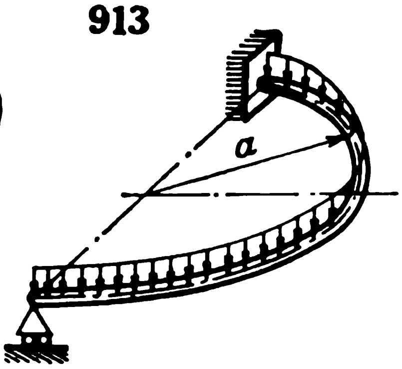
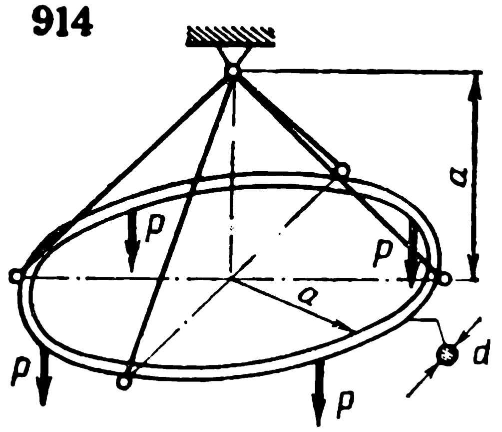


Problems 914 and 915. Determine the required dimensions of the cross sections of the round rings.

The permissible normal stress of the material of the rings is $[\sigma]$.

914



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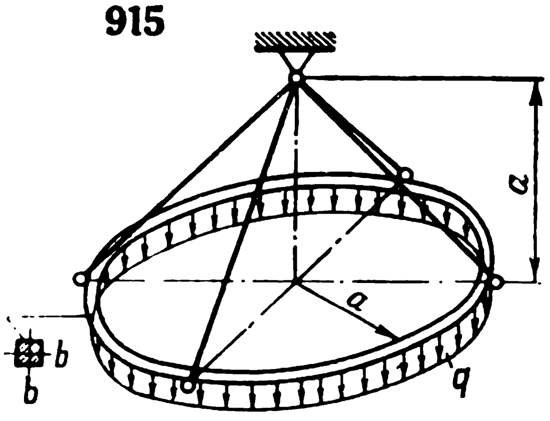


CHAPTER 13. THICK-WALLED TUBES

13.1.

Cylindrical Tubes

For a tube of infinite length without bottoms, subject to the action of the internal p_1 and external p_2 uniform radial pressures (Fig. 203), at an arbitrary point of the wall at a distance ρ from the centre, the

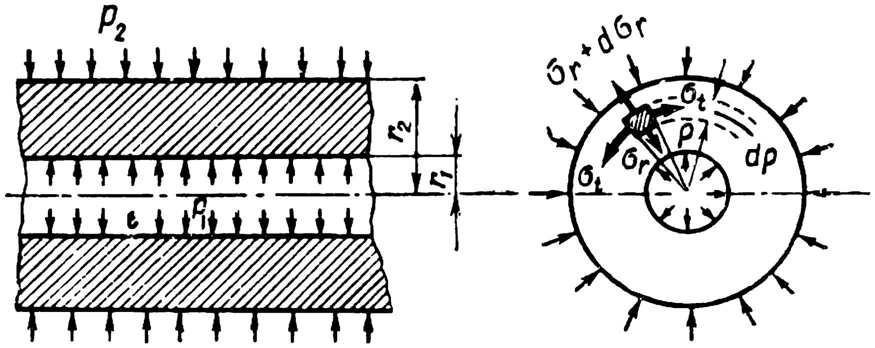


Fig. 203

circumferential (tangential) σ_t and radial σ_r normal stresses are found from the formulas

$$\sigma_t=\frac{p_1r_1^2\left(1+\frac{r_2^2}{\rho^2}\right)-p_2r_2^2\left(1+\frac{r_1^2}{\rho^2}\right)}{r_2^2-r_1^2};\qquad (202)$$

$$\sigma_r=\frac{p_1r_1^2\left(1-\frac{r_2^2}{\rho^2}\right)-p_2r_2^2\left(1-\frac{r_1^2}{\rho^2}\right)}{r_2^2-r_1^2}\qquad (203)$$

The stress $\sigma_r<0$ at any values of p_1 and p_2, whereas σ_t can be more or less than zero, depending on the ratio of values p_1 and p_2.

If $p_1>\frac{p_2}{2}\left(\frac{r_2^2}{r_1^2}+1\right)$, then $\sigma_t>0$. In this case the diagrams for σ_r and σ_t over the thickness of the tube are of the shape shown in Fig. 204.

The maximum and minimum stresses are

$$\left.\begin{aligned}
\max\sigma_r&=\sigma_{r_{\rho=r_2}}=-p_2;\\
\min\sigma_r&=\sigma_{r_{\rho=r_1}}=-p_1;\\
\max\sigma_t&=\sigma_{t_{\rho=r_1}}=\frac{p_1(r_1^2+r_2^2)-2p_2r_2^2}{r_2^2-r_1^2};\\
\min\sigma_t&=\sigma_{t_{\rho=r_2}}=\frac{2p_1r_1^2-p_2(r_2^2+r_1^2)}{r_2^2-r_1^2}
\end{aligned}\right\}\qquad (204)$$

If $p_1 = \frac{p_2}{2}\left(\frac{r_2^2}{r_1^2} + 1\right)$, then $\min \sigma_t = 0$.

At the dangerous points of the internal surface of the tube (Fig. 205) the design equation, in accordance with the fifth strength theory (O. Mohr's hypothesis), is as follows:

$$\max \sigma_t - \nu \min \sigma_r \leqslant [\sigma_t]$$

from which

$$\frac{r_2}{r_1} = \sqrt{\frac{[\sigma_t] + (1-\nu)\, p_1}{[\sigma_t] - (1+\nu)\, p_1 + 2p_2}} \qquad (205)$$

in which

$$\nu = \frac{[\sigma_t]}{[\sigma_c]}$$

For a material of equal tensile and compressive strength $[\sigma_t] = [\sigma_c] = [\sigma]$, $\nu = 1$ and the design equation (205) becomes:

$$\frac{r_2}{r_1} = \sqrt{\frac{[\sigma]}{[\sigma] - 2p_1 + 2p_2}} \qquad (206)$$

If $p_1 < \frac{2p_2}{1 + \frac{r_1^2}{r_2^2}}$, then $\sigma_t < 0$. In this case the diagrams for σ_r and σ_t are of the form shown in Fig. 206.

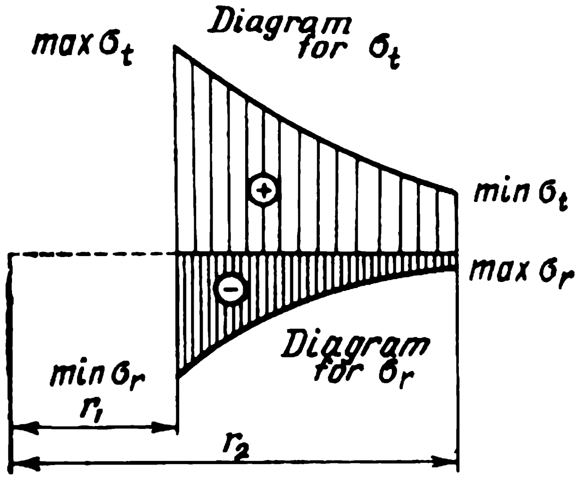


Fig. 204

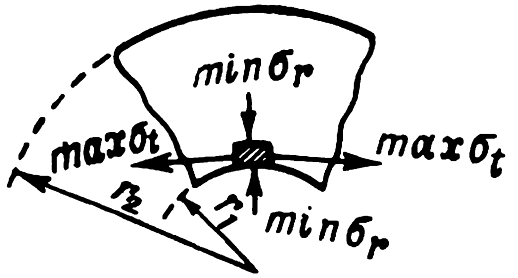


Fig. 205

The maximum and minimum values for the stresses are

$$\left.\begin{aligned}
\max \sigma_r &= \sigma_{r_{\rho=r_1}} = -p_1;\\
\min \sigma_r &= \sigma_{r_{\rho=r_2}} = -p_2;\\
\max \sigma_t &= \sigma_{t_{\rho=r_2}} = \frac{2p_1 r_1^2 - p_2 (r_2^2 + r_1^2)}{r_2^2 - r_1^2};\\
\min \sigma_t &= \sigma_{t_{\rho=r_1}} = \frac{p_1 (r_2^2 + r_1^2) - 2p_2 r_2^2}{r_2^2 - r_1^2}
\end{aligned}\right\} \qquad (207)$$

If $p_1 = \frac{2p_2}{1 + \frac{r_1^2}{r_2^2}}$, then $\max \sigma_t = 0$.

For the dangerous points of the internal surface of the tube (Fig. 207) the design equation, in accordance with the fifth strength theory is as follows:

$$-\nu \min \sigma_t = [\sigma_t]$$

from which

$$\frac{r_2}{r_1} = \sqrt{\frac{[\sigma_c] - p_1}{[\sigma_c] + p_1 - 2p_2}} \tag{208}$$

If the ratio of the pressure is within the range

$$\frac{1}{2}\left(\frac{r_2^2}{r_1^2} + 1\right) < \frac{p_1}{p_2} < \frac{1}{2}\left(\frac{r_1^2}{r_2^2} + 1\right)$$

the values of σ_t change their signs along the thickness of the tube wall. If $p_1 = p_2 = p$, then $\sigma_t = \sigma_r = -p$.

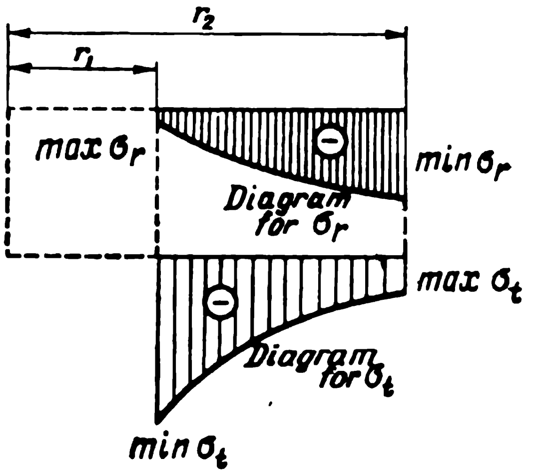


Fig. 206

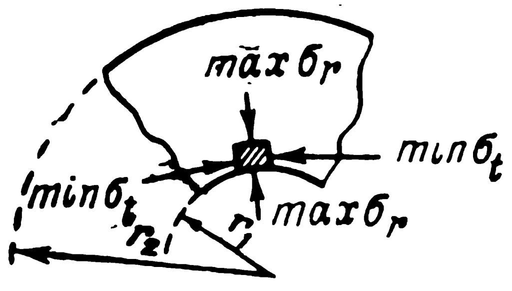


Fig. 207

The radial displacement δ of an arbitrary point of the tube wall is found from the formula

$$\delta = \frac{\rho}{E}(\sigma_t - \mu\sigma_r) = \frac{\rho}{E(r_2^2 - r_1^2)} \left\{ p_1 r_1^2 \left[1 + \frac{r_2^2}{\rho^2} - \mu\left(1 - \frac{r_2^2}{\rho^2}\right)\right] - p_2 r_2^2 \left[1 + \frac{r_1^2}{\rho^2} - \mu\left(1 - \frac{r_1^2}{\rho^2}\right)\right]\right\} \tag{209}$$

in which E and μ are Young's modulus and Poisson's ratio for the tube material.

The inside radius of the tube changes according to the following equation:

$$\Delta r_1 = \delta_{\rho = r_1} = \frac{r_1}{E}\left[p_1\left(\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} + \mu\right) - 2p_2 \frac{r_2^2}{r_2^2 - r_1^2}\right] \tag{210}$$

And the change in the outside radius of the tube is

$$\Delta r_2 = \delta_{\rho = r_2} = \frac{r_2}{E}\left[2p_1 \frac{r_1^2}{r_2^2 - r_1^2} - p_2\left(\frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} - \mu\right)\right] \tag{211}$$

If the tube is subject only to an internal pressure p_1, p_2 in formulas (202 through 211) should be equated to 0.

The diagrams for σ_t and σ_r will be similar, in this case, to the diagrams shown in Fig. 204 but at $p_2 = 0$.

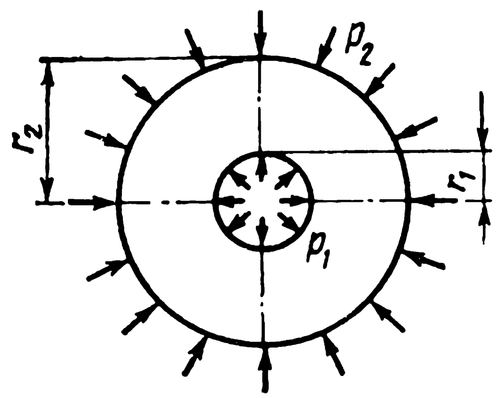


Fig. 208

If the tube is subject only to an external pressure p_2, in formulas (202 through 211) p_1 should be equated to 0.

The diagrams for σ_t and σ_r are similar, in this case, to the diagrams shown in Fig. 206 but at $p_1 = 0$.

Example 112. Let $p_2 = 1$ MN/m², $r_1 = 4$ cm, $r_2 = 8$ cm, $[\sigma_t] = 30$ MN/m², $[\sigma_c] = 120$ MN/m², $E = 1.2 \times 10^5$ MN/m² and $\mu = 0.24$ (Fig. 208).

Determine p_1, Δr_1 and Δr_2, assuming that $p_1 > p_2$.

Solution. From the design formula (205) we find

$$\frac{r_2}{r_1} = \sqrt{\frac{[\sigma_t] + (1-\nu)\, p_1}{[\sigma_t] - (1+\nu)\, p_1 + 2p_2}}$$

Since $\nu = \frac{[\sigma_t]}{[\sigma_c]} = 0.25$ and $\frac{r_2}{r_1} = 2$, substituting the numerical values and squaring both sides of the equation we obtain

$$4 = \frac{30 + 0.75p_1}{30 - 1.25p_1 + 2}$$

from which $p_1 = \frac{98}{5.75} \cong 17$ MN/m².

From formula (210)

$$\Delta r_1 = \frac{4}{1.2 \times 10^5}\left[17\left(\frac{64+16}{64-16} + 0.24\right) - 2\,\frac{64}{64-16}\right]$$

$$\cong 0.99 \times 10^{-3} \text{ cm} \cong 0.01 \text{ mm}$$

From formula (211)

$$\Delta r_2 = \frac{8}{1.2 \times 10^5}\left[2 \times 17 \times \frac{16}{64-16} - \left(\frac{64+16}{64-16} - 0.24\right)\right]$$

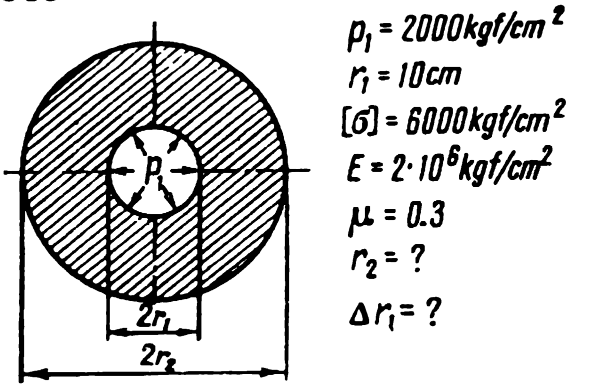
$$\cong 0.66 \times 10^{-3} \text{ cm} = 0.007 \text{ mm}$$

Problems 916, 917 and 918. Determine the required values.

In Problems 916, 917 and 918*a* use the third strength theory and in Problem 918*b*, the fifth strength theory.

Problem 919. A perfectly rigid tapered punch with a small angle of taper β is driven into a thick-walled ring of height a. Determine the

916



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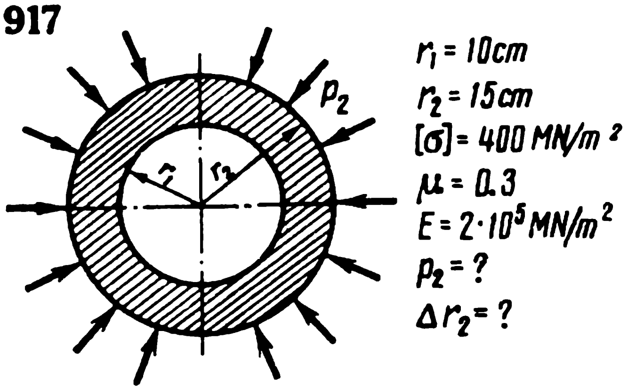

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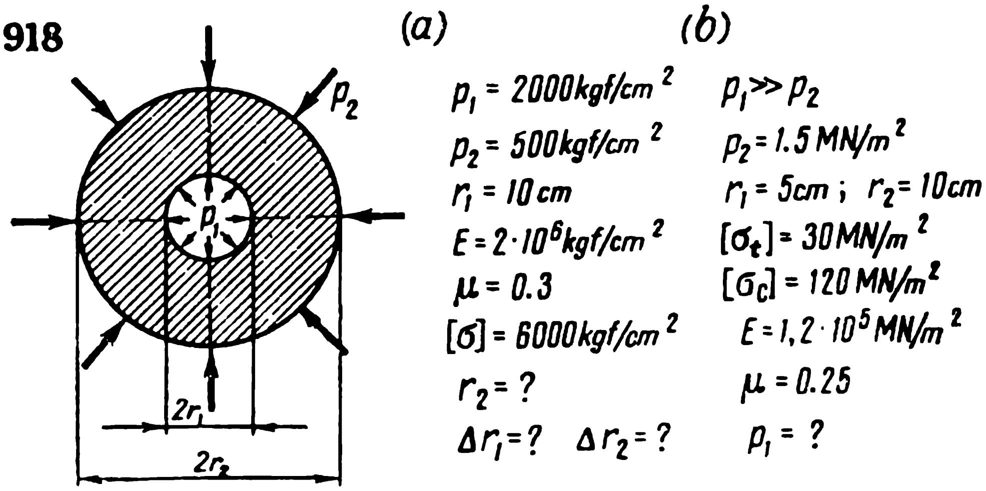
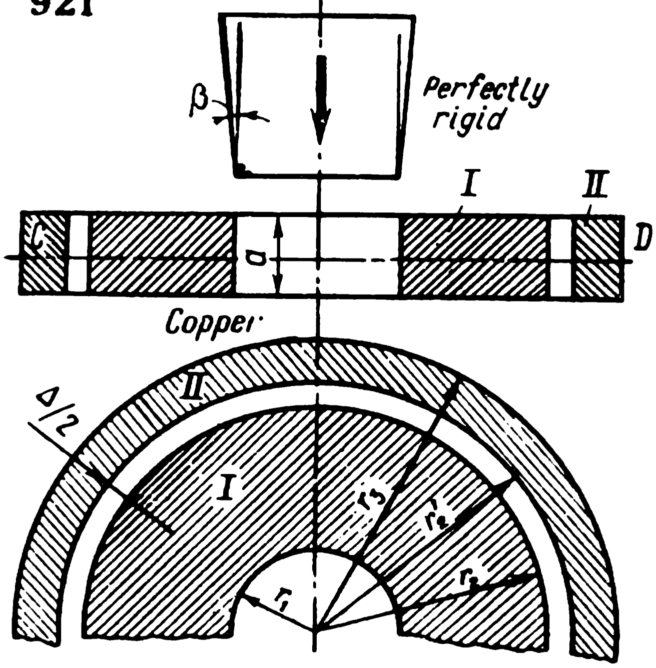


pressure p_0 developed between the taper and ring in the middle section CD of the ring when the punch is driven in to the whole height a of the ring. Also find the maximum stresses developed in section CD of the ring.

Problem 920. Determine at what angle of taper β of the tapered punch in the preceding problem the radius of the outside surface increases $\Delta r_2 = 0.2$ mm at the middle of the ring, if $r_1 = 10$ cm, $r_2 = 30$ cm, $E = 10^6$ kgf/cm^2, $\mu = 0.34$ and $a = 5$ cm.

Problem 921. Determine at what angle of taper β of the tapered punch, driven into the inner

921



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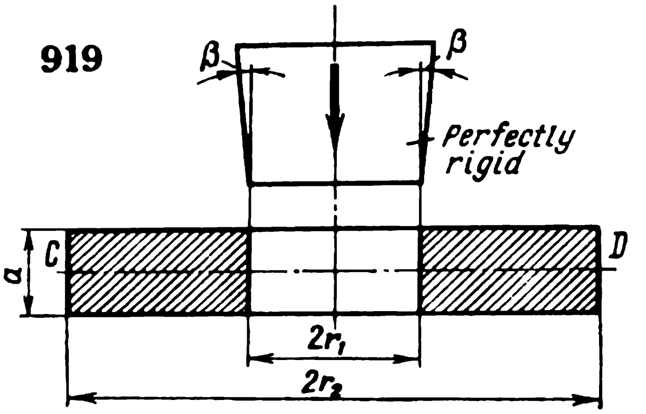
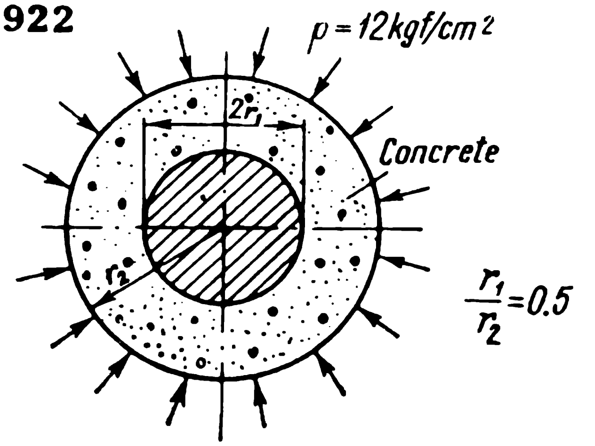


hole to the height a, the clearance between tubes I and II disappears and a mutual pressure of $p = 100$ kgf/cm^2 is developed between the tubes.

Calculations should be performed for the middle section CD, assuming that $r_1 = 10$ cm, $r_2 = 30$ cm, $r_3 = 40$ cm, $E = 10^6$ kgf/cm^2, $\mu = 0.34$, $a = 5$ cm and $\Delta = 0.4$ mm.

Problem 922. Determine the pressure p_0 between the concrete tube and perfectly rigid core; also check the strength of the tube, using the fifth strength theory. Assume that $E_c = 2 \times 10^5$ kgf/cm^2, $\mu_c = 0.16$, $[\sigma_c]_c = 20$ kgf/cm^2 and $[\sigma_t]_c = 4$ kgf/cm^2.

922



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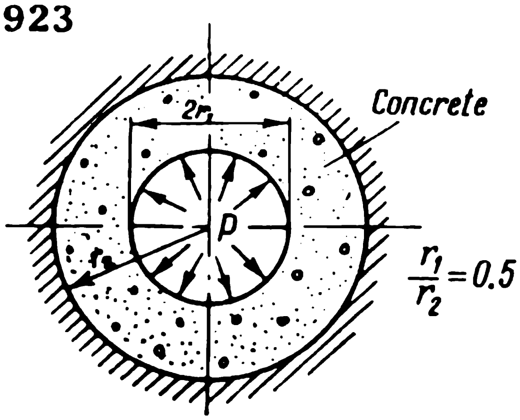
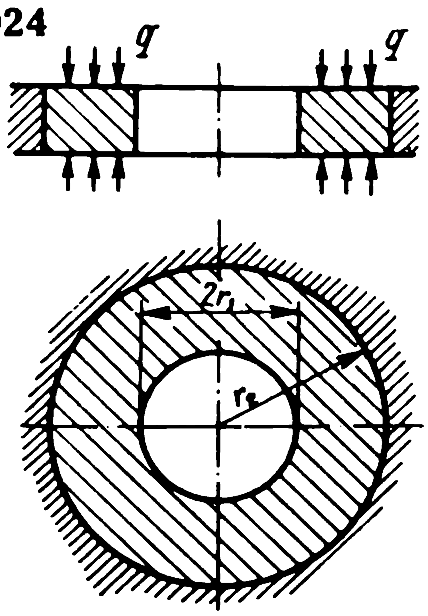


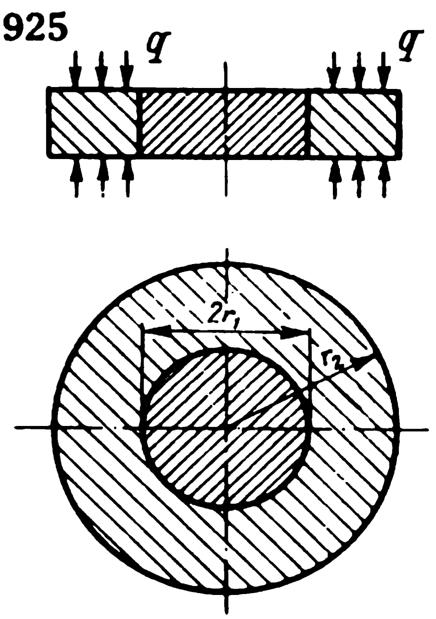
Problem 923. Determine the maximum internal pressure p which can be applied to the concrete tube surrounded by perfectly rigid shell, also find the boundary pressure p_0, if $E_c = 2 \times 10^5$ kgf/cm^2, $\mu_c = 0.16$, $[\sigma_c]_c = 50$ kgf/cm^2 and $[\sigma_t]_c = 5$ kgf/cm^2.

Problem 924. Determine the boundary pressure p_0 between the ring and a perfectly rigid shell, if the values of q, r_1, r_2 and μ are known. Assume that the modulus of elasticity of the ring is small.

924



925



Problem 925. Determine the pressure p_0 between the ring and a perfectly rigid core, if the values of q, r_1, r_2 and μ are known.

13.2.

Composite (Built-up) Cylindrical Tubes

Shrink fit or heavy-force fit composite tubes made from two or more cylinders are used for high internal pressures. The strength of the joint between the cylinders and the initial stresses developed in their walls are due to the fact that the inside diameter of the outer cylinder

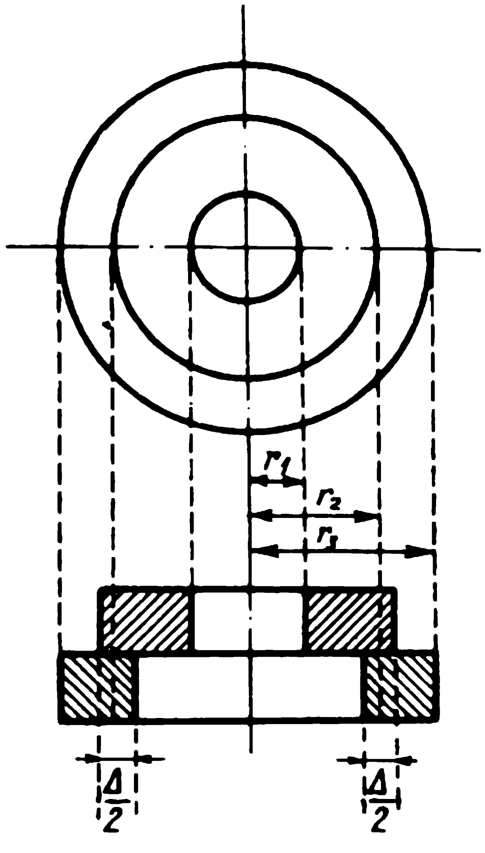


Fig. 209

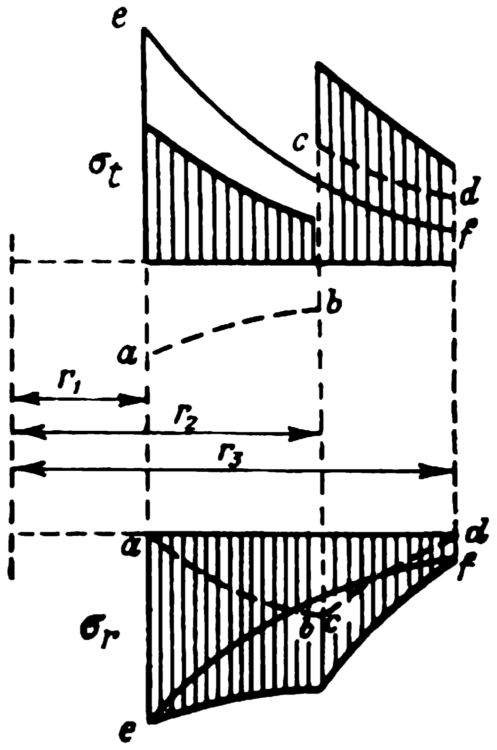


Fig. 210

is less than the outside diameter of the inner cylinder by the interference Δ (Fig. 209).

When the fitted cylinders are of equal length, the contact pressure p_0 is uniformly distributed over the fit surface and equals

$$p_0=\frac{\dfrac{\Delta}{2r_2}}{\dfrac{1}{E_1}\left(\dfrac{r_2^2+r_1^2}{r_2^2-r_1^2}-\mu_1\right)+\dfrac{1}{E_2}\left(\dfrac{r_3^2+r_2^2}{r_3^2-r_2^2}+\mu_2\right)} \qquad (212)$$

in which E_1, E_2, μ_1 and μ_2 are Young's moduli and Poisson's ratios of the materials of the inner and outer cylinders.

If the cylinders are made of the same material, then

$$E_1=E_2=E; \quad \mu_1=\mu_2$$

and

$$p_0=\frac{\Delta E}{4r_2^3}\times\frac{(r_3^2-r_2^2)(r_2^2-r_1^2)}{r_3^2-r_1^2} \qquad (213)$$

In the walls of the fitted cylinders the initial stresses due to the pressure p_0 are determined by formulas (202) and (203). It should be taken into account that for the inner cylinder with radii r_1 and r_2, the contact pressure p_0 is the external pressure (the internal pressure is zero) and for the outer cylinder with radii r_2 and r_3 the contact pressure p_0 is the internal pressure (the external pressure is zero).

Approximate diagrams for σ_t and σ_r due to pressure p_0 in the fitted cylinders are shown in Fig. 210 by the dash lines *ab* and *cd*.

The stresses σ_t and σ_r, developed in the composite tube due to a high internal working pressure p_1 and a low external pressure p_2, are found by formulas (202) and (203) as for a solid tube with radii r_1 and r_3.

Approximate diagrams for these stresses are shown in Fig. 210 by the thin solid lines *ef*.

The resultant stresses in the composite tube are found by the method of stress superposition (see lines *ab*, *cd* and *ef*).

The approximate diagrams for the resultant stresses σ_t and σ_r are hatched in Fig. 210.

If the cylinders making up the tube are of the same material, of equal tensile and compressive strength, the optimum ratios between the radii and interference are determined in accordance with the third strength theory by the formulas

$$r_2 = \sqrt{r_1 r_3} \tag{214}$$

$$\Delta = \frac{2r_2}{E}(p_1 - p_2) \tag{215}$$

The maximum excess internal pressure is

$$\max(p_1 - p_2) = \frac{r_3 - r_1}{r_3}[\sigma] \tag{216}$$

The design methods are the same for tubes made up of three or more cylinders. The contact pressure between the first two cylinders is determined from their interference; it is used to find the initial stresses in these cylinders. Then, using the second interference between the unit, consisting of the first two cylinders (assuming it to be one cylinder of a size which is the sum of the two), and the third cylinder the new contact pressure over the fit surface is similarly determined. This second contact pressure is used to determine the initial stresses in the unit of two cylinders and in the third cylinder.

The resultant initial stresses in the unit consisting of three cylinders are obtained by the method of superposition of the obtained initial stresses due to the first and second interference fits. The stresses due to the working pressure in the unit consisting of the three cylinders are determined as for one solid tube of a size which is the sum of the three. The algebraic sum of the obtained stresses gives the design values used for checking the strength of the system.

With the addition of the fourth and subsequent cylinders, a new contact pressure and additional initial stresses are to be determined in a similar way.

In fitting a hollow cylinder on a solid one the contact pressure is determined by formula (212) or (213) with the provision that $r_1 = 0$.

Example 113. Let $p_1 = 2000$ kgf/cm², $p_2 = 0$, $r_1 = 8$ cm, $E = 2 \times 10^6$ kgf/cm² and $[\sigma] = 3000$ kgf/cm² (Fig. 211*a*).

It is necessary to design a two-piece composite tube of optimum dimensions and to check its strength by the use of the third strength theory.

Solution. By using formula (216) we first determine the outside radius of the composite tube:

$$r_3 = \frac{[\sigma]\, r_1}{[\sigma] - p_1} = \frac{3000 \times 8}{3000 - 2000} = 24 \text{ cm}$$

By formula (214) we find the radius of the fit surface:

$$r_2 = \sqrt{r_1 r_3} = \sqrt{8 \times 24} = 8\sqrt{3} \cong 13.856 \text{ cm}$$

For simplicity of calculation we shall take $r_2 = 14$ cm. Now by formula (215) we find the required interference

$$\Delta = \frac{2r_2}{E} p_1 = \frac{2 \times 14}{2 \times 10^6} \times 2 \times 10^3 = 0.028 \text{ cm}$$

and by formula (213), the contact pressure

$$p_0 = \frac{\Delta E}{4r_2} \times \frac{\left[1 - \left(\frac{r_2}{r_3}\right)^2\right]\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]}{1 - \left(\frac{r_1}{r_3}\right)^2}$$

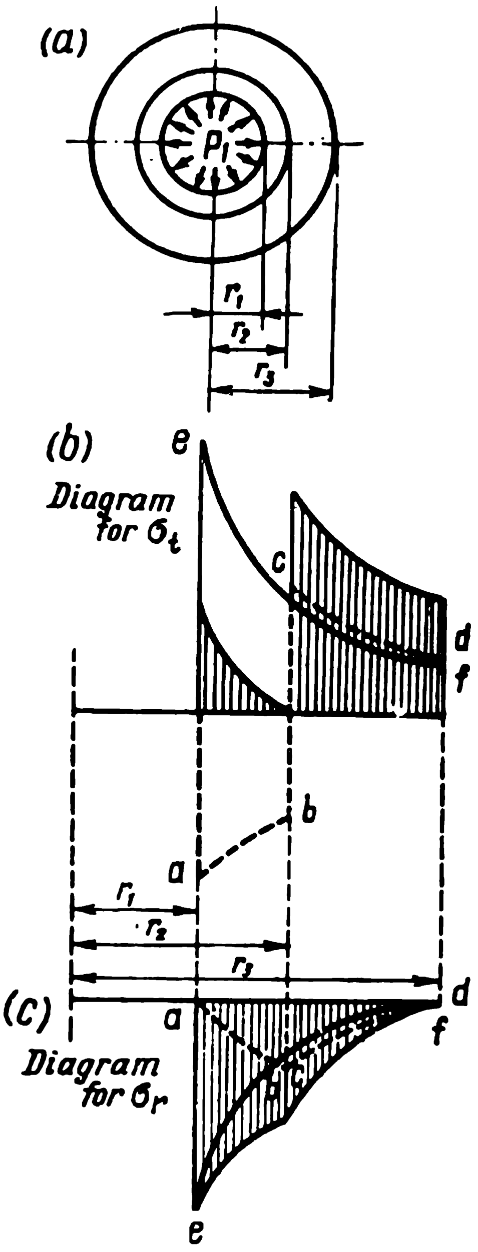


Fig. 211

$$= \frac{0.028 \times 2 \times 10^6}{4 \times 14} \times \frac{\left[1 - \left(\frac{7}{12}\right)^2\right]\left[1 - \left(\frac{4}{7}\right)^2\right]}{1 - \left(\frac{1}{3}\right)^2} \cong 500 \text{ kgf/cm}^2$$

Next we can determine the initial stresses due to the contact pressure p_0:

In the wall of the inner cylinder:

According to formulas (207) at $p_1 = 0$ and $p_2 = p_0$

$$\max \sigma_t = \sigma_{t_{\rho=r_2}} = -p_0 \frac{1+\left(\frac{r_1}{r_2}\right)^2}{1-\left(\frac{r_1}{r_2}\right)^2} = -500 \frac{1+\left(\frac{4}{7}\right)^2}{1-\left(\frac{4}{7}\right)^2} \cong -985 \text{ kgf/cm}^2;$$

$$\min \sigma_t = \sigma_{t_{\rho=r_1}} = -2p_0 \frac{1}{1-\left(\frac{r_1}{r_2}\right)^2} = -10^3 \frac{1}{1-\left(\frac{4}{7}\right)^2} \cong -1485 \text{ kgf/cm}^2;$$

$$\max \sigma_r = \sigma_{r_{\rho=r_1}} = 0;$$

$$\min \sigma_r = \sigma_{r_{\rho=r_2}} = -p_0 = -500 \text{ kgf/cm}^2$$

The diagrams for σ_t and σ_r are shown in Fig. 211*b* and *c* by the dash lines *ab*.

In the wall of the outer cylinder:

According to formulas (204) at $p_1 = p_0$ and $p_2 = 0$ assuming that $r_1 = r_2$ and $r_2 = r_3$ we find

$$\max \sigma_t = \sigma_{t_{\rho=r_2}} = p_0 \frac{1+\left(\frac{r_2}{r_3}\right)^2}{1-\left(\frac{r_2}{r_3}\right)^2} = 500 \frac{1+\left(\frac{7}{12}\right)^2}{1-\left(\frac{7}{12}\right)^2} \cong 1016 \text{ kgf/cm}^2$$

$$\min \sigma_t = \sigma_{t_{\rho=r_3}} = 2p_0 \frac{1}{\left(\frac{r_3}{r_2}\right)^2 - 1} = 10^3 \frac{1}{\left(\frac{12}{7}\right)^2 - 1} \cong 516 \text{ kgf/cm}^2;$$

$$\max \sigma_r = \sigma_{r_{\rho=r_3}} = 0;$$

$$\min \sigma_r = \sigma_{r_{\rho=r_2}} = -p_0 = -500 \text{ kgf/cm}^2$$

The diagrams for σ_t and σ_r are illustrated in Fig. 211*b* and *c* by the dash lines *cd*.

Next we determine the stresses in the wall of the unit consisting of two cylinders (like a solid tube) due to the action of the internal pressure p_1.

According to formulas (204) at $p_1 = 2000$ kgf/cm² and $p_2 = 0$, assuming $r_1 = r_1$ and $r_2 = r_3$ we find

$$\max \sigma_t = \sigma_{t_{\rho=r_1}} = p_1 \frac{1+\left(\frac{r_1}{r_3}\right)^2}{1-\left(\frac{r_1}{r_3}\right)^2} = 2 \times 10^3 \frac{1+\left(\frac{1}{3}\right)^2}{1-\left(\frac{1}{3}\right)^2} = 2500 \text{ kgf/cm}^2;$$

$$\min \sigma_t = \sigma_{t_{\rho=r_3}} = 2p_1 \frac{1}{\left(\frac{r_3}{r_1}\right)^2 - 1} = 2 \times 2 \times 10^3 \frac{1}{3^2 - 1} = 500 \text{ kgf/cm}^2;$$

$$\max \sigma_r = \sigma_{r_{\rho=r_3}} = 0;$$

$$\min \sigma_r = \sigma_{r_{\rho=r_1}} = -p_1 = -2000 \text{ kgf/cm}^2$$

According to formulas (202) and (203) at $p_1 = 2000$ kgf/cm² and $p_2 = 0$, assuming that $r_1 = r_1$ and $r_2 = r_3$ for the points of the fit surface we find

$$\sigma_{t_{\rho=r_2}} = p_1 \frac{1+\left(\frac{r_3}{r_2}\right)^2}{\left(\frac{r_3}{r_1}\right)^2} = 2\times 10^3 \frac{1+\left(\frac{12}{7}\right)^2}{3^2-1} \cong 985 \text{ kgf/cm}^2;$$

$$\sigma_{r_{\rho=r_2}} = p_1 \frac{1-\left(\frac{r_3}{r_2}\right)^2}{\left(\frac{r_3}{r_1}\right)^2-1} = 2\times 10^3 \frac{1-\left(\frac{12}{7}\right)^2}{3^2-1} \cong -485 \text{ kgf/cm}^2$$

The diagrams for σ_t and σ_r are shown in Fig. 211*b* and *c* by the solid lines *ef*.

Next we find the resultant stresses in the wall of the inner cylinder:

$\sigma_{t_{\rho=r_1}} = -1485+2500 = 1015$ kgf/cm²; $\sigma_{t_{\rho=r_2}} = -985+985 = 0$;

$\sigma_{r_{\rho=r_1}} = -2000$ kgf/cm²; $\sigma_{r_{\rho=r_2}} = -500-485 = -985$ kgf/cm²

and in the wall of the outer cylinder:

$$\sigma_{t_{\rho=r_2}} = 1016+985 = 2001 \text{ kgf/cm}^2;$$

$$\sigma_{t_{\rho=r_3}} = 516+500 = 1016 \text{ kgf/cm}^2;$$

$$\sigma_{r_{\rho=r_2}} = -500-485 = -985 \text{ kgf/cm}^2;$$

$$\sigma_{r_{\rho=r_3}} = 0$$

The diagrams for the resultant stresses σ_t and σ_r are shown hatched in Fig. 211*b* and *c*.

The equivalent stresses, in accordance with the third strength theory are as follows:

In points of the internal surface of the inner cylinder:

$$\sigma_{eq_{III}} = \sigma_{t_{\rho=r_1}} - \sigma_{r_{\rho=r_1}} = 1015+2000 = 3015 \text{ kgf/cm}^2$$

In points of the internal surface of the outer cylinder:

$$\sigma_{eq_{III}} = \sigma_{t_{\rho=r_2}} - \sigma_{r_{\rho=r_2}} = 2001+985 = 2986 \text{ kgf/cm}^2$$

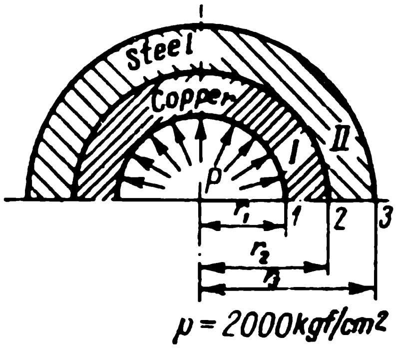
The small difference from equal strength of the material at the dangerous points of the cylinders is due to the rounding off of dimension r_2.

The condition of strength $\sigma_{eq_{III}} = [\sigma]$ can be considered to be complied with, since the overstress is only 0.5%.

Problems 926 through 930. Determine the radial σ_r and tangential σ_t normal stresses as well as the boundary pressures p_0 in the built-up tubes due to the internal pressure p_1 (Problems 926 and 927), due to heating to raise the temperature by Δt °C (Problem 928) and due to the interference Δ of a heavy-force fit (Problems 929 and 930).

In Problems 926, 927 and 928 assume that $E_{st} = 2 \times 10^6$ kgf/cm²; $\mu_{st} = 0.3$; $\alpha_{st} = 12.5 \times 10^{-6}$; $E_{cu} = 1 \times 10^6$ kgf/cm²;

926



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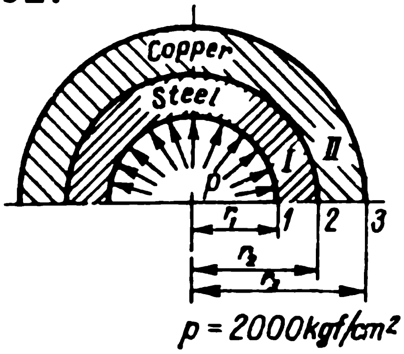

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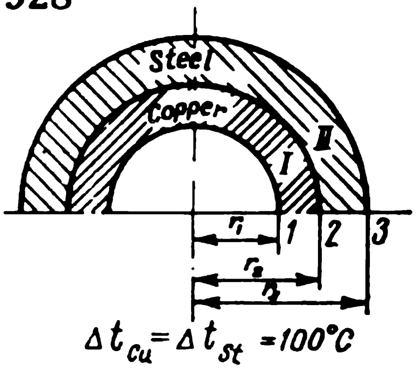

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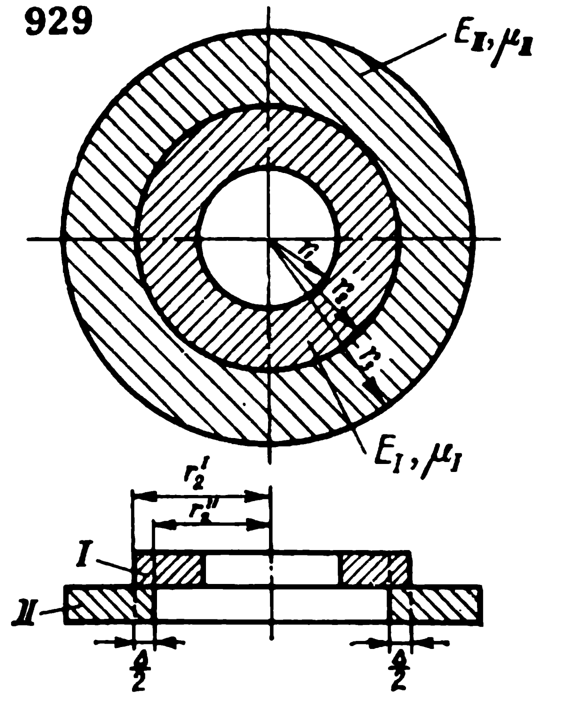

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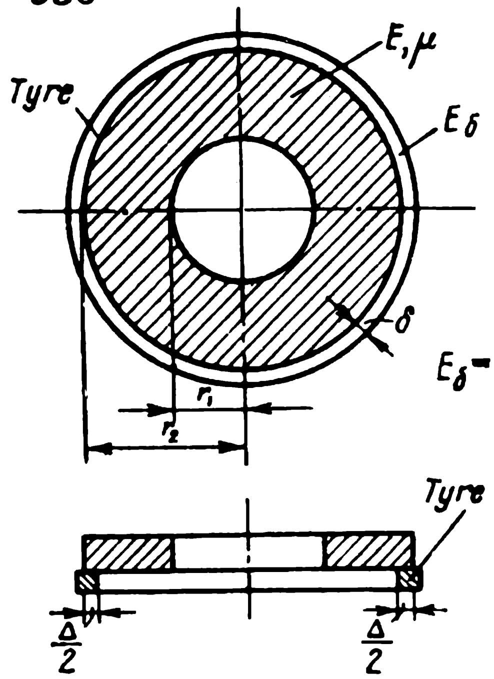


$\mu_{cu} = 0.34$; $\alpha_{cu} = 16.5 \times 10^{-6}$; $r_1 = 10$ cm; $r_2 = 20$ cm and $r_3 =$ $= 40$ cm.

In Problems 929 and 930 determine only p_0.

Problem 931. Determine for the composite (built-up) two-piece tube:

(1) Stresses due to the heavy-force fit with an interference of $\Delta =$ $= 0.4$ mm.

(2) Stresses due to the internal pressure $p_0 = 2000$ kgf/cm².

(3) The total stresses.

(4) Percentage reduction of the design stress (using the third strength theory) as compared with a solid tube of the same dimensions. Let $r_1 = 10$ cm; $r_2 = 25$ cm and $r_3 = 50$ cm.

Problem 932. Design a composite two-piece tube with rational dimensions, using the third strength theory. Let $r_1 = 10$ cm; $E =$ $= 2 \times 10^6$ kgf/cm²; $\mu = 0.3$; $[\sigma] = 4000$ kgf/cm² and $p = 3000$ kgf/cm².

Problem 933. Determine for the composite three-piece tube:

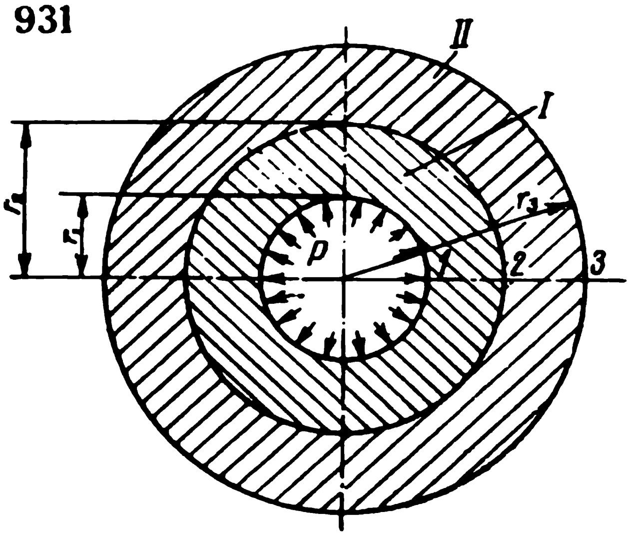
(1) Stresses due to the heavy-force fits with interferences of Δ_1 and Δ_2.

(2) Stresses due to the internal pressure p.

(3) Design stresses according to the third strength theory.

(4) Percentage reduction of the design stresses in the composite tube as compared with a solid tube of the same dimensions.

931



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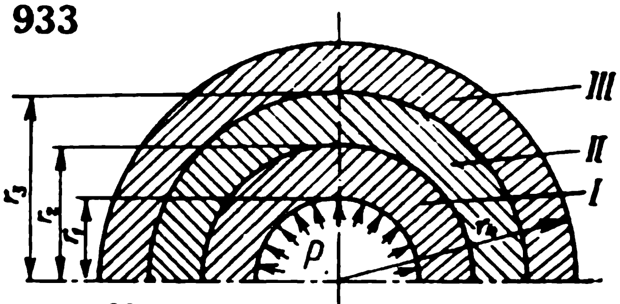


$r_1 = 80\,mm$; $r_2 = 100\,mm$

$r_3 = 140\,mm$

$r_4 = 200\,mm$

$\Delta_1 = 0.06\,mm$; $\Delta_2 = 0.12\,mm$

$p = 2400\,kgf/cm^2$

$E = 2.2\cdot 10^6\,kgf/cm^2$

CHAPTER 14. DYNAMIC LOADS

14.1.

Design of Moving Bodies (Systems) Taking Inertia Into Account

The dynamic action of forces is characterized by acceleration of the elements of the body (system) being considered. The nature of the deformation (strain) and failure of the body depend on the kind of the acceleration produced.

Acceleration is associated with inertial forces directed opposite to the direction of the acceleration. The element of inertial force dP_i equals the product of the mass dm of an element of volume dV of the body by the acceleration a, i.e.

$$dP_i = dma = \frac{\gamma}{g} a\, dV$$

in which γ = weight of unit volume of the body material
g = acceleration of gravity.

Figure 212*a* illustrates nonuniform translatory motion, and Fig. 212*b*, uniform rotary motion of a body.

If, in motion, the elements of a body undergo uniform acceleration, no specific features occur in the behaviour of its material. In this case the stresses and strains can be determined as for a static load due to external and internal forces.

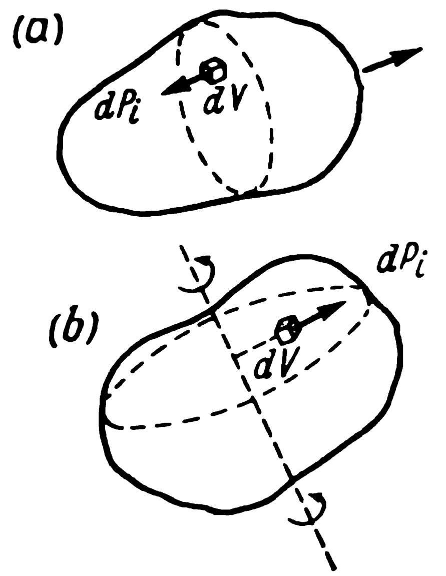


Fig. 212

For a sufficiently rigid moving body (or system), carrying certain loads, the forces of inertia are determined from the acceleration of the elements of the body and of the loads. This acceleration, in turn, is established on the basis of the motion. Adding the forces of inertia to the loads, the dead weight of the body and all external forces acting on the body, the latter is considered as being in a state of rest and is analysed by the rules valid for static loading.

If the weight of the loads carried by a body is large compared to its dead weight and the inertial forces due the to latter are small compared to those due to the loads, then, in analyses of the strength and rigidity of the body, the effect of its dead weight can be neglected.

In cases when the direction of acceleration a of the given motion of a body coincides with acceleration g due to the earth's gravity, the kind of strain of the body due to the inertial forces corresponds to the kind due to its dead weight and to the loads carried by the body. If no other loads, except the latter, are applied to the body, the generalized dynamic forces P_d, stresses p_d (either normal σ_d or shear τ_d) and displacements δ_d (taking inertial forces into account) are readily determined in terms of the quantities P, p and δ, which correspond to the static loading (without forces of inertia), and the dynamic factor

$$k_d = 1 + \frac{a}{g} \tag{217}$$

according to the formulas

$$\left.\begin{aligned} P_d &= k_d P; \\ p_d &= k_d p; \\ \delta_d &= k_d \delta \end{aligned}\right\} \tag{218}$$

In cases when the direction of acceleration a of the given motion of the body does not coincide with acceleration g due to the earth's gravity, the kind of strain of the body due to the inertial forces does not correspond to that due to its dead weight and to the loads carried by the body. In designing the body the stresses and strains due to each kind of strain should be taken into account similarly to static loading.

Example 114. A prismatic bar of length l carries two loads Q_1 and Q_2 and travels vertically upwards with uniform acceleration, covering the distance S during the first t seconds (Fig. 213).

Determine the required cross-sectional area F of the bar and its dynamic elongation, if the weight of the unit volume of the material of the bar is γ, Young's modulus is E and the permissible tensile stress is $[\sigma]$.

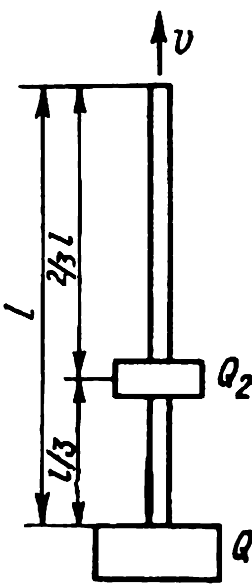


Fig. 213

Solution. The maximum static axial internal force in the upper section of the bar is

$$\max N = Q_1 + Q_2 + \gamma F l$$

The maximum static normal stress is

$$\max \sigma = \frac{\max N}{F} = \frac{Q_1 + Q_2}{F} + \gamma l$$

Since the acceleration of the given motion, $a = \frac{2S}{t^2}$ is directed vertically upwards, the dynamic coefficient is

$$k_d = 1 + \frac{a}{g}$$

The maximum dynamic stress is

$$\max \sigma_d = k_d \max \sigma = k_d \left(\frac{Q_1 + Q_2}{F} + \gamma l \right)$$

In accordance with the strength condition

$$\max \sigma_d = k_d \max \sigma \leqslant [\sigma]$$

or

$$\max \sigma = \frac{Q_1 + Q_2}{F} + \gamma l \leqslant \frac{[\sigma]}{k_d}$$

From which the required cross-sectional area of the bar is

$$F \geqslant \frac{Q_1 + Q_2}{\frac{[\sigma]}{k_d} - \gamma l}$$

The elongation of a bar of cross-sectional area F due to the static effect of the forces is

$$\Delta l = \left(Q_1 + \frac{2}{3} Q_2 + \frac{1}{2} \gamma F l \right) \frac{1}{EF}$$

and the dynamic elongation of the bar is $\Delta l_d = k_d \Delta l$.

Example 115. Let a coiled cylindrical spring of small pitch, length $l = 30$ cm, coil radius $R = 2$ cm, the radius of the wire cross section $r = 0.2$ cm and a number of turns $n_0 = 10$, be loaded with a weight $Q = 1$ kgf applied at its end and be rotating in a vertical plane about a stationary hinge at a speed of $n = 200$ rpm (Fig. 214).

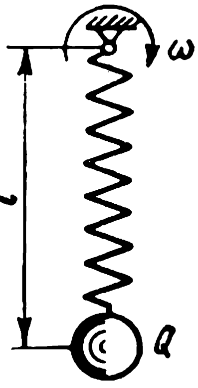


Fig. 214

Determine the maximum dynamic shearing stress $\max \tau_d$ in a section of the spring wire and the maximum displacement Δl of load Q, if the shear modulus of the wire material $G = 8 \times 10^5$ kgf/cm² and the weight of the spring is very small as compared to load Q.

Solution. When load Q is in its lowest position, the spring will be subject to the action of the largest dynamic tensile force $P_d = k_d Q$.

Owing to the low rigidity of the spring, the stretching of its axis should be taken into account in determining the centrifugal force. Therefore the dynamic coefficient is

$$k_d = 1 + \frac{a}{g} = 1 + \frac{\omega^2}{g} (l + \Delta l)$$

in which the angular velocity is

$$\omega = \frac{\pi n}{30} = \frac{\pi \times 200}{30} = 21 \text{ rad/sec}; \quad \frac{\omega^2}{g} = \frac{21^2}{981} \cong 0.45 \text{ 1/sec}$$

and

$$\Delta l = \frac{4 P_d R^3 n_0}{G r^4} = P_d \frac{4 \times 8 \times 10}{8 \times 10^5 \times 16 \times 10^{-4}} = \frac{P_d}{4} \text{ cm}$$

Thus

$$P_d = Q\left[1+\frac{\omega^2}{g}\left(l+\frac{4P_dR^3n_0}{Gr^4}\right)\right] = Q\,[1+0.45\,(30+0.25P_d)]$$

$$\cong Q\,(14.5+0.11P_d)$$

from which

$$P_d = \frac{14.5Q}{1-0.11Q} \cong 17.1 \text{ kgf}$$

The maximum dynamic shearing stress in the cross section of the spring wire is

$$\max \tau_d = \frac{2P_dR}{\pi r^3} = \frac{2\times 17.1\times 2}{\pi\times 8\times 10^{-3}} \cong 2720 \text{ kgf/cm}^2$$

and the maximum dynamic displacement of the load is

$$\Delta l = \frac{1}{4}P_d = \frac{17.1}{4} \cong 4.3 \text{ cm}$$

Example 116. A prismatic bar of length l carries a load weighing Q applied at its top end. The lower end of the bar is fixed in a perfectly rigid element which travels at an angle α to the horizontal plane with a uniform acceleration a (Fig. 215a).

Determine the maximum and minimum dynamic normal stresses in the dangerous section of the bar, if the area and section modulus

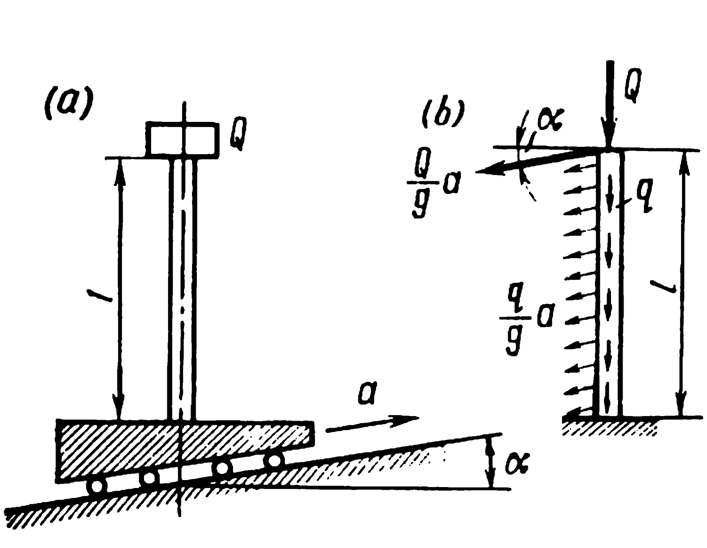


Fig. 215

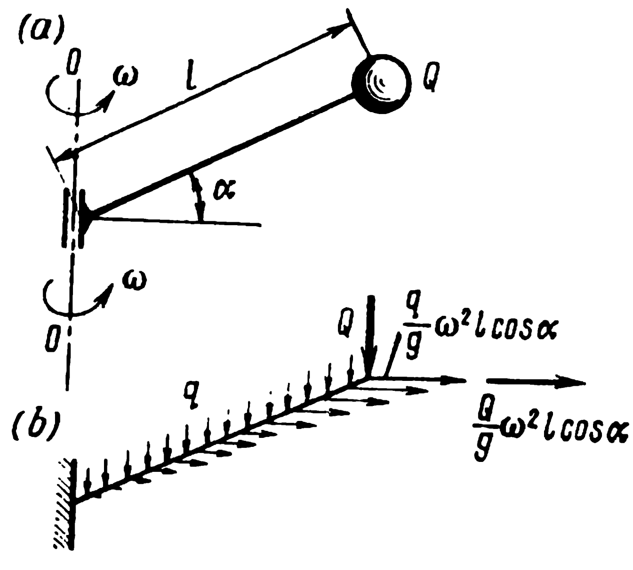


Fig. 216

in bending are F and W, respectively, and the weight of unit length is q.

Solution. In motion, the bar is subject to the vertical downward load Q and its dead weight q, as well as the concentrated inertial force $\frac{Qa}{g}$ and a uniformly distributed inertial force $\frac{qa}{g}$, both at an angle of $\frac{\pi}{2}-\alpha$ to the geometric axis and opposite to the direction of the motion (Fig. 215b).

The dangerous fixed section of the bar is subject to the action of normal stresses: (1) due to the gravity forces Q and q which cause axial compression:

$$\sigma' = -\frac{Q+ql}{F}$$

(2) due to the component inertial forces acting along the geometric axis of the bar and causing axial compression:

$$\sigma'' = -\frac{(Q+ql)\,a\sin\alpha}{gF}$$

(3) due to the component inertial forces perpendicular to the geometric axis of the bar and causing bending of the bar:

$$\sigma''' = \pm\frac{\left(Ql+\frac{1}{2}ql^2\right)a\cos\alpha}{gW}$$

The maximum and minimum dynamic normal stresses in the dangerous section of the bar are

$$\sigma_{\substack{\max\\ \min}} = \pm\frac{\left(Q+\frac{1}{2}ql\right)al\cos\alpha}{gW} - \frac{Q+ql}{F}\left(1+\frac{a}{g}\sin\alpha\right)$$

Example 117. A bar of length l at an angle α to the horizontal plane carries a load Q attached to its free end and rotates about the vertical axis OO at a constant angular velocity ω (Fig. 216*a*).

Determine the maximum dynamic normal stress in the dangerous section of the bar, if the weight of unit length of the bar is q, its cross sectional area is F and the section modulus in bending is W.

Solution. Taking into account the centrifugal forces, use is made of a design scheme of the form shown in Fig. 216*b*.

The dangerous fixed section of the bar is subject to the following:

(1) Bending moments:

(a) due to the dead weight of the bar (q)

$$M_1 = \frac{ql^2}{2}\cos\alpha$$

(b) due to the load (Q)

$$M_2 = Ql\cos\alpha$$

(c) due to the concentrated centrifugal force $\left(\frac{Q}{g}\omega^2 l\cos\alpha\right)$

$$M_3 = \frac{Q\omega^2}{2g}l^2\sin 2\alpha$$

(d) due to the centrifugal force distributed according to the triangle law, of maximum intensity $\frac{q}{g}\omega^2 l\cos\alpha$

$$M_4 = \frac{q\omega^2}{6g}l^3\sin 2\alpha$$

(2) Compressive forces:

(a) due to its dead weight (q)

$$N_1 = -ql \sin \alpha$$

(b) due to the load (Q)

$$N_2 = -Q \sin \alpha$$

(3) Tensile forces:

(a) due to the concentrated centrifugal force $\frac{Q}{g}\omega^2 l \cos \alpha$

$$N_3 = \frac{Q\omega^2}{g} l \cos^2 \alpha$$

(b) due to the centrifugal force, distributed according to the triangle law, of maximum intensity $\frac{q}{g}\omega^2 l \cos \alpha$

$$N_4 = \frac{q\omega^2}{2g} l^2 \cos^2 \alpha$$

The resultant bending moment is

$$M_d = M_1 + M_2 + M_3 + M_4$$

$$= \left(Q + \frac{1}{2} ql\right) l \cos \alpha$$

$$+ \left(Q + \frac{1}{3} ql\right) \frac{\omega^2 l^2}{2g} \sin 2\alpha$$

The resultant axial internal force is

$$N_d = N_1 + N_2 + N_3 + N_4$$

$$= \left(Q + \frac{1}{2} ql\right) \frac{\omega^2 l}{g} \cos^2 \alpha - (Q + ql) \sin \alpha$$

The maximum dynamic normal stresses in the dangerous section of the bar are

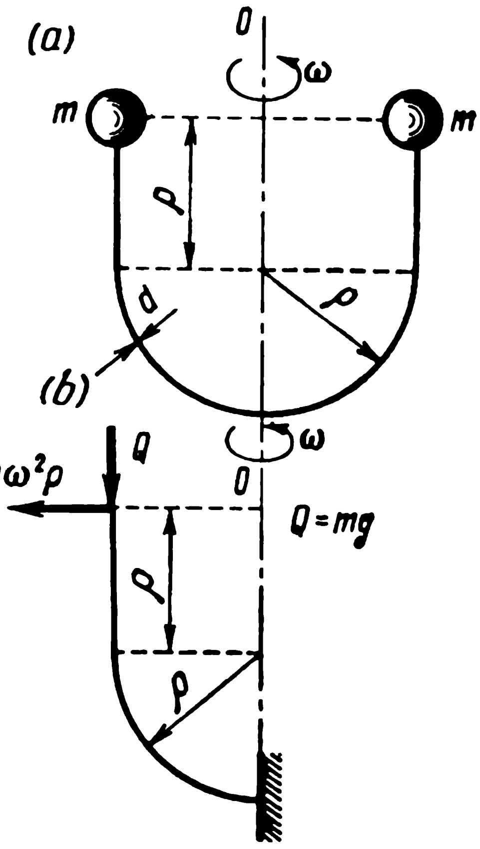


Fig. 217

$$\max \sigma_d = \frac{N_d}{F} + \frac{M_d}{W} = \frac{1}{F}\left[\left(Q + \frac{1}{2} ql\right) \frac{\omega^2 l}{g} \cos^2 \alpha - (Q + ql) \sin \alpha\right] + \frac{l}{W}\left[\left(Q + \frac{1}{3} ql\right) \frac{\omega^2 l}{2g} \sin 2\alpha + \left(Q + \frac{ql}{2}\right) \cos \alpha\right]$$

Example 118. The system illustrated in Fig. 217a rotates about the vertical axis OO at a constant angular velocity ω.

Determine the permissible rotational speed n (in rpm), if $m = 1$ kg, $\rho = 10$ cm, $d = 1$ cm and $[\sigma] = 160$ MN/m^2.

The dead weight of the system and the effect on the centrifugal force of the change in the distance between the loads may be neglected.

Solution. Taking the centrifugal force into account, the design diagram can be taken as shown in Fig. 217*b*.

The dangerous fixed section is subject to the axial internal tensile force $N_d = m\omega^2\rho$ and the bending moment

$$M_d = mg\rho + m\omega^2\rho 2\rho = m\rho\,(g + 2\omega^2\rho)$$

The maximum dynamic tensile stress in the internal fibre of the fixed section is

$$\max \sigma_d = \frac{N_d}{F} + \frac{M_d}{W}\,\alpha_{int} = \frac{m\omega^2\rho}{F} + \frac{m\rho\,(g+2\omega^2\rho)}{W}\,\alpha_{int}$$

Since at $\frac{d}{\rho} = \frac{1}{10}$ the coefficient $\alpha_{int} = \dfrac{1-\dfrac{d}{8\rho}}{1-\dfrac{d}{2\rho}} \cong 1$, the strength condition in the dangerous section is

$$\max \sigma_d = m\omega^2\rho\left(\frac{1}{F} + \frac{2\rho}{W}\right) + \frac{m\rho g}{W} \leqslant [\sigma]$$

from which

$$\omega = \frac{\pi n}{30} \leqslant \sqrt{\frac{[\sigma] - \dfrac{m\rho g}{W}}{m\rho\left(\dfrac{1}{F} + \dfrac{2\rho}{W}\right)}} \cong \sqrt{\frac{160\times 10^6 - \dfrac{1\times 0.1\times 9.81}{0.1\times 10^{-6}}}{1\times 0.1\left(\dfrac{1}{0.8\times 10^{-4}} + \dfrac{2\times 0.1}{0.1\times 10^{-6}}\right)}}$$

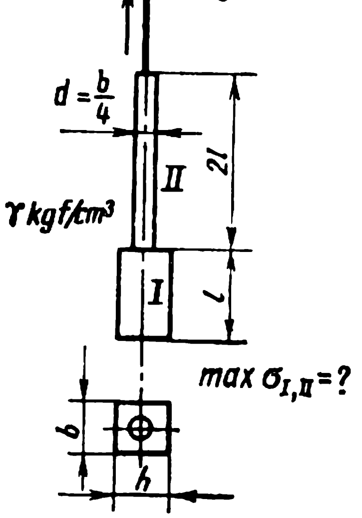
$$\cong \sqrt{750}\,;$$

$$n \cong \frac{30}{\pi}\sqrt{750} \cong 270 \text{ rpm}$$

Problems 934 through 936. Determine the normal stresses σ in the cross sections of the given elements of moving systems due to the joint action of the forces of gravity and inertia.

In the figures the direction of motion is indicated by the arrow with a symbol a.

934



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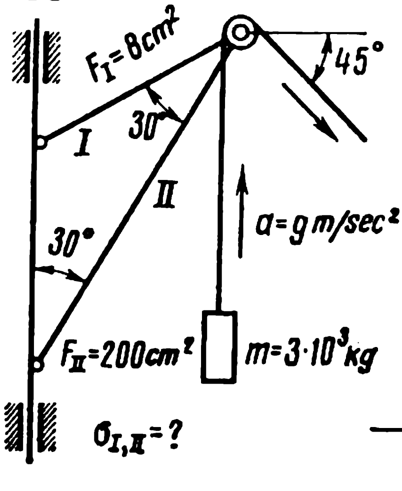

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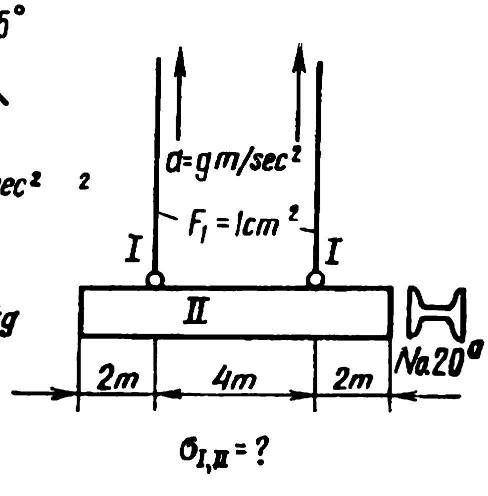
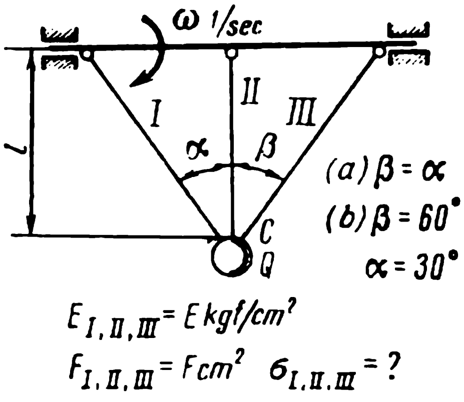


Problems 937 through 950. Determine the values given in the statements of the problems due to the forces of inertia developed in the rotating systems.

In the drawings the axis about which the system rotates is indicated by the arrow with the symbol ω or n. If the cross-sectional area and E are not given for the axis of rotation, the latter is assumed to be perfectly rigid. The other perfectly rigid elements of the systems (besides the axis) are drawn with two lines and hatched.

937



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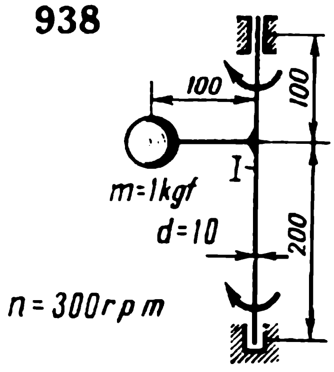

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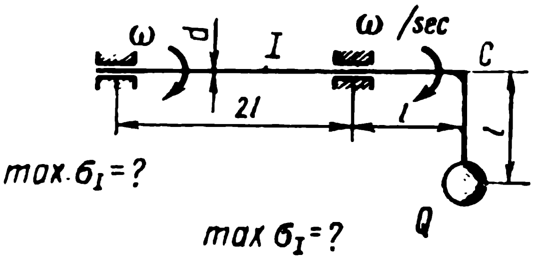

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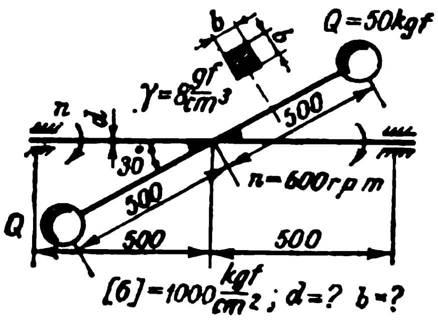

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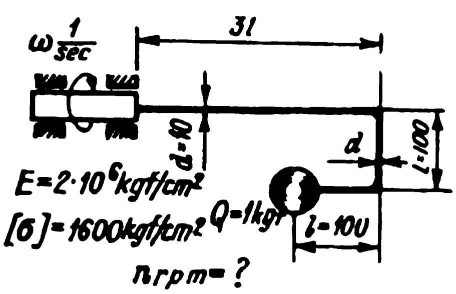

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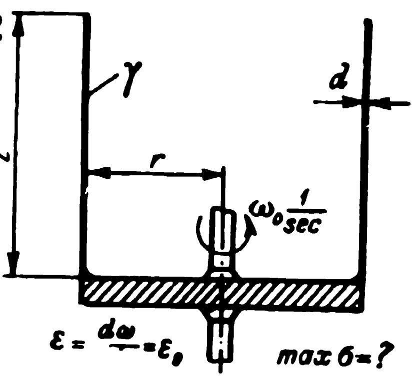

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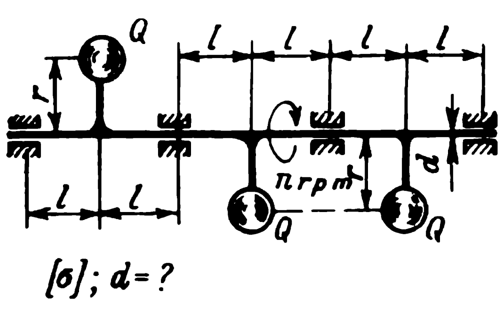

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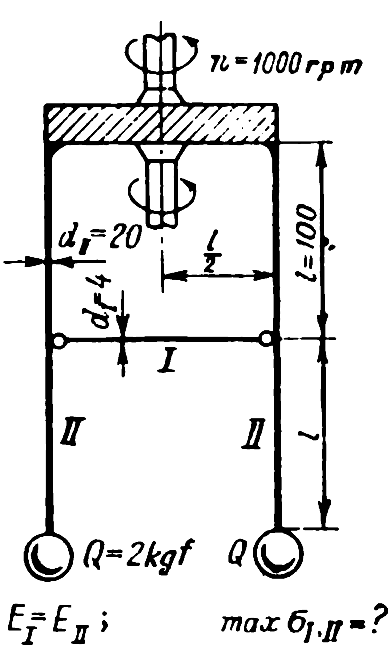

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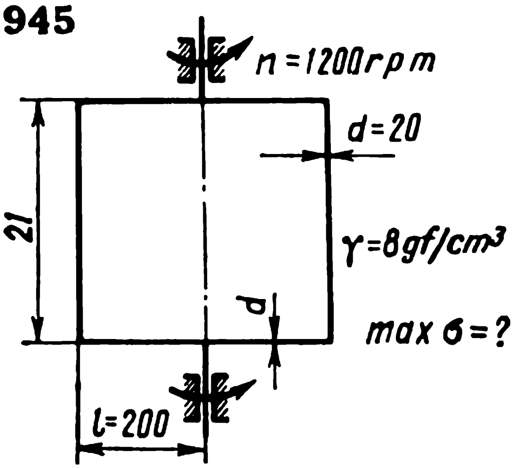

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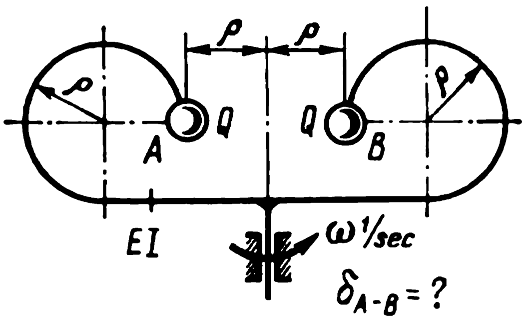

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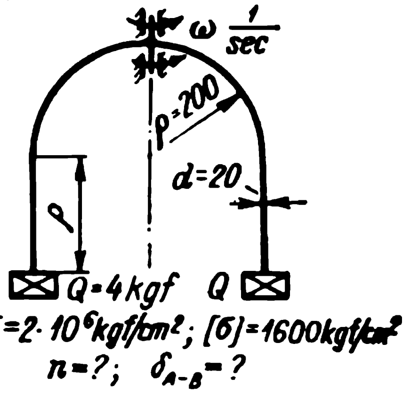

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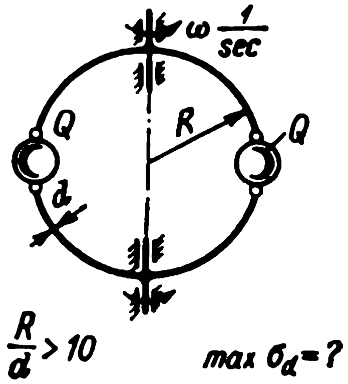

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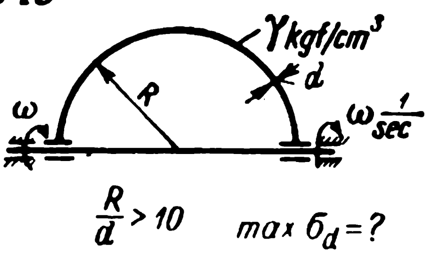

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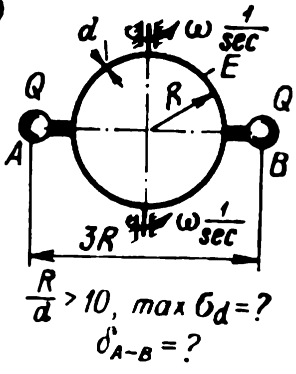


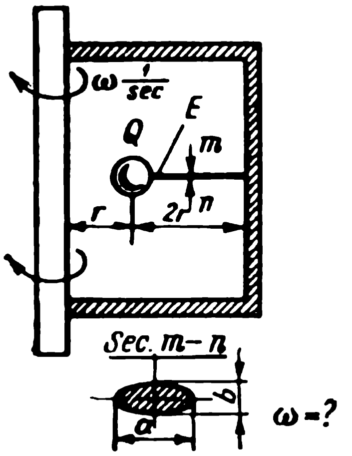
In all the problems, except 942, the rotation is uniform. In determining the forces of inertia the strain of the elements of the systems are not to be taken into consideration.

In Problems 937 and 938, in addition to the forces of inertia, also take the forces of gravity into account.

Problems 951, 952 and 953. Determine the angular velocity ω in 1/sec (or the revolutions per minute n) of the axis, at which the compressed element of the system loses its stability.

Assume that the critical stresses do not exceed the proportional limit of the material.

951



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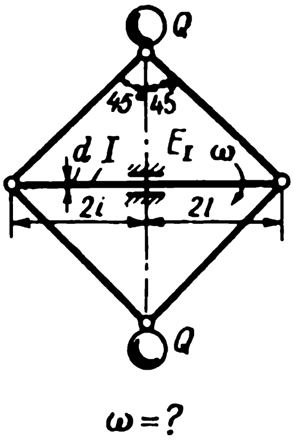

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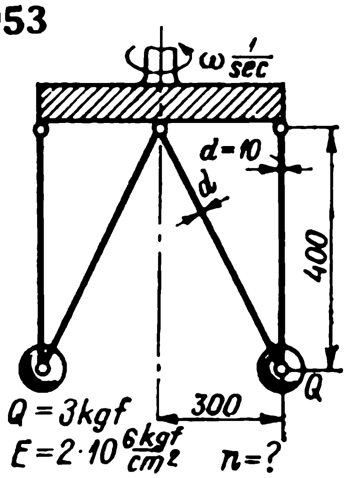


14.2.

Oscillations (Vibrations) of Elastic Bodies

Elastic oscillations (*vibrations*) are the periodic deviations of an elastic body (system) from the stable equilibrium position. If a system is brought out of the equilibrium position by a single power impulse, the resulting oscillations are said to be *free* or *natural*. If a system is subject to the action of a generalized force, periodically changing with time (disturbing force), the resulting oscillations are said to be *forced*.

Owing to resisting forces natural oscillations are damped, i.e., their amplitude decreases with time. If the frequency of natural oscillations is equal to that of the disturbing force, resonance occurs. This leads to rapid and continuous increase of the amplitude with time. Since an increase of the amplitude of oscillations is associated with an increase of stresses, resonance may end in premature destruction of the system.

Any elastic system has an infinite number of degrees of freedom, since the number of independent coordinates specifying the position in space of the masses distributed in the elements of the system is infinitely great.

If the dead mass of a system is either neglected or taken into account approximately by reducing it to one or several points, the system can be conventionally considered to have one or several degrees of freedom. Without taking into account its dead weight, an elastic system can be considered to have one degree of freedom, if the system carries one load whose position in space is determined by a single coordinate. If the dead mass is taken into account, a system can be conventionally considered to have one degree of freedom, if the dead mass can be reduced with sufficient accuracy to the point of suspension of the load.

This chapter deals only with sustained oscillations of systems with one degree of freedom without taking their dead mass into account. The latter will be taken into account only in the simplest problems concerning the oscillations of prismatic bars.

In all cases the oscillating mass is considered to be constant and the elastic system, linear. In this system the generalized restoring force P, at an arbitrary moment of time, is proportional to the respective generalized displacement δ, i.e.

$$P = C\delta \tag{219}$$

in which C is the rigidity of the system, i.e. the generalized force which causes a generalized displacement equal to unity.

The strain of a system produced by vibrations is considered similar to that due to a static loading of this system by an appropriate generalized force applied at the point of suspension of the load and acting in the direction of the oscillation.

NATURAL OSCILLATIONS OF ELASTIC SYSTEMS (WITH THE DEAD MASS NEGLECTED)

Since for linearly deformed systems the generalized displacement δ of the point of load suspension, when the respective generalized force P is also applied at this point, can be readily found, using any of the known methods, the rigidity of the system C can be obtained from the expression

$$C = \frac{P}{\delta} \tag{220}$$

The rigidity of systems with elastic elements connected in parallel, consecutively or in some combination of the two is readily determined if the following well-known conditions are used.

If elastic elements of rigidities C_i are connected in parallel in a system (diagrams are shown in Fig. 218*a* and *b*), the rigidity of the system is

$$C = \sum C_i \tag{221}$$

If elastic elements of rigidity C_i are connected consecutively (Fig. 218*c*), the rigidity of the system is

$$C = \frac{1}{\sum \frac{1}{C_i}} \tag{222}$$

In case of a combination of connections of the elastic elements (Fig. 218*d*), some of rigidity C_i being connected in parallel and others,

of rigidity C'_i being connected consecutively, the rigidity of the system is

$$C = \frac{1}{\frac{1}{\sum C_i} + \sum \frac{1}{C'_i}} \tag{223}$$

For systems for which the dead mass can be neglected, the cyclic frequency of natural oscillations ω (number of oscillations in 2π sec), vibration frequency N (number of oscillations per sec) and period of oscillations T (time for one full oscillation) are determined from the following formulas:

(a) In the case of reciprocating motion of the load

$$\left.\begin{aligned} \omega &= \sqrt{\frac{C}{m}} = \sqrt{\frac{g}{\delta}}; \\ N &= \frac{1}{2\pi}\sqrt{\frac{C}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}; \\ T &= 2\pi\sqrt{\frac{m}{C}} = 2\pi\sqrt{\frac{\delta}{g}} \end{aligned}\right\} \tag{224}$$

in which $m = \frac{Q}{g}$ = mass of a load weighing Q, kgf-sec^2/cm (in the International System of Units [SI] the mass of load m is found by weighing, i.e. by comparing with a standard mass of 1 kg; weight of load Q in newtons, $Q = mg$)

δ = linear displacement of the point of suspension of the load due to the statical effect of force Q in the direction of oscillations, cm (m)

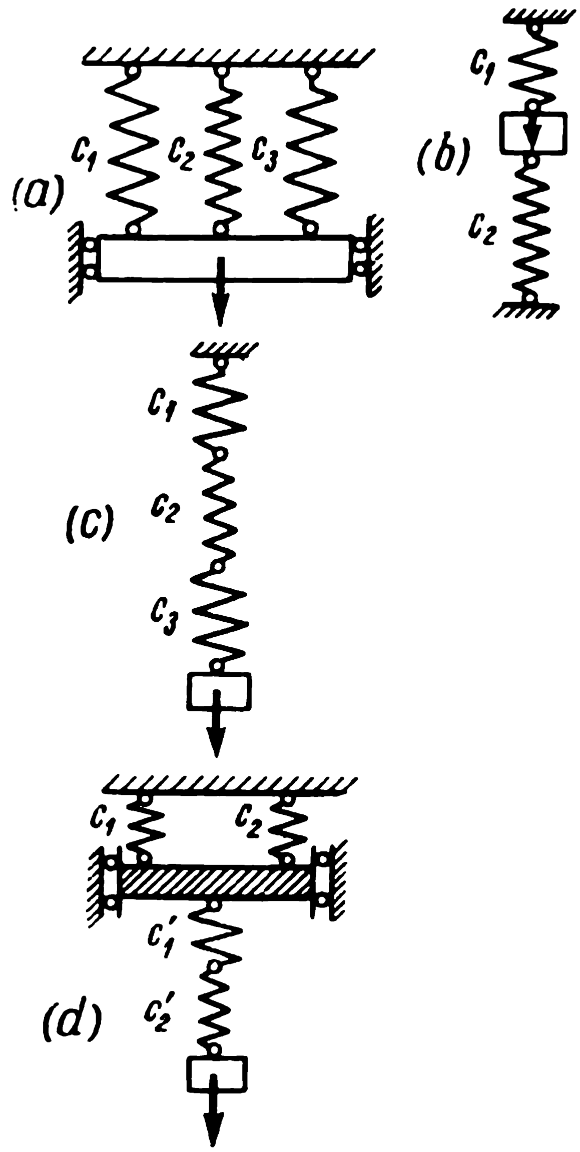


Fig. 218

$C = \frac{Q}{\delta}$ = rigidity of the system, kgf/cm (N/m or kN/m).

(b) In the case of swinging motion of the load

$$\left.\begin{aligned} \omega &= \sqrt{\frac{C}{I_m}}; \\ N &= \frac{1}{2\pi}\sqrt{\frac{C}{I_m}}; \\ T &= 2\pi\sqrt{\frac{I_m}{C}} \end{aligned}\right\} \tag{225}$$

in which $I_m = \int\limits_m \rho^2 dm = \frac{\gamma}{g}\int\limits_V \rho^2\, dV$ = moment of inertia of the mass of the load with respect to its axis of rotation, kgf-cm-sec² (in the International System of Units [SI] $I_m = \int\limits_m \rho^2\, dm = \rho_0 \int\limits_V \rho^2\, dV$ kg-m²)

γ = weight of unit volume V of the load, kgf/cm³

ρ_0 = specific gravity, mass of unit volume V, kg/m³ (in SI units)

ρ = distance of an element of volume dV from the axis of rotation, cm (m)

$C = \frac{M}{\varphi}$ = rigidity (stiffness) of the system, kgf-cm (N-m)

where M is the moment, acting statically in the cross section of the load suspension in the direction of oscillation, kgf-cm (N-m)

φ = angular displacement of the cross section at which the moment M is applied due to the statical effect of this moment, rad.

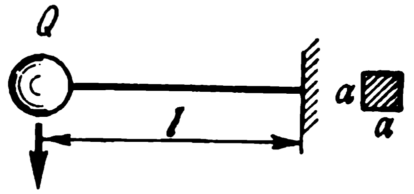


Fig. 219

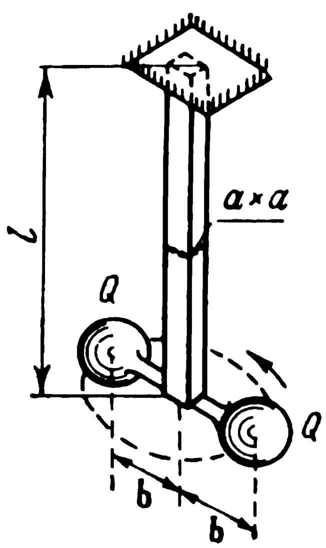


Fig. 220

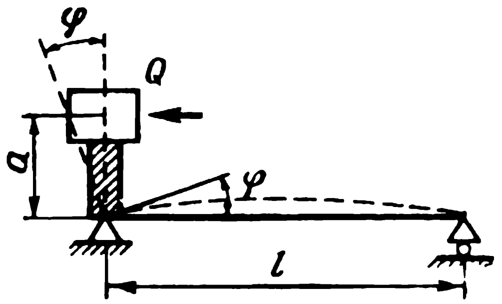


Fig. 221

Example 119. Let $Q = 16$ kgf, $a = 2$ cm, $E = 2 \times 10^6$ kgf/cm² and $T = 0.1$ sec (Fig. 219).

Determine l.

Solution. Since the period of oscillation $T = 2\pi\sqrt{\frac{\delta}{g}}$, then $\delta = \frac{T^2 g}{4\pi^2}$.

On the other hand, the deflection of the free end of the beam due to the statical effect of force Q is $\delta = \frac{Ql^3}{3EI}$. Therefore

$$\frac{Ql^3}{3EI} = \frac{T^2 g}{4\pi} \text{ and } l = \sqrt[3]{\frac{T^2 g \times 3EI}{4\pi^2 Q}} = \sqrt[3]{\frac{1 \times 981 \times 3 \times 2 \times 10^6 \times 2^4}{10^2\pi^2 \times 4 \times 16 \times 12}}$$

$$= 50 \text{ cm}$$

Example 120. Let $m = 20$ kg, $b = 8$ cm, $l = 40$ cm, $N = 20$ Hz (1 hertz [Hz] = 1 cycle/sec [cps]) and $G = 8 \times 10^4$ MN/m^2 (Fig. 220). Determine a.

Solution. Since $N = \frac{1}{2\pi}\sqrt{\frac{C}{I_m}}$ and $I_m = 2mb^2$, it follows that

$$C = 4\pi^2 N^2 I_m = 8\pi^2 N^2 mb^2$$

On the other hand, the rigidity of the bar in torsion is

$$C = \frac{GI_t}{l} \cong \frac{G \times 0.14a^4}{l}$$

in which $I_t = 0.14a^4$ is the moment of inertia in torsion of a square section with the side a. Therefore

$$\frac{0.14a^4 G}{l} = 8\pi^2 N^2 mb^2 \text{ and } a = \sqrt[4]{\frac{8\pi^2 N^2 mb^2 l}{0.14G}}$$

$$= \sqrt[4]{\frac{8 \times 10 \times 4 \times 10^2 \times 20 \times 64 \times 10^{-4} \times 40 \times 10^{-2}}{0.14 \times 8 \times 10^{11}}}$$

$$= 1.95 \times 10^{-2} = 1.95 \text{ cm}$$

Example 121. Given: Q, a, l, E and I (Fig. 221). Determine ω, N and T.

Solution. Since the angle of rotation of load Q about the hinge of the left-hand support $\varphi = \frac{Ml}{3EI}$, the rigidity of the beam is

$$C = \frac{M}{\varphi} = \frac{3EI}{l}$$

The moment of inertia of the mass of the load with respect to the axis of rotation is

$$I_m = \frac{Q}{g}a^2$$

According to formulas (225) the circular (cyclic) frequency of oscillation is

$$\omega = \sqrt{\frac{C}{I_m}} = \frac{1}{a}\sqrt{\frac{3EIg}{Ql}}$$

the frequency of oscillation is

$$N = \frac{\omega}{2\pi} = \frac{1}{2\pi a}\sqrt{\frac{3EIg}{Ql}}$$

and the period of oscillation is

$$T = \frac{1}{N} = 2\pi a\sqrt{\frac{Ql}{3EIg}}$$

Example 122. Given: Q, l; for the spring: D, d, n (n = number of turns) and G. Bar AB is perfectly rigid and weightless (Fig. 222).

Determine the circular (cyclic) frequency of oscillation ω as a function of the position of load Q, i.e. of the distance x.

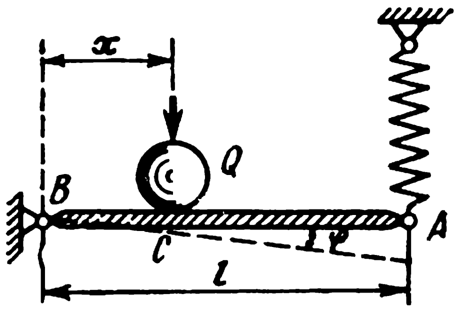


Fig. 222

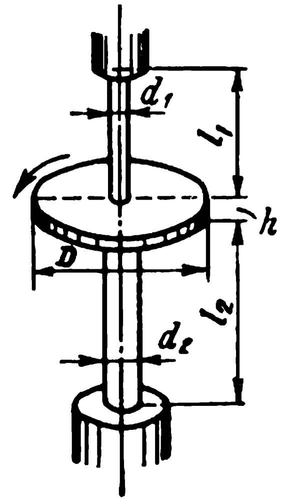


Fig. 223

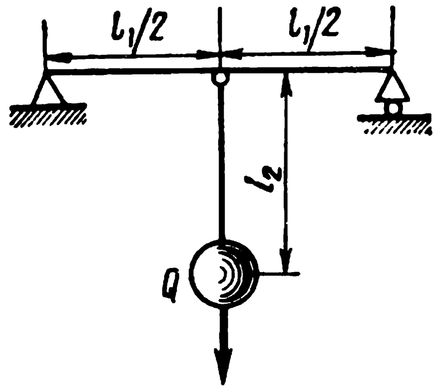


Fig. 224

Solution. The stretching force of the spring $P = Q\frac{x}{l}$.

The vertical displacement of point A is

$$\delta_A = \frac{8PD^3n}{Gd^4} = \frac{8QD^3n}{Gd^4} \times \frac{x}{l}$$

The vertical displacement of point C is

$$\delta_C = \delta_A \frac{x}{l} = \frac{8QD^3n}{Gd^4} \times \frac{x^2}{l^2}$$

The circular frequency of load oscillation is

$$\omega = \sqrt{\frac{g}{\delta_C}} = \frac{ld^2}{2Dx}\sqrt{\frac{Gg}{2QDn}}$$

Another way of solving this problem is based on rotary rather than translatory motion of load Q. Thus

the angular displacement of the load $\varphi = \frac{\delta_A}{l}$

the restoring moment $M = Pl$

the rigidity of the system $C = \frac{M}{\varphi} = \frac{Pl^2}{\delta_A}$

the moment of inertia of the mass of the load with respect to its axis of rotation is

$$I_m = \frac{Q}{g}x^2$$

the circular frequency of load oscillation is

$$\omega = \sqrt{\frac{C}{I_m}} = \frac{l}{x}\sqrt{\frac{Pg}{Q\delta_A}} = \frac{la^2}{2Dx}\sqrt{\frac{Gg}{2QDn}}$$

Example 123. Given: for the bars l_1, d_1, G_1, l_2, d_2 and G_2; for the disk γ (weight of unit volume), D and h (Fig. 223).

Determine ω, the circular frequency of oscillation of the disk.

Solution. The rigidities of the bars in torsion are

$$C_1 = \frac{G_1 I_{p1}}{l_1} = \frac{G_1 \pi d_1^4}{32 l_1}; \quad C_2 = \frac{G_2 I_{p2}}{l_2} = \frac{G_2 \pi d_2^4}{32 l_2}$$

Since the bars are connected in parallel, the rigidity of the oscillating system can be found by formula (221)

$$C = C_1 + C_2 = \frac{\pi}{32}\left(\frac{G_1 d_1^4}{l_1} + \frac{G_2 d_2^4}{l_2}\right)$$

The moment of inertia of the mass of the cylindrical disk about its axis of rotation is

$$I_m = \frac{\gamma}{g}\int_V \rho^2 \, dV = \frac{\gamma}{g}\int_0^{D/2} \rho^2 2\pi\rho h \, d\rho = 2\pi \frac{\gamma}{g} h \int_0^{D/2} \rho^3 \, d\rho = \frac{\pi}{32} \times \frac{\gamma h D^4}{g}$$

From formula (225) the circular frequency of oscillation of the disk is

$$\omega = \sqrt{\frac{C}{I_m}} = \frac{1}{D^2}\sqrt{\frac{\frac{G_1 d_1^4}{l_1} + \frac{G_2 d_2^4}{l_2}}{\frac{\gamma}{g} h}}$$

Example 124. Given: for the beam l_1, E_1 and I_1; for the bar l_2, E_2, F_2 and Q (Fig. 224).

Determine T.

Solution. The rigidity of the beam resting on two supports with the bending force in the middle is

$$C_1 = \frac{48 E_1 I_1}{l_1^3}$$

The rigidity of the bar in tension is

$$C_2 = \frac{E_2 F_2}{l_2}$$

Since the beam and bar are connected consecutively, the rigidity of the oscillating system can be found by formula (224)

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{1}{\frac{l_1^3}{48 E_1 I_1} + \frac{l_2}{E_2 F_2}}$$

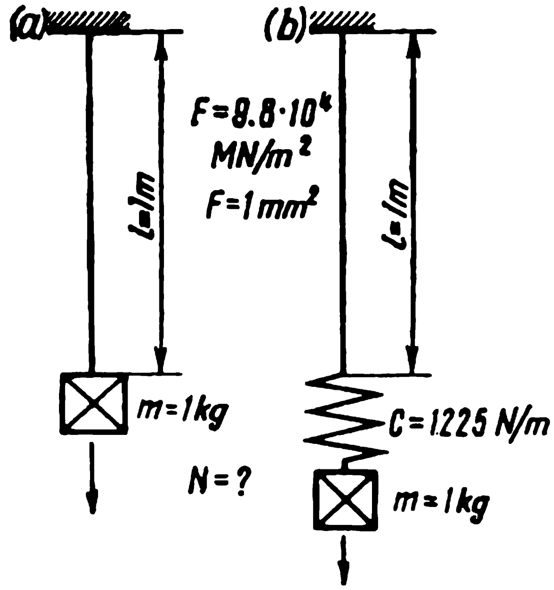
Then the period of oscillation of the system can be obtained by formula (224)

$$T = 2\pi \sqrt{\frac{m}{C}} = 2\pi \sqrt{\frac{Q}{g}\left(\frac{l_1^3}{48 E_1 I_1} + \frac{l_2}{E_2 F_2}\right)}$$

Problems 954 through 965. Determine the quantities given in the problems for oscillating systems.

The arrow at the load indicates the direction of oscillation.

954



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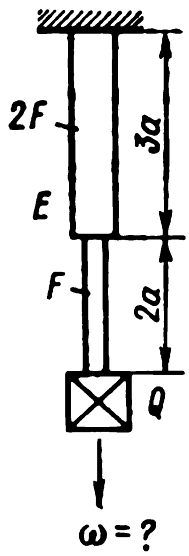

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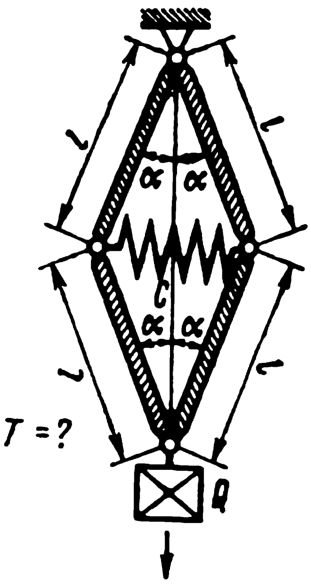

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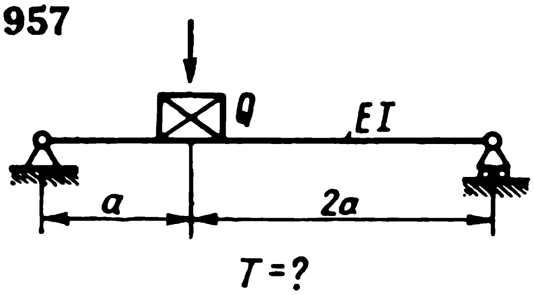

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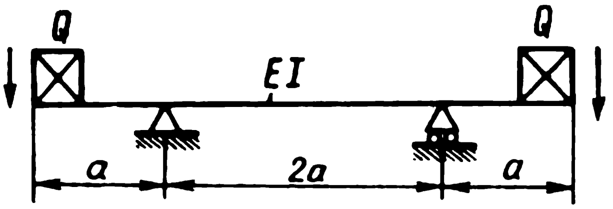

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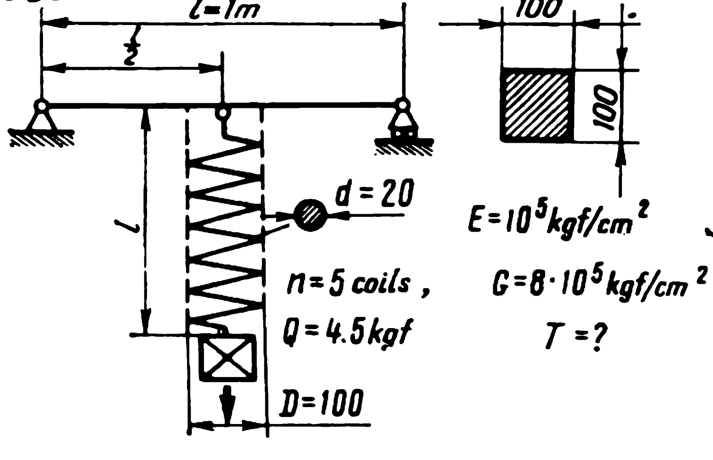

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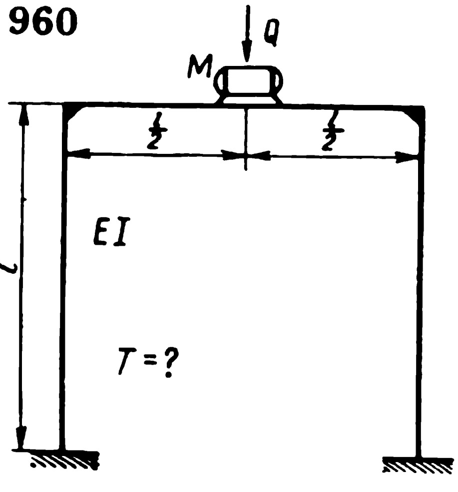

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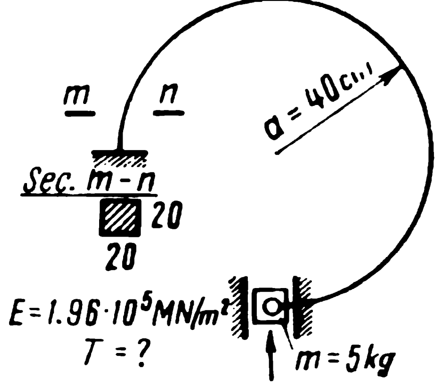

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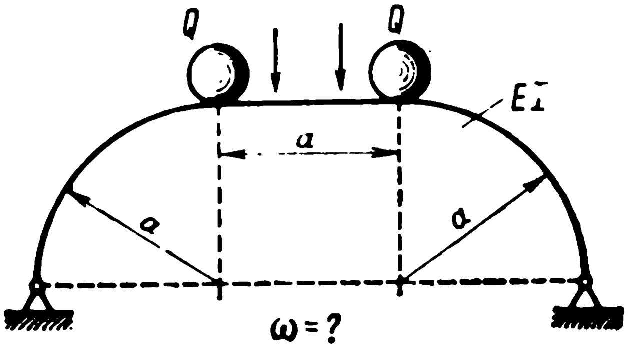

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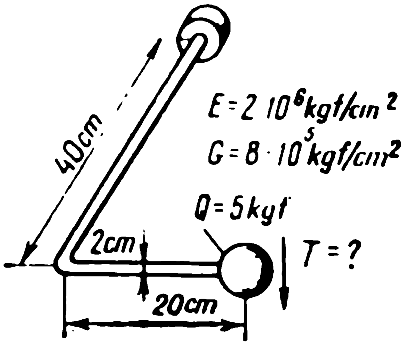

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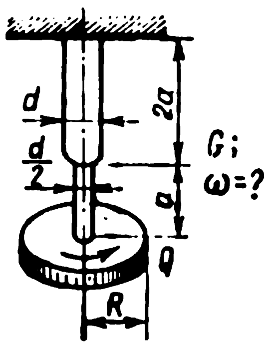

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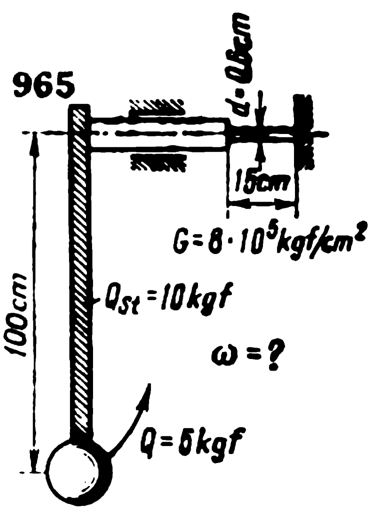


NATURAL OSCILLATIONS OF ELASTIC SYSTEMS (WITH THE DEAD MASS TAKEN INTO ACCOUNT)

The distributed dead mass m_0 of an elastic system subject to natural oscillations can be approximately accounted for by transferring this mass to the point of suspension of the load and adding it to the mass m of the latter.

This reduced mass m_r is a mass, concentrated at the point of suspension of the load, whose kinetic energy is equal to the kinetic energy of motion of the mass m_0 of the system. The reduced mass is proportional to the true mass and is found by formula

$$m_r = k_m m_0 \tag{226}$$

The mass reduction factor k_m, depending on the law of velocity variation of the elements of the mass m_0, is determined from the condition of equality of the kinetic energies m_r and m_0 from which the following expression can be obtained

$$k_m = \frac{1}{m_0} \int\limits_V \left(\frac{\delta_x}{\delta}\right)^2 dm_0 \tag{227}$$

in which δ and δ_x are the generalized displacements of the load suspension point and an arbitrary point of the system due to the statical

effect of a generalized force, corresponding to the kind of strain of the system in oscillation and applied at the suspension point in the direction of oscillation.

For straight bars of constant cross section, the mass reduction factors can be found by the formula

$$k_m = \frac{1}{l}\int_0^l \left(\frac{\delta_x}{\delta}\right)^2 dx \tag{228}$$

in which dx = element of length of the bar
l = length of the bar.

The circular frequency ω, frequency N and period of natural oscillations T in the elastic system are determined (taking its dead weight into account) by the formulas:

(a) In the case of reciprocating motion

$$\left.\begin{aligned}
\omega &= \sqrt{\frac{C}{m+k_m m_0}} = \sqrt{\frac{g}{\delta}\times\frac{1}{1+k_m\dfrac{Q_0}{Q}}} = \sqrt{\frac{g}{\delta}\times\frac{1}{1+k_m\dfrac{m_0}{m}}};\\
N &= \frac{1}{2\pi}\sqrt{\frac{C}{m+k_m m_0}} = \frac{1}{2\pi}\sqrt{\frac{q}{\delta}\times\frac{1}{\left(1+k_m\dfrac{Q_0}{Q}\right)}}\\
&= \frac{1}{2\pi}\sqrt{\frac{q}{\delta}\times\frac{1}{1+k_m\dfrac{m_0}{m}}};\\
T &= 2\pi\sqrt{\frac{m+k_m m_0}{C}} = 2\pi\sqrt{\frac{\delta}{g}\left(1+k_m\frac{Q_0}{Q}\right)}\\
&= 2\pi\sqrt{\frac{\delta}{g}\left(1+k_m\frac{m_0}{m}\right)}
\end{aligned}\right\} \tag{229}$$

in which $Q = mg$ and $Q_0 = m_0 g$ = weight of the load and system
$\delta = \frac{Q}{C}$ = linear displacement of the point of load suspension due to the statical effect of force Q on the system in the direction of oscillation.

(b) In the case of swinging motion

$$\left.\begin{aligned}
\omega &= \sqrt{\frac{C}{I_m+k_m I_{m_0}}};\\
N &= \frac{1}{2\pi}\sqrt{\frac{C}{I_m+k_m I_{m_0}}};\\
T &= 2\pi\sqrt{\frac{I_m+k_m I_{m_0}}{C}}
\end{aligned}\right\} \tag{230}$$

in which I_m and I_{m_0} are the moments of inertia of mass m of the load and mass m_0 of the system with respect to the axis of rotation.

Let us consider the following examples which take into account the dead mass of prismatic bars for the simplest cases of their natural oscillations.

Example 125. Longitudinal oscillations. Given: Q, γ, F, a, b, E (given in SI units are mass m of the load in kg, density of the bar ρ_0, kg/m^3, F, a, b and E, Fig. 225a).

Determine T.

Solution. Upon the statical effect of load Q, the ratios of the linear displacements (lengthening and shortening) of arbitrary sections (specified by coordinates x_1 and x_2) to that of the section at which the load is located are respectively equal. Thus

$$\frac{\delta_{x_1}}{\delta}=\frac{x_1}{a} \text{ and } \frac{\delta_{x_2}}{\delta}=\frac{x_2}{b}$$

On the basis of expression (228) the mass reduction factor is

$$k_m=\frac{1}{a+b}\left[\int_0^a\left(\frac{x_1}{a}\right)^2dx_1+\int_0^b\left(\frac{x_2}{b}\right)^2dx_2\right]=\frac{1}{3}$$

Since the portions of the bar are connected in parallel, the rigidity of the system can be found by formula (221):

$$C=\frac{EF}{a}+\frac{EF}{b}=EF\,\frac{a+b}{ab}$$

The dead mass of the bar is $m_0=\frac{\gamma F}{g}(a+b)$ and the mass of the load is

$$m=\frac{Q}{g}\ [\text{in SI units, } m_0=\rho_0F(a+b)]$$

Next we obtain the period of natural oscillations of the system using formula (229)

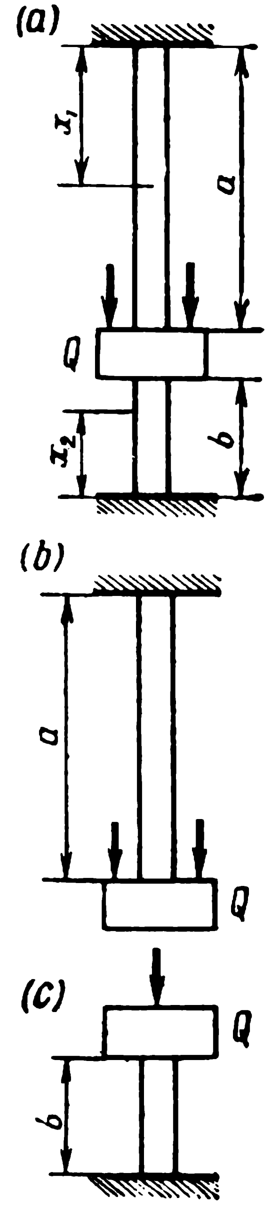


Fig. 225

$$T=2\pi\sqrt{\frac{m+k_mm_0}{C}}=2\pi\sqrt{\frac{Q}{EFg}\times\frac{ab}{a+b}\left(1+\frac{\gamma F}{3}\times\frac{a+b}{Q}\right)}$$

in SI units:

$$T=2\pi\sqrt{\frac{m+k_mm_0}{C}}=2\pi\sqrt{\frac{m}{EF}\times\frac{ab}{a+b}\left(1+\frac{\rho_0F}{3}\times\frac{a+b}{m}\right)}$$

Particular cases (in the metre-kilogram (force)-second system).

If load Q is suspended from a bar of length a, then, if there is no bar of length b and assuming that $\frac{EF}{b}=0$, we obtain (Fig 225b)

$$C=\frac{EF}{a};\quad m_0=\frac{\gamma F}{g}a \text{ and } T=2\pi\sqrt{\frac{Qa}{EFg}\left(1+\frac{\gamma Fa}{3Q}\right)}$$

If load Q rests on a bar of length b and there is no bar of length a (Fig. 225c) then

$$\frac{EF}{a}=0;\quad C=\frac{EF}{b};\quad m_0=\frac{\gamma F}{g}b \text{ and } T=2\pi\sqrt{\frac{Qb}{EFg}\left(1+\frac{\gamma Fb}{3Q}\right)}$$

Example 126. Transverse vibrations. Given: Q, γ, F, l, E and I (Fig. 226).

Determine T.

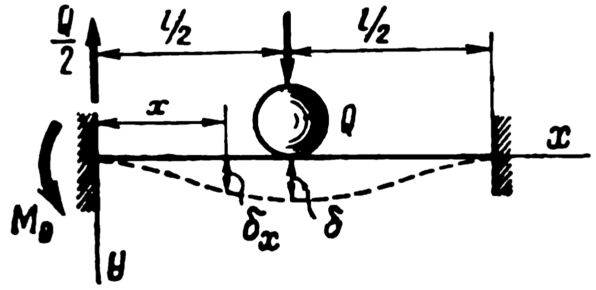


Fig. 226

Solution. Upon the statical effect of load Q, the linear displacement (deflection) of an arbitrary section at a distance x from the left-hand fixed end is found by using the method of initial parameters

$$\delta_x=\frac{1}{EI}\left(M_0\frac{x^2}{2}-\frac{Q}{2}\times\frac{x^3}{6}\right)$$

From the condition of symmetry of the beam we have

$$\left(\frac{d\delta_x}{dx}\right)_{x=\frac{l}{2}}=\frac{1}{EI}\left(\frac{M_0 l}{2}-\frac{Ql^2}{16}\right)=0 \text{ and } M_0=\frac{Ql}{8}$$

Therefore

$$\delta_x=\frac{Ql^3}{48EI}\left(3\frac{x^2}{l^2}-4\frac{x^3}{l^3}\right);\quad \delta=\delta_{x=\frac{l}{2}}=\frac{Ql^3}{192EI}$$

$$\text{and } \frac{\delta_x}{\delta}=4\left(3\frac{x^2}{l^2}-4\frac{x^3}{l^3}\right)$$

Using formula (228) we find the mass reduction factor of the beam:

$$k_m=\frac{2}{l}\int_0^{l/2}\left(\frac{\delta_x}{\delta}\right)^2dx=\frac{32}{l}\int_0^{l/2}\left(3\frac{x^2}{l^2}-4\frac{x^3}{l^3}\right)^2=\frac{13}{35}$$

The dead weight of the beam is $Q=\gamma Fl$.

The period of natural oscillations of the beam is then found from expression (229):

$$T=2\pi\sqrt{\frac{\delta}{g}\left(1+k_m\frac{Q_0}{Q}\right)}=\frac{\pi}{4}\sqrt{\frac{Ql^3}{3EIg}\left(1+\frac{13}{35}\frac{\gamma Fl}{Q}\right)}$$

Example 127. Torsional oscillations. Given: Q, D, γ, d, a, b and G (Fig. 227a).

Determine T.

Solution. Upon the statical effect of torque in the suspension section of the disk the ratios of the angular displacements (angles of twist) of arbitrary sections (specified by coordinates x_1 and x_2) and the section of disk suspension are equal respectively to

$$\frac{\delta_{x_1}}{\delta}=\frac{\varphi_{x_1}}{\varphi}=\frac{x_1}{a} \text{ and } \frac{\delta_{x_2}}{\delta}=\frac{\varphi_{x_2}}{\varphi}=\frac{x_2}{b}$$

In accordance with formula (228), the mass reduction factor of the bar is

$$k_m=\frac{1}{a+b}\left[\int_0^a\left(\frac{x_1}{a}\right)^2dx_1+\int_0^b\left(\frac{x_2}{b}\right)^2dx_2\right]=\frac{1}{3}$$

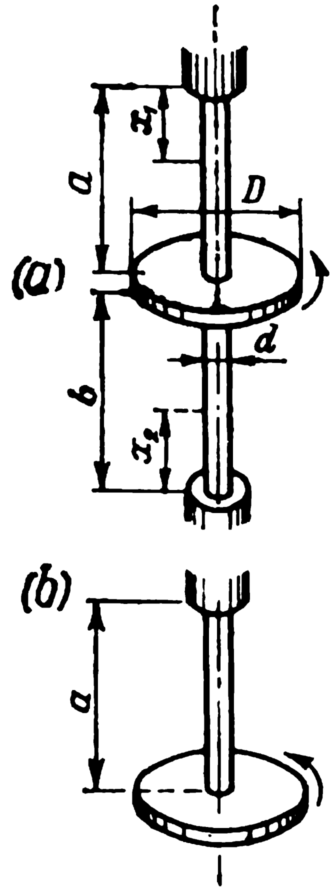


Fig. 227

Since the portions of the bar are connected in parallel, the rigidity of the system can be found by formula (221)

$$C=\frac{GI_p}{a}+\frac{GI_p}{b}=GI_p\frac{a+b}{ab}=G\frac{\pi d^4}{32}\times\frac{a+b}{ab}$$

The moment of inertia of the mass of the cylindrical bar with respect to its geometric axis is

$$I_{m_0}=\frac{\pi}{32}\times\frac{\gamma}{g}d^4(a+b)$$

The moment of inertia of the mass of the cylindrical disk of weight Q and diameter D is

$$I_m=\frac{Q}{g}\times\frac{D^2}{8}$$

Using formula (230) we obtain the period of natural oscillations of the system

$$T=2\pi\sqrt{\frac{I_m+k_mI_{m_0}}{C}}=2\pi\sqrt{\frac{I_mab}{GI_p(a+b)}\left(1+\frac{1}{3}\frac{I_{m_0}}{I_m}\right)}$$
$$=4\pi\frac{D}{d^2}\sqrt{\frac{Q}{\pi Gg}\times\frac{ab}{a+b}\left(1+\frac{\pi d^4\gamma}{12Q}\times\frac{a+b}{D^2}\right)}$$

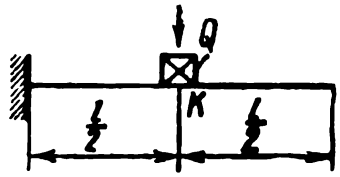
A particular case. If $\frac{GI_p}{b}=0$ (Fig. 227*b*), then

$$C=\frac{GI_p}{a};\quad I_{m_0}=\frac{\pi d^4\gamma a}{32g}$$

and $T=2\pi\sqrt{\frac{I_ma}{GI_p}\left(1+\frac{1}{3}\frac{I_{m_0}}{I_m}\right)}=4\pi\frac{D}{d^2}\sqrt{\frac{Qa}{\pi Gg}\left(1+\frac{\pi d^4\gamma a}{12QD^2}\right)}$

Problems 966 through 970. Determine the mass reduction factors k_m of the beams, assuming section K to be the place to which the mass is transferred.

966



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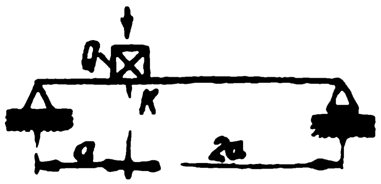

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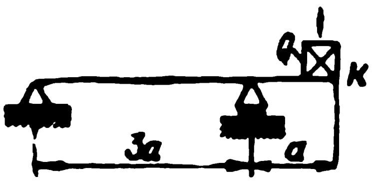
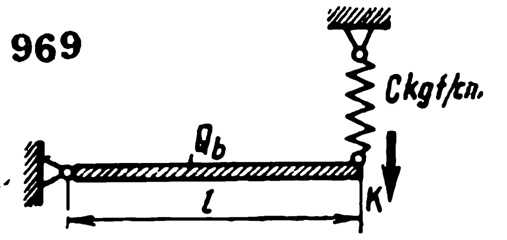


$a = \nu l$

969



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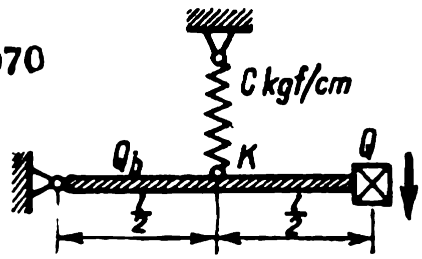
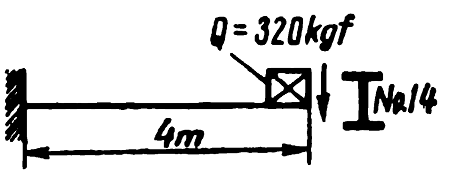


Problems 971 through 975. Determine the periods T of natural oscillations of the systems.

971



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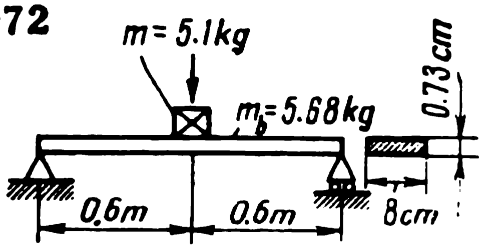
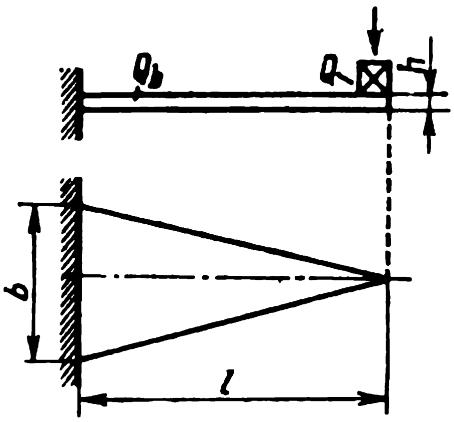


$E = 1.96 \cdot 10^5$ MN/m^2

973



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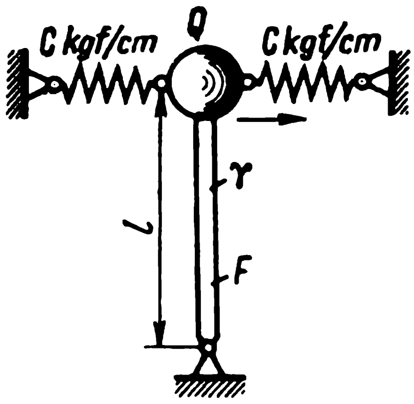

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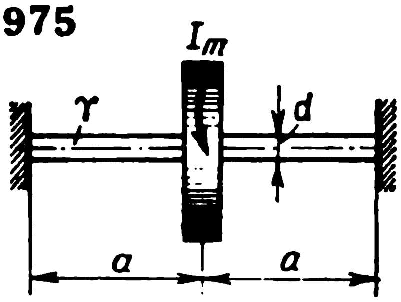


FORCED OSCILLATIONS, NEGLECTING DAMPING

If forced oscillations of a system with one degree of freedom are caused by a harmonic disturbing force applied to the load, then

$$P_{dis} = P_0 \sin \omega_0 t \tag{231}$$

in which P_0 = maximum value of the disturbing force P_{dis}

ω_0 = circular frequency of force P_{dis}

t = time

then the amplitude of forced oscillations can be found from the formula

$$A = \frac{\delta_0}{1 - \frac{\omega_0^2}{\omega^2}} = \beta\delta_0 \tag{232}$$

In formula (232) δ_0 is the generalized displacement of the point of load suspension due to the statical effect of force P_0, ω is the circular frequency of natural oscillations of the system and

$$\beta = \frac{1}{1 - \frac{\omega_0^2}{\omega^2}} \text{ is the oscillation build-up factor.} \tag{233}$$

If $\omega_0 \gg \omega$, then $A \to 0$ and the load on the elastic system will be practically motionless.

If $\omega_0 \ll \omega$, then $A \to \delta_0$ and the oscillations of the load will occur with an amplitude equal to the load displacement due to the statical effect of force P_0.

If ω_0 approaches ω, amplitude A will increase rapidly.

When $\omega_0 = \omega$ resonance appears and $A = \infty$. Actually, A will not become infinite owing to the forces of resistance, but may reach an extremely high value.

In the resonance zone when $0.7 \leqslant \frac{\omega_0}{\omega} \leqslant 1.3$, comparatively small disturbing forces can induce considerable strains in the elements of the oscillating system and hence, considerable stresses.

If, in increasing, the frequency of the disturbing force quickly passes through the resonance zone, the amplitude will not have time to reach the maximum level.

Example 128. A load $Q_1 = 2$ kgf is pivoted on a load $Q = 100$ kgf which is attached at the end of a prismatic bar of length $l = 1$ m and cross-sectional area $F = 1$ cm^2; the load $Q_1 = 2$ kgf rotates at the end of an arm $\rho = 8$ cm long at a speed of 2400 rpm (Fig. 228*a*). Young's modulus of the bar material $E = 2 \times 10^6$ kgf/cm^2.

Determine the amplitude of the forced oscillations of the large load, neglecting the mass of the bar and the forces of resistance.

Solution. Since the linear displacement (elongation) of the point of load suspension due to the statical effect of force Q in the direction of oscillation is

$$\delta = \frac{Ql}{EF} = \frac{100 \times 100}{2 \times 10^6 \times 1} = \frac{1}{2 \times 10^2} = 0.005 \text{ cm}$$

we can use formula (224) to obtain the circular frequency of natural longitudinal oscillations

$$\omega = \sqrt{\frac{g}{\delta}} = \sqrt{981 \times 2 \times 10^2} \cong 443 \frac{1}{\text{sec}}$$

Upon rotation of the unbalanced load Q_1 a centrifugal force $P_0 = \frac{Q_1}{g}\omega_0^2\rho$ is developed whose component in the direction of oscilla-

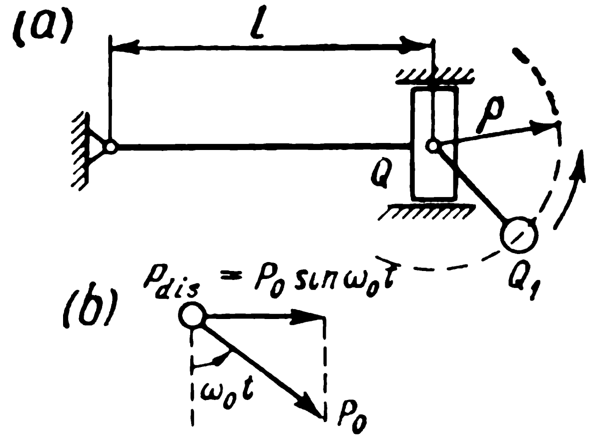


Fig. 228

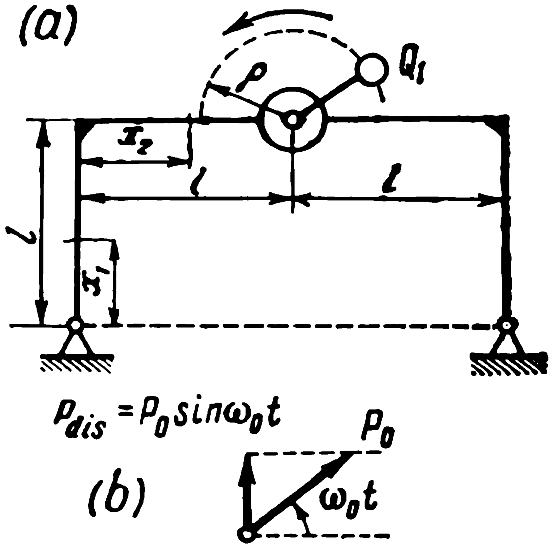


Fig. 229

tion is the harmonic disturbing force (Fig. 228*b*) $P_{dis} = P_0 \sin \omega_0 t$ which causes forced vibrations.

The angular velocity of rotation of load Q_1 is the circular frequency of the forced oscillations and equals

$$\omega_0 = \frac{\pi n}{30} = \frac{\pi \times 2400}{30} \cong 251 \ \ 1/\text{sec}$$

The maximum value of the disturbing force is

$$P_0 = \frac{Q_1}{g}\omega_0^2\rho = \frac{2 \times 251^2 \times 8}{981} \cong 1030 \ \text{kgf}$$

The elongation of the bar (in the direction of oscillation) due to the statical effect of force P_0 is

$$\delta_0 = \frac{P_0 l}{EF} = \frac{1030 \times 100}{2 \times 10^6 \times 1} \cong 0.052 \ \text{cm}$$

Since the oscillation build-up factor, found from formula (233), is

$$\beta = \frac{1}{1 - \frac{\omega_0^2}{\omega^2}} = \frac{1}{1 - \frac{251^2}{443^2}} \cong 1.47$$

the amplitude of the forced oscillations of the system will reach the value

$$A = \beta\delta_0 = 1.47 \times 0.052 \cong 0.08 \ \text{cm}$$

Example 129. An electric motor of mass $m = 96$ kg is mounted in the middle of the collar beam of the frame shown in Fig. 229*a*. The unbalanced portion of the motor has the form of a concentrated load

of mass $m_1 = 4$ kg, rotating at the end of an arm $\rho = 4$ cm long about the motor axis at a speed of $n = 1500$ rpm.

Determine the amplitude of forced oscillations of the frame, if $l = 1$ m, the moment of inertia of the sectional area of the collar beam and struts $I = 400$ cm^4, and $E = 2 \times 10^5$ MN/m^2.

The dead weight of the frame can be neglected.

Solution. The gravity force of the electric motor and its unbalanced portion is

$$Q = mg = 940 \text{ N}; \quad Q_1 = m_1 g = 39.2 \text{ N}$$

The horizontal reactions in the supports of the frame due to the statical effect of load Q can be found by formula (200) of the canonical equation of the force method, viz. $X_1 = \frac{-\delta_{1P}}{\delta_{11}}$. Since

$$EI\delta_{1P} = -2 \int_0^l \frac{Q}{2} l x_2 \, dx_2 = -\frac{Ql^3}{2}$$

and

$$EI\delta_{11} = 2 \int_0^l x_1^2 \, dx_1 + 2l^2 l = \frac{8}{3} l^3$$

then

$$X_1 = \frac{Ql^3 \times 3}{2 \times 8l^3} = \frac{3}{16} Q = \frac{3}{16} 940 = 176 \text{ N}$$

The deflection at the point of suspension of load Q due to its statical effect can be found by the method of the unit fictitious force. Thus

$$\delta = \frac{2}{EI} \int_0^l \left(-X_1 l + \frac{Q}{2} x_2\right) \frac{x_2}{2} \, dx_2 = \frac{7}{96} \times \frac{Ql^3}{EI}$$

$$= \frac{7 \times 940 \times 1^3}{96 \times 2 \times 10^{11} \times 400 \times 10^{-8}} = 0.857 \times 10^{-4} \text{ m} = 0.857 \times 10^{-2} \text{ cm}$$

The circular frequency of natural oscillations of the frame is

$$\omega = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.857 \times 10^{-4}}} = 338 \ \text{1/sec}$$

In the rotation of the unbalanced load Q_1, a centrifugal force $P_0 = m_1 \omega_0^2 \rho$ is developed. Its component in the direction of oscillations (Fig. 229*b*) is the harmonic disturbing force $P_{dis} = P_0 \sin \omega_0 t$ which causes forced oscillations of circular frequency

$$\omega_0 = \frac{\pi n}{30} = \frac{\pi \times 1500}{30} = 157 \ \text{1/sec}$$

The maximum value of the disturbing force is

$$P_0 = 4 \times 157^2 \times 4 \times 10^{-2} = 3940 \text{ N}$$

The deflection in the middle of the collar beam due to the statical effect of this force is

$$\delta_0 = \frac{7}{96} \times \frac{P_0 l^3}{EI} = \frac{7 \times 3940 \times 1}{96 \times 2 \times 10^{11} \times 400 \times 10^{-8}} = 3.7 \times 10^{-4} \text{ m}$$

Since the oscillation build-up factor is

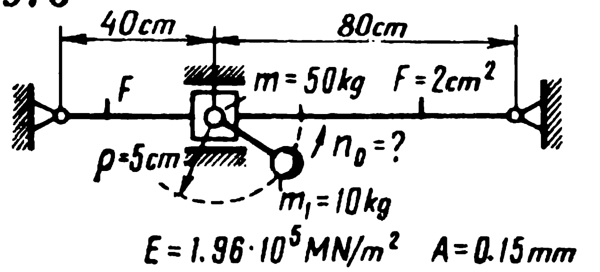
$$\beta = \frac{1}{1 - \frac{\omega_0^2}{\omega^2}} = \frac{1}{1 - \frac{157^2}{338^2}} \approx 1.28$$

the amplitude of forced oscillations of the system equals

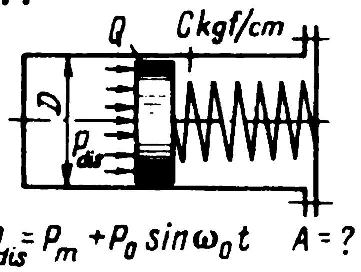
$$A = \beta\delta_0 = 1.28 \times 3.7 \times 10^{-4} = 4.7 \times 10^{-4} \text{ m} = 0.047 \text{ cm}$$

Problem 976. Determine the speed of rotation n_0 of a motor (which can move along guides) with an eccentric weighing Q_1 mounted on its shaft, if the amplitude A of forced oscillations of the motor is known. Weight Q includes both the weight of the motor and of the eccentric (Q_1). The mass of the rods can be neglected.

976



977



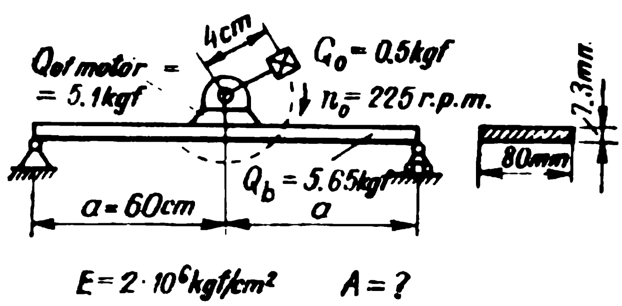
Problem 977. Determine the amplitude A of oscillations of the piston of a steam engine pressure indicator, if the piston travels without friction inside a cylinder of diameter D and the pressure on the piston varies according to the function

$$P_{dis} = P_m + P_0 \sin \omega_0 t$$

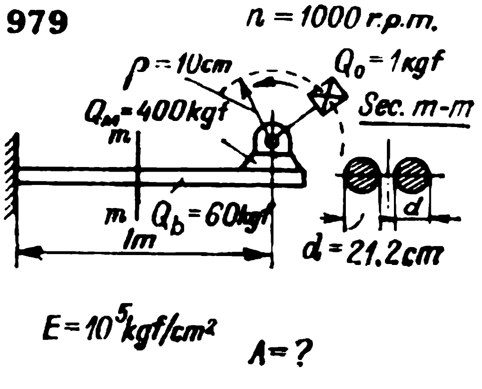
Problems 978, 979 and 980. Determine amplitudes A of forced oscillations of the systems.

In Problems 978 and 979, take the mass of the beams into account.

978



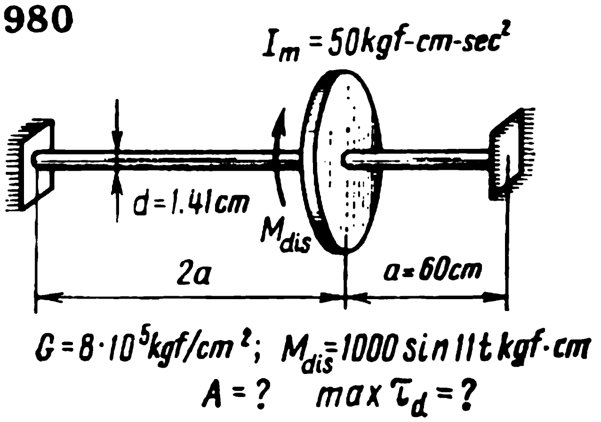
979



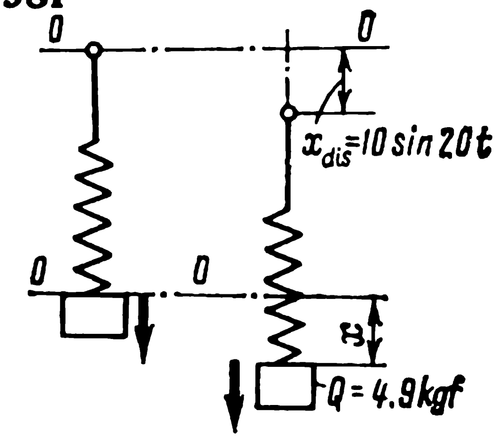
Problem 981. The point of suspension of a coiled spring has a harmonic oscillatory motion in accordance with the equation $x_{dis} = 10 \sin 20t$.

Determine amplitude A of the forced oscillations of the load and the dynamic stresses max τ_d, if the diameter of the spring coil $D = 10$ cm, the diameter of the spring wire $d = 1$ cm, number of coils $n = 25$ and $G = 8 \times 10^5$ kgf/cm^2.

980



981



STRESSES AND STRENGTH CALCULATIONS

The dynamic generalized displacement δ_d of some point of the system at an arbitrary instant of time t of the oscillatory motion is the sum of the constant generalized displacement δ corresponding to the kind of strain of the system during the oscillations due to the statical effect of load Q and the dead weight of the system, and of the variable generalized displacement caused by the disturbing force

$$P_{dis} = P_0 \sin \omega_0 t, \quad \text{i.e.} \quad \delta_d = \delta + \beta\delta_0 \sin \omega_0 t \qquad (234)$$

For linear systems, the generalized dynamic stresses p_d (σ_d or τ_d) at points of the elements in the system are found in a similar manner, viz.

$$p_d = p + \beta p_0 \sin \omega_0 t \qquad (235)$$

in which p is the generalized stress (σ or τ), corresponding to the kind of strain in oscillation due to the statical effect of load Q and the dead weight of the system, and p_0 is the generalized stress due to the statical effect of the maximum disturbing force P_0.

The extremal values of the stresses occur at the moments of maximum deviations of the system from the position of statical equilibrium and equal

$$\substack{\max \\ \min} p_d = p \pm \beta p_0 \qquad (236)$$

Since in oscillation the stresses vary periodically in magnitude (Fig. 230), in the case of a prolonged process the strength analysis of

oscillating systems should be carried out by methods used for cyclic loads.

In short-term oscillating processes, when the amplitude of alternating stresses $\left(\frac{\max p_d - \min p_d}{2}\right)$ is not large, the strength analysis

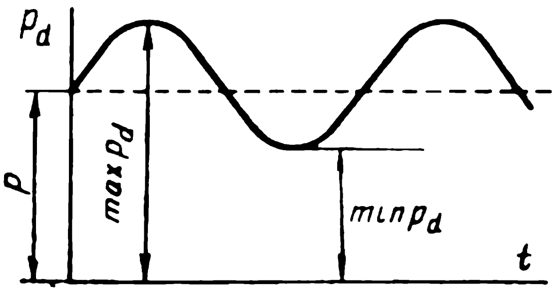


Fig. 230

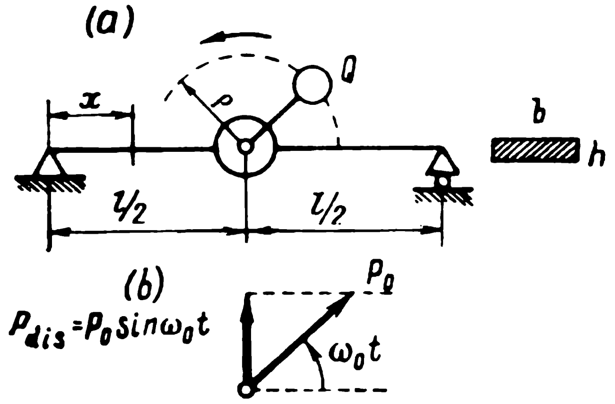


Fig. 231

may be carried out on the basis of the maximum stresses as for a constant load.

Since the maximum dynamic generalized displacement is

$$\max \delta_d = \delta + \beta\delta_0 = \delta\left(1 + \beta\frac{\delta_0}{\delta}\right) = \delta k_d \tag{237}$$

in which

$$k_d = 1 + \beta\frac{\delta_0}{\delta} = 1 + \frac{A}{\delta} \tag{238}$$

is the dynamic factor in oscillation, the maximum dynamic generalized stress is

$$\max p_d = pk_d \tag{239}$$

The strength condition can be written in the following form:

$$\max p_d = pk_d \leqslant [p] \tag{240}$$

or

$$p \leqslant \frac{[p]}{k_d} \tag{241}$$

in which $[p]$ is the allowable generalized stress ($[\sigma]$ or $[\tau]$).

Example 130. An electric motor of weight $Q = 20$ kgf is located in the middle of a beam of rectangular cross section $F = b \times h = 12 \times 1$ cm^2 and $l = 1$ m long. The unbalanced rotary part of the motor is a concentrated load $Q_1 = 1$ kgf mounted on the motor axle at a distance $\rho = 4$ cm (Fig. 231*a*).

Determine the motor speed n (in rpm) at which the maximum normal stresses in the beam reach the value $\max \sigma_d = 2000$ kgf/cm^2. The weight per unit volume of the beam material $\gamma = 8$ gf/cm^3 and Young's modulus $E = 2 \times 10^6$ kgf/cm^2.

Solution. The dead weight of the beam is

$$Q_0 = \gamma F l = 8 \times 10^{-3} \times 12 \times 1 \times 10^2 = 9.6 \text{ kgf}$$

The moment of inertia of the beam cross section with respect to the neutral axis and the section modulus are

$$I = \frac{bh^3}{12} = \frac{12 \times 1}{12} = 1 \text{ cm}^4; \quad W = \frac{bh^2}{6} = \frac{12 \times 1}{6} = 2 \text{ cm}^3$$

Upon the statical effect of the concentrated force Q and distributed force Q_0, the maximum normal stress σ_{max} and the maximum deflection δ_{max} of the beam are

$$\sigma_{max} = \frac{M_{max}}{W} = \frac{\frac{Ql}{4} + \frac{Q_0 l}{8}}{W} = \frac{l}{4W}\left(Q + \frac{Q_0}{2}\right) = \frac{100}{4 \times 2}\left(20 + \frac{9.6}{2}\right)$$
$$= 310 \text{ kgf/cm}^2$$

$$\delta_{max} = \frac{Ql^3}{48EI} + \frac{5}{384}\frac{Q_0 l^3}{EI} = \frac{l^3}{48EI}\left(Q + \frac{5}{8}Q_0\right)$$
$$= \frac{10^6}{48 \times 2 \times 10^6 \times 1}\left(20 + \frac{5 \times 9.6}{8}\right) \cong 0.27 \text{ cm}$$

Since the dynamic factor found by formula (238) is

$$k_d = 1 + \frac{A}{\delta_{max}} = \frac{\max \sigma_d}{\sigma_{max}} = \frac{2000}{310} \cong 6.45$$

the permissible amplitude of oscillation

$$A = \left(\frac{\max \sigma_d}{\sigma_{max}} - 1\right)\delta_{max} = 5.45 \times 0.27 = 1.47 \text{ cm}$$

The deflection δ_x in an arbitrary section at a distance x from the left-hand support and the deflection δ in the middle of the beam due to the statical effect of the concentrated force Q equal

$$\delta_x = \frac{Ql^3}{48EI}\left(3\frac{x}{l} - 4\frac{x^3}{l^3}\right) \text{ and } \delta = \frac{Ql^3}{48EI} = \frac{20 \times 10^6}{48 \times 2 \times 10^6 \times 1} = \frac{5}{24} \text{ cm}$$

Therefore

$$\frac{\delta_x}{\delta} = 3\frac{x}{l} - 4\frac{x^3}{l^3}$$

and the mass reduction factor found by formula (238) is

$$k_m = \frac{2}{l}\int_0^{l/2}\left(\frac{\delta_x}{\delta}\right)^2 dx = \frac{2}{l}\int_0^{l/2}\left(3\frac{x}{l} - 4\frac{x^3}{l^3}\right)^2 dx = \frac{17}{35}$$

The circular frequency of natural oscillations of the beam is determined by formula (229):

$$\omega=\sqrt{\frac{g}{\delta}\times\frac{1}{1+k_m\frac{Q_0}{Q}}}=\sqrt{\frac{981\times 24}{5}\times\frac{1}{1+\frac{17}{35}\times\frac{9.6}{20}}}\cong 61.8\ \text{1/sec}$$

Since the rotating load Q_1 is unbalanced, a centrifugal force $P_0 = \frac{Q_1}{g}\omega_0^2\rho$ is developed. Its vertical component (Fig. 231*b*) $P_{dis} = P_0 \sin\omega_0 t$ is the harmonic disturbing force causing forced oscillations of the beam. Here $\omega_0 = \frac{\pi n}{30}$ is the circular frequency of the disturbing force and P_0 is the maximum value of this force.

Upon the statical effect of force P_0 the maximum deflection of the beam is

$$\delta_0=\frac{P_0 l^3}{48EI}=\frac{Q_1 l^3\rho}{48EIg}\omega_0^2=\frac{1\times 10^6\times 4}{48\times 2\times 10^6\times 1\times 981}\omega_0^2=\frac{\omega_0^2}{24\times 981}\ \text{cm}$$

According to formula (232) we can write

$$1.47=\frac{\omega_0^2}{24\times 981\left(1-\frac{\omega_0^2}{61.8^2}\right)}=\frac{61.8^2}{24\times 981}\times\frac{\omega_0^2}{61.8^2-\omega_0^2}$$

$$\cong 0.162\frac{\omega_0^2}{61.8^2-\omega_0^2}$$

from which

$$\omega_0^2=\frac{61.8^2\times 1.47}{1.63}\quad\text{and}\quad\omega_0=61.8\sqrt{\frac{147}{163}}\cong 58.6\ \text{1/sec}$$

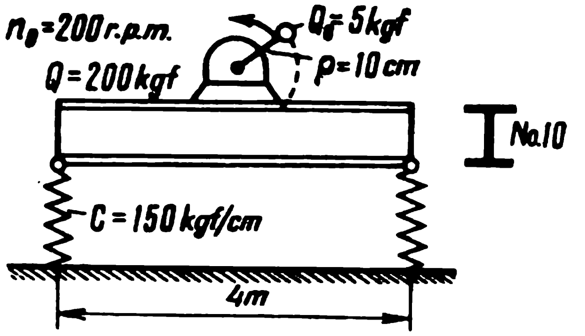
Thus the required number of revolutions per min is

$$n=\frac{30}{\pi}\omega_0=\frac{30\times 58.6}{\pi}\cong 560\ \text{rpm}$$

Problems 982 and 983. Determine the maximum dynamic stresses in the systems subject to forced oscillations.

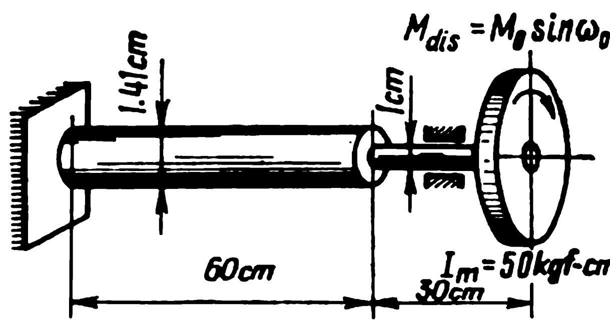
Weight Q_0 is included in weight Q.

982



max σ_d = ?

983



$\omega_0=10\frac{1}{sec}$; M_0=2000kgf-cm

$G=8\cdot 10^5$kgf/cm²; max τ_d=?

14.3.

Impact

Impact phenomena occur in cases of an abrupt change in the speeds of bodies, systems or their parts when they come into contact.

We shall deal here with only the simplest cases of collision of a moving body (incident body) with a stationary body or system (struck body), assuming that:

1. The incident body is perfectly rigid.
2. The struck body has one degree of freedom and its generalized displacements are proportional to the corresponding generalized forces in statical or dynamic action.
3. The impact is inelastic and the incident body is not separated from the struck body, but the total strains of the latter are elastic.
4. The kind of deformation experienced by the struck body is the same as that due to static loading by a corresponding generalized force applied at the point of and in the direction of collision.
5. The speed of the incident body is small as compared to the speed of propagation of shock waves, and the collision time greatly exceeds the time of propagation of these waves throughout the volume of the struck body.

With these assumptions the generalized dynamic internal forces P_d, stresses p_d and displacements δ_d in the struck body can be determined approximately from the formulas:

$$\left.\begin{aligned} P_d &= k_d P; \\ p_d &= k_d p; \\ \delta_d &= k_d \delta \end{aligned}\right\} \qquad (242)$$

Here P, p and δ are taken for the case of the statical effect on the struck body of a generalized force applied at the point of and in the direction of collision, and k_d is a nondimensional dynamic factor greater than unity.

If, an incident body of weight Q upon impact with a struck body of weight Q_0, travels in the direction of the gravity force at a velocity v_0 and causes translatory displacements in the elements of the struck body, the dynamic factor is

$$k_d = 1 + \sqrt{1 + \frac{v_0^2}{g\delta} \times \frac{1}{1 + k_m \frac{Q_0}{Q}}} = 1 + \sqrt{1 + \frac{v_0^2 C}{Qg} \times \frac{1}{1 + k_m \frac{Q_0}{Q}}} = 1 + \sqrt{1 + \frac{T}{U} \times \frac{1}{1 + k_m \frac{Q_0}{Q}}} \qquad (243)$$

in which $\delta = \frac{Q}{C}$ = linear displacement of the point of collision due to the statical effect of load Q in the direction of impact

C = rigidity of the struck body for the given type of strain

k_m = mass reduction factor of the struck body, transferring its mass to the point of collision and determined in the same way as for oscillations

$T = \frac{Qv_0^2}{2g}$ = kinetic energy of motion of load Q up to the moment of impact

$U = \frac{Q\delta}{2}$ = elastic strain energy of the struck body upon the statical effect of load Q.

In SI units the kinetic energy $T = \frac{mv^2}{2}$ and the ratio $\frac{Q_0}{Q}$ is replaced by the ratio $\frac{m_0}{m}$.

It is evident from formula (243) that the greater the rigidity C of the struck body, the greater the dynamic factor k_d.

Upon a sudden application of load Q, when $v_0 = 0$

$$k_d = 2 \tag{244}$$

For an impact when the mass of the struck body is neglected ($Q_0 \ll Q$)

$$k_d = 1 + \sqrt{1 + \frac{v_0^2}{g\delta}} = 1 + \sqrt{1 + \frac{v_0^2 C}{Qg}} = 1 + \sqrt{1 + \frac{T}{U}} \tag{245}$$

If the mass of the struck body is neglected, the dynamic factor increases as compared to its general value (243) and calculations are in favour of the factor of safety.

If $\frac{v_0^2}{g\delta} = \frac{v_0^2 C}{Qg} = \frac{T}{U} \geqslant 10$, then the dynamic factor can be calculated to an accuracy within 5% by the following formula:

$$k_d = 1 + \sqrt{\frac{v_0^2}{g\delta}} = 1 + \sqrt{\frac{v_0^2 C}{Qg}} = 1 + \sqrt{\frac{T}{U}} \tag{246}$$

If, on the other hand, $\frac{v_0^2}{g\delta} \geqslant 110$, then the following equation is accurate within 10%:

$$k_d = \sqrt{\frac{v_0^2}{g\delta}} = \sqrt{\frac{v_0^2 C}{Qg}} = \sqrt{\frac{T}{U}} \tag{247}$$

It should be kept in mind that in using formula (246) and (247) the calculations are not in favour of the factor of safety.

In cases when the dead weight of the struck body is very large ($Q_0 \to \infty$), $k_d \to 2$.

In case of a horizontal impact, equation (243) should be replaced by

$$k_d=\sqrt{\frac{v_0^2}{g\delta}\times\frac{1}{1+k_m\frac{Q_0}{Q}}}=\sqrt{\frac{v_0^2C}{Qg}\times\frac{1}{1+k_m\frac{Q_0}{Q}}}$$

$$=\sqrt{\frac{T}{U}\times\frac{1}{1+k_m\frac{Q_0}{Q}}} \tag{248}$$

In this case neglection of the mass of the struck body will favour the factor of safety, even though k_d is found by formula (247).

If the dead weight of the struck body is small, but it carries a heavy load Q_1 which is hit by load Q, in formulas (243) or (248) the quantity k_mQ_0 should be replaced by the quantity Q_1.

In calculating colliding systems with parallel, consecutive or combined connection of their elements, the rigidity of the system (C) can be found by formulas (221), (222) or (223), respectively.

If the incident body is in rotary motion and causes strains in the struck body which are specified by angular displacements, then use can be made of the above formulas, after replacing the linear velocity v_0 with the angular velocity ω_0 and masses $\frac{Q}{g}$ and $\frac{Q_0}{g}$ with the moments of inertia of the masses I_m and I_{m_0} with respect to their axes of rotation.

Strength calculations for impacts are carried out using the formulas valid for statical loading.

The strength condition can be written in the following form:

$$\max p_d = k_d p_{max} \leqslant [p_d] \tag{249}$$

in which $\max p_d$ = maximum dynamic generalized design stress
p_{max} = maximum generalized stress due to the statical effect of load Q
$[p_d]$ = corresponding generalized dynamic allowable stress.

In practical design calculations, one should take into account the fact that upon impact loads a body tends to brittle failure which is dependent to a great extent on the composition and structure of the material of the body, loading rate, temperature and stress concentration.

In the problems which follow it is presumed that the above basic factors affecting the impact strength have been taken into consideration in the given allowable stresses.

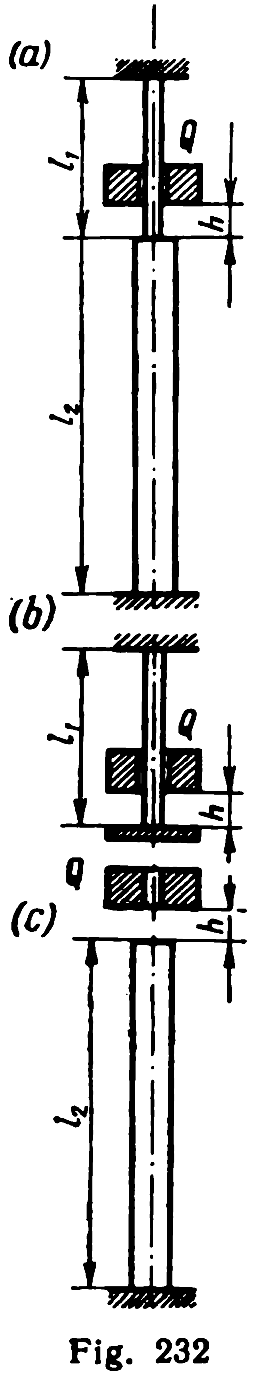
Example 131. Let $m = 10$ kg; h = 4 cm; $l_1 = 20$ cm; $F_1 = 2$ cm^2; $l_2 = 40$ cm; $F_2 = 4$ cm^2 and $E_1 = E_2 = E = 2 \times 10^5$ MN/m^2 (Fig. 232*a*). The dead weight of the bars can be neglected.

Determine σ_{d1}, σ_{d2} and δ_d.

Solution. First we find the weight of the incident load

$$Q = mg = 10 \times 9.81 = 98.1 \text{ N}$$

Since bars of rigidity $C_1 = \frac{EF_1}{l_1}$ and $C_2 = \frac{EF_2}{l_2}$ are connected in parallel, the rigidity of the system, according to formula (221), is



$$C = C_1 + C_2 = \frac{EF_1}{l_1} + \frac{EF_2}{l_2} = E\left(\frac{F_1}{l_1} + \frac{F_2}{l_2}\right)$$

$$= 2 \times 10^{11}\left(\frac{2 \times 10^{-4}}{0.20} + \frac{4 \times 10^{-4}}{0.40}\right) = 4 \times 10^8 \text{ N/m}$$

Taking into account that by the moment of impact the velocity of load Q, freely falling from height h, is $v_0 = \sqrt{2gh}$, we obtain

$$\frac{v_0^2 C}{Qg} = \frac{2hC}{Q} = \frac{2 \times 4 \times 10^{-2} \times 4 \times 10^8}{98.1} = 32.6 \times 10^4$$

This quantity is so great that the dynamic factor can be safely found by formula (247)

$$k_d = \sqrt{\frac{v_0^2 C}{Qg}} = \sqrt{32.6 \times 10^4} = 571$$

Since due to the statical effect of load Q, the displacement of its point of application is $\delta = \frac{Q}{C} = \frac{N_1}{C_1} = \frac{N_2}{C_2}$, the axial internal forces in the cross sections of the bars are

$$N_1 = \frac{Q}{C} C_1 = \frac{Q}{1 + \frac{C_2}{C_1}} \quad \text{and} \quad N_2 = \frac{Q}{C} C_2 = \frac{Q}{1 + \frac{C_1}{C_2}}$$

The dynamic normal stresses in the cross sections of the bars become

$$\sigma_{d_1} = \frac{N_1}{F_1} k_d = k_d \frac{Q}{F_1\left(1 + \frac{F_2 l_1}{F_1 l_2}\right)} = 571$$

$$\times \frac{98.1}{2 \times 10^{-4}\left(1 + \frac{4 \times 10^{-4}}{2 \times 10^{-4}} \times \frac{0.20}{0.40}\right)}$$

$$= 140 \times 10^6 \text{ N/m}^2 = 140 \text{ MN/m}^2$$

$$\sigma_{d_2} = \frac{N_2}{F_2} k_d = \frac{Q}{F_2\left(1 + \frac{F_1 l_2}{F_2 l_1}\right)} k_d$$

$$= \frac{98.1}{4 \times 10^{-4}\left(1 + \frac{2 \times 10^{-4} \times 0.40}{4 \times 10^{-4} \times 0.20}\right)} 571 = 70 \times 10^6 \text{ N/m}^2 = 70 \text{ MN/m}^2$$

The dynamic displacement of the point of collision is

$$\delta_d=\frac{Q}{C}k_d=\frac{10\times 9.81}{4\times 108}571\cong 1.4\times 10^{-4}\ \text{m}=0.014\ \text{cm}$$

Particular cases.

(1) If $C_2=0$ (Fig. 232*b*), then

$$C=C_1=\frac{EF_1}{l_1}=\frac{2\times 10^{11}}{0.20}\times 0.02\times 10^{-2}\cong 2\times 10^8\ \text{N/m};$$

$$k_d=\sqrt{\frac{2hC_1}{Q}}=\sqrt{\frac{2\times 4\times 10^{-2}\times 2\times 10^8}{98.1}}\cong 4\times 10^2;$$

$$\sigma_{d_1}=\frac{Q}{F_1}k_d=\frac{98.1}{2\times 10^{-4}}\times 4\times 10^2\cong 200\times 10^6\ \text{N/m}^2\cong 200\ \text{MN/m}^2;$$

$$\delta_d=\frac{Q}{C}k_d=\frac{98.1}{2\times 10^8}\times 4\times 10^2\cong 2\times 10^{-4}\ \text{m}\cong 0.02\ \text{cm}$$

(2) If $C_1=0$ (Fig. 232*c*), then

$$C=C_2=\frac{EF_2}{l_2}=\frac{2\times 10^{11}\times 4\times 10^{-4}}{0.40}=2\times 10^8\ \text{N/m};\quad k_d=4\times 10^2;$$

$$\sigma_{d_2}=\frac{Q}{F_2}k_d=\frac{10\times 9.81}{4}\times 4\times 10^2\cong 2\times 10^8\ \text{N/m}^2=200\ \text{MN/m}^2;$$

$$\delta_d=\frac{Q}{C}k_d=2\times 10^{-4}\ \text{m}=2\times 10^{-2}\ \text{cm}$$

Example 132. Let $Q=4$ kgf; $h=4$ cm; for small-pitch springs: $D=2$ cm, $d=0.4$ cm, $n=4$ coils and $G=8\times 10^5$ kgf/cm², for

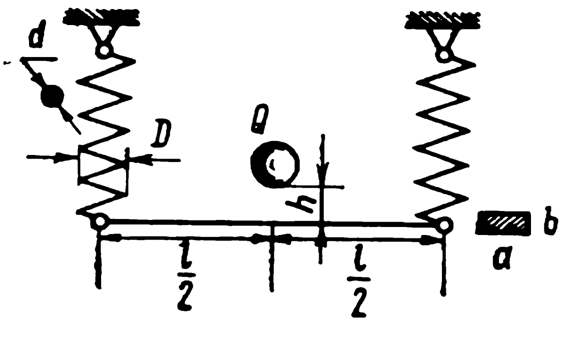


Fig. 233

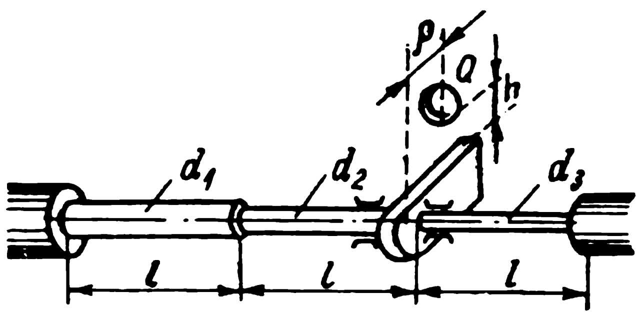


Fig. 234

the beam: $l=40$ cm, $a=3$ cm, $b=1$ cm and $E=2\times 10^6$ kgf/cm² (Fig. 233). The dead weight of the springs and the beam can be neglected.

Determine max τ_d in the spring, max σ_d in the beam, and the displacement δ_d of the point of collision.

Solution. Since the springs of rigidity $C_1=\frac{Gd^4}{8D^3n}$ are connected in parallel and the beam of rigidity $C_2=\frac{48EI}{l^3}$ is connected with

them consecutively, according to formula (223) the rigidity of the system, as one with combined connections of its elements, is

$$C=\frac{1}{\frac{1}{2C_1}+\frac{1}{C_2}}=\frac{1}{\frac{4D^3n}{Gd^4}+\frac{l^3}{48EI}}$$

$$=\frac{1}{\frac{4\times8\times4}{8\times10^5\times256\times10^{-4}}+\frac{64\times10^3\times12}{48\times2\times10^6\times3\times1}}\cong 112 \text{ kgf/cm}$$

The dynamic factor is found from formula (245)

$$k_d=1+\sqrt{1+\frac{2hC}{Q}}=1+\sqrt{1+\frac{2\times4\times112}{4}}\cong 16$$

Since $\frac{D}{d}=\frac{2}{0.4}=5$, the stress build-up factor in the spring, due to the curvature of the coil, is

$$k=\frac{\frac{D}{d}-0.25}{\frac{D}{d}-1}+\frac{0.615}{\frac{D}{d}}=\frac{4.75}{4}+\frac{0.615}{5}\cong 1.31$$

The maximum dynamic stresses in the springs and beam are

$$\max\tau_d=k\frac{4QD}{\pi d^3}k_d\cong 1.31\frac{4\times4\times2}{\pi\times64\times10^{-3}}16\cong 3340 \text{ kgf/cm}^2;$$

$$\max\sigma_d=\frac{Ql}{4W}k_d=\frac{4\times40\times6}{4\times3\times1}16\cong 1280 \text{ kgf/cm}^2$$

The dynamic displacement of the point of collision is

$$\delta_d=\frac{Q}{C}k_d=\frac{4}{112}16\cong 0.57 \text{ cm}$$

If there were no springs, then

$$C=C_2=\frac{48EI}{l^3}=375 \text{ kgf/cm};$$

$$k_d=1+\sqrt{1+\frac{2hC}{Q}}=1+\sqrt{1+\frac{2\times4\times375}{4}}\cong 28.4;$$

$$\sigma_{max}=\frac{4\times40\times6}{4\times3\times1}\times28.4\cong 2270 \text{ kgf/cm}^2$$

Example 133. Let $Q = 20$ kgf, $h = 2$ cm, $\rho = 4$ cm; $l_1 = l_2 = l_3 = l = 20$ cm, $d_1 = 1.4$ cm, $d_2 = 1.2$ cm, $d_3 = 1$ cm and $G = 2.8 \times 10^5$ kgf/cm^2 (Fig. 234).

The dead weight of the stepped bar can be neglected.

Determine δ_d (dynamic displacement of the point of application of load Q) and $\max\tau_{d_{1,2,3}}$ (maximum dynamic shearing stresses in the portions of the bar).

Solution. The torsional rigidities of the bar steps are given, respectively, by

$$C_1=\frac{GI_{p1}}{l};\quad C_2=\frac{GI_{p2}}{l};\quad C_3=\frac{GI_{p3}}{l}$$

The first two steps of the bar are connected consecutively and their total rigidity is

$$C_0=\frac{1}{\frac{1}{C_1}+\frac{1}{C_2}}$$

Since in the section at which the external couple is applied, the third step of the bar is connected in parallel with the first two steps, the total rigidity of the entire bar, as a system with the elements connected in parallel is

$$C=C_0+C_3=\frac{1}{\frac{1}{C_1}+\frac{1}{C_2}}+C_3=\frac{1}{\frac{l}{GI_{p1}}+\frac{l}{GI_{p2}}}+\frac{GI_{p3}}{l}$$

$$=\frac{G}{l}\left(\frac{1}{\frac{1}{I_{p1}}+\frac{1}{I_{p2}}}+I_{p3}\right)\cong 0.1\,\frac{G}{l}\left(\frac{1}{\frac{1}{d_1^4}+\frac{1}{d_2^4}}+d_3^4\right)$$

$$=0.1\,\frac{2.8\times 10^5}{20}\left(\frac{1}{\frac{1}{1.4^4}+\frac{1}{1.2^4}}+1\right)\cong 3290 \text{ kgf/cm}$$

The kinetic energy of the incident load at the moment of impact is

$$T=\frac{Qv_0^2}{2g}=Qh \text{ kgf-cm}$$

The angle of statical rotation of the section at which the couple of forces (moment $M = Q\rho$) is applied can be found by

$$\varphi=\frac{M}{C}=\frac{Q\rho}{C}=\frac{20\times 4}{3290}=\frac{8}{329}\text{ rad}$$

The elastic strain energy upon statical torsion of the bar is

$$U=\frac{M\varphi}{2}=\frac{Q^2\rho^2}{2C}$$

Since the ratio

$$\frac{T}{U}=\frac{Qh}{Q^2\rho^2}2C=\frac{2hC}{Q\rho^2}=\frac{2\times 2\times 3290}{20\times 16}\cong 41.1$$

is not large, the dynamic factor can be found by formula (246). Thus

$$k_d=1+\sqrt{1+\frac{T}{U}}=1+\sqrt{42.1}\cong 7.5$$

The dynamic angle of rotation of the section of the bar, at which the external moment due to the falling load is applied, is

$$\varphi_d = \varphi k_d = \frac{8 \times 7.5}{329} \cong 0.182 \text{ rad}$$

The dynamic linear displacement of load Q is

$$\delta_d = \rho\varphi_d = 0.182 \times 4 = 0.73 \text{ cm}$$

The support moment at the right-hand fixed end of the bar, due to the statical effect of an external couple of moment $M = Q\rho$, is

$$M_{rh} = \varphi C_3 = \varphi \frac{GI_{p3}}{l} = \frac{8}{329} \times \frac{2.8 \times 10^5 \times 0.1}{20} \cong 34 \text{ kgf/cm}$$

The support moment at the left-hand fixed end of the bar is

$$M_{lh} = M - M_{rh} = 20 \times 4 - 34 = 46 \text{ kgf-cm}$$

The maximum dynamic shearing stresses in the portions of the bar are

$$\max \tau_{d_1} = \frac{M_{lh}}{W_{p_1}} k_d \cong \frac{46}{0.2 \times 1.4^3} 7.5 \cong 628 \text{ kgf/cm}^2;$$

$$\max \tau_{d_2} = \frac{M_{lh}}{W_{p_2}} k_d \cong \frac{46}{0.2 \times 1.2^3} 7.5 \cong 986 \text{ kgf/cm}^2;$$

$$\max \tau_{d_3} = \frac{M_{rh}}{W_{p_3}} k_d \cong \frac{34}{0.2 \times 1} 7.5 \cong 1230 \text{ kgf/cm}^2$$

Example 134. Let $m = 1$ kg, $v_0 = 4$ m/sec, $m_1 = 20$ kg, $d = 2$ cm, $l = 0.4$ m and $E = \frac{5}{2}G = 2 \times 10^5$ MN/m^2 (Fig. 235). The impact is horizontal. The dead weight of the bar can be neglected.

Determine $\max \sigma_d$, $\max \tau_d$ and δ_d.

Solution. First we find the weight corresponding to mass m: $Q = mg = 9.81$ N. Since the system is subject to an impact by a mass m, travelling horizontally, the dynamic factor should be determined by formula (248)*, replacing value $k_m m_0$ by the magnitude of mass m_1, i.e.

$$k_d = \sqrt{\frac{v_0^2 C}{Qg} \times \frac{1}{1 + \frac{m_1}{m}}}$$

Since the horizontal displacement of the point of collision, due to the twisting of the vertical bar by the statical effect of force Q in

---

* Since the problem is solved in International System (SI) units, in formula (248) the ratio $\frac{k_m Q_0}{Q}$ should be replaced by $\frac{k_m m_0}{m}$.

the direction of impact, is

$$\delta_0 \cong \varphi l = \frac{Ml}{GI_p} l = \frac{Ql^3}{GI_p}$$

the struck system can be considered as one with consecutive connection of the horizontal element being bent whose rigidity is $C_1 = \frac{3EI}{l^3}$, to the vertical twisted element whose rigidity with respect to the linear displacement of the point of collision is $C_2 = \frac{GI_p}{l^3}$.

According to formula (222) the rigidity of the entire system is

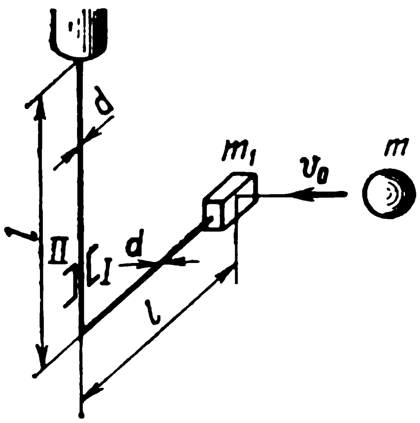


Fig. 235

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{1}{\frac{l^3}{3EI} + \frac{l^3}{GI_p}}$$

$$= \frac{1}{\frac{l^3}{3 \times \frac{5}{2} G \times \frac{1}{2} \times I_p} + \frac{l^3}{GI_p}}$$

$$\cong \frac{GI_p}{1.27 l^3} \cong \frac{8 \times 10^{10} \times 0.1 \times 0.02^4}{1.27 \times 0.4^3} \cong 16 \times 10^3 \ \text{N/m}$$

and the dynamic factor is

$$k_d = \sqrt{\frac{16 \times 16 \times 10^3 \times 1}{1 \times 9.81} \times \frac{1}{1 + \frac{20}{1}}} \cong 11.2$$

With the statical effect of force Q in the direction of impact, the maximum normal stress σ_{max} in the horizontal bar, the maximum shearing stress τ_{max} in the vertical bar and the linear displacement δ of the point of collision are

$$\sigma_{max} = \frac{Ql}{W} \cong \frac{9.81 \times 0.4}{0.1 \times 0.02^3} \cong 5 \times 10^6 \ \text{N/m}^2 = 5 \ \text{MN/m}^2;$$

$$\tau_{max} = \frac{Ql}{W_p} \cong \frac{9.81 \times 0.4}{0.2 \times 0.02^3} \cong 2.5 \times 10^6 \ \text{N/m}^2 = 2.5 \ \text{MN/m}^2$$

and

$$\delta = \frac{Q}{C} = \frac{9.81}{16 \times 10^3} \cong 0.613 \times 10^{-3} \ \text{m} = 0.0613 \ \text{cm}$$

The required dynamic quantities are

$$\max \sigma_d = k_d \sigma_{max} = 11.2 \times 5 = 56 \ \text{MN/m}^2;$$

$$\max \tau_d = k_d \tau_{max} = 11.2 \times 2.5 = 28 \ \text{MN/m}^2;$$

$$\delta_d = k_d \delta = 11.2 \times 0.0613 \cong 0.69 \ \text{cm}$$

Assuming the system to be without mass m_1, then

$$k_d = \sqrt{\frac{v_0^2 C}{Qg}} = \sqrt{\frac{16 \times 16 \times 10^3}{9.81 \times 9.81}} \cong 51.6;$$

$\max \sigma_d = 51.6 \times 5 = 258$ MN/m^2;

$\max \tau_d = 51.6 \times 2.5 = 129$ MN/m^2; $\quad \delta_d = 51.6 \times 0.0613 \cong 3.16$ cm

Example 135. Longitudinal impact. Let $Q = 8$ kgf, $\gamma = 8$ gf/cm^3, $l = 2$ m, $F = 4$ cm^2, $E = 2 \times 10^6$ kgf/cm^2 and $[\sigma_d] = 1000$ kgf/cm^2 (Fig. 236).

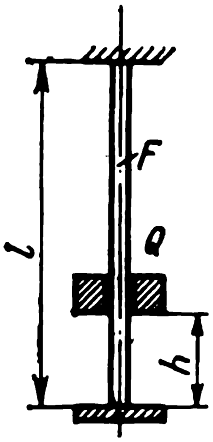


Fig. 236

Determine h: (1) taking the mass of the bar into account and (2) neglecting this mass.

Solution. Since, upon the statical effect of load Q,

$$\sigma = \frac{Q}{F} = \frac{8}{4} = 2 \text{ kgf/cm}^2$$

the dynamic factor according to formula (249) is

$$k_d = \frac{[\sigma_d]}{\sigma} = \frac{1000}{2} = 500$$

On the other hand, since the velocity of load Q at the moment of impact is $v_0 = \sqrt{2gh}$, the rigidity of the rod in tension is $C = \frac{EF}{l}$, the dead weight of the rod is $Q_0 = \gamma F l$ and the mass reduction factor, transferring the mass of the bar to the point of collision is $k_m = \frac{1}{3}$ (see Example 125), then, using formula (243), we obtain

$$k_d = 1 + \sqrt{1 + \frac{2hEF}{Ql} \times \frac{1}{1 + \frac{1}{3} \times \frac{Q_0}{Q}}}$$

$$= 1 + \sqrt{1 + \frac{2h \times 2 \times 10^6 \times 4}{8 \times 2 \times 10^2} \times \frac{1}{1 + \frac{1}{3} \times \frac{64}{8}}} = 1 + \sqrt{1 + \frac{3 \times 10^4}{3.8} h}$$

Therefore

$$500 = 1 + \sqrt{1 + \frac{3 \times 10^4}{3.8} h}$$

from which

$$h = \frac{(499^2 - 1) \times 3.8}{3 \times 10^4} \cong 31.5 \text{ cm}$$

Neglecting the mass of the bar, we have

$$k_d = 500 = \sqrt{\frac{v_0^2 C}{Qg}} = \sqrt{\frac{2hEF}{Ql}} = \sqrt{\frac{2h \times 2 \times 10^6 \times 4}{8 \times 2 \times 10^2}} = 10^2 \sqrt{h}$$

Hence $h = 25$ cm.

It follows that the calculation in which the dead weight of the bar is neglected, yields a safe height from which the load can be dropped that is reduced by

$$\frac{31.5-25}{31.5} \times 100 \cong 21\%$$

Example 136. Transverse impact. Let $h = 5$ cm, $l = 1$ m, $I = 2000$ cm^4, $W = 200$ cm^3, $Q_0 = 25$ kgf, $E = 2 \times 10^6$ kgf/cm^2 and $[\sigma_d] = 1200$ kgf/cm^2 (Fig. 237).

Determine Q: (1) taking the mass of the beam into account and (2) neglecting this mass.

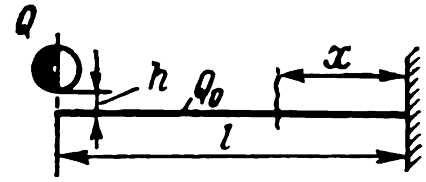


Fig. 237

Solution. Upon the statical effect of load Q the deflection of the free end of the beam is $\delta = \frac{Ql^3}{3EI}$; the deflection of an arbitrary section of the beam at a distance x from the fixed end, according to the formula of the method of initial parameters, is

$$\delta_x = \frac{1}{EI}\left(\frac{Qlx^2}{2} - \frac{Qx^3}{6}\right) = \frac{Ql^3}{3EI}\left(\frac{3}{2} \times \frac{x^2}{l^2} - \frac{1}{2} \times \frac{x^3}{l^3}\right)$$

Therefore, the mass reduction factor, transferring the mass of the beam to the point of collision, is

$$k_m = \frac{1}{l}\int_0^l \left(\frac{\delta_x}{\delta}\right)^2 dx = \frac{1}{l}\int_0^l \left(\frac{3x^2}{2l^2} - \frac{x^3}{2l^3}\right)^2 dx = \frac{33}{140}$$

The dynamic factor found from formula (243)

$$k_d = 1 + \sqrt{1 + \frac{2h3EI}{Ql^3} \times \frac{1}{1 + \frac{33}{140} \times \frac{Q_0}{Q}}}$$

$$= 1 + \sqrt{1 + \frac{2 \times 5 \times 3 \times 2 \times 10^6 \times 2 \times 10^3}{Q \times 10^6} \times \frac{1}{1 + \frac{33}{140} \times \frac{25}{Q}}}$$

$$\cong 1 + \sqrt{1 + \frac{12 \times 10^4}{Q+6}}$$

On the other hand, since, upon the statical effect of load Q, the maximum normal stress in the beam is

$$\sigma_{max} = \frac{Ql}{W} = \frac{Q \times 100}{200} = \frac{Q}{2} \text{ kgf/cm}^2$$

the dynamic factor, according to formula (249), is

$$k_d = \frac{[\sigma_d]}{\sigma_{max}} = \frac{1200 \times 2}{Q} = \frac{2400}{Q}$$

Hence

$$\frac{2400}{Q}=1+\sqrt{1+\frac{12\times 10^4}{Q+6}} \text{ or}$$

$$13Q^2-6\times 10^2Q-36\times 10^2=0$$

from which

$$Q=\frac{1}{13}(3\times 10^2\pm\sqrt{9\times 10^4+36\times 10^2})\cong\frac{3\pm 3.7}{13}10^2$$

Only one of these roots satisfies the problem. Thus

$$Q=\frac{6.7\times 10^2}{13}\cong 52 \text{ kgf}$$

Neglecting the mass of the beam we obtain

$$\frac{2400}{Q}=\sqrt{\frac{2h\times 3EI}{Ql^3}}$$

Therefore

$$Q=\frac{24^2\times 10^4\times l^3}{2h\times 3\times EI}=\frac{24^2\times 10^4\times 10^6}{2\times 5\times 3\times 2\times 10^6\times 2\times 10^3}=48 \text{ kgf}$$

Thus, if we neglect the mass of the beam, the safe weight of the load being dropped is reduced by $\frac{52-48}{52}100\cong 8\%$.

Example 137. Torsional impact. Let load $Q = 3$ kgf rotate on an arm $\rho = 8$ cm long about the horizontal x-axis at a constant angular

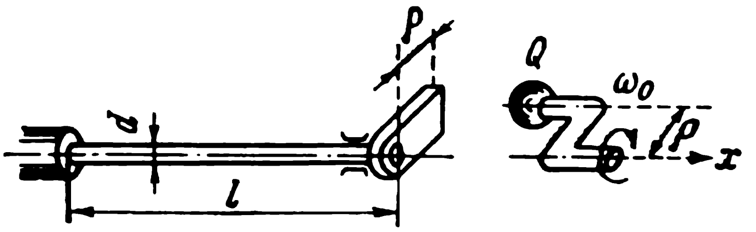


Fig. 238

velocity ω_0 and strike the lug of a cylindrical rod of length $l = 40$ cm and diameter $d = 6$ cm (Fig. 238).

Determine the permissible rotational speed n of load Q (in rpm), if the specific weight of the rod material is $\gamma = 8$ gf/cm^3, the shear modulus is $G = 8 \times 10^5$ kgf/cm^2 and the permissible shearing stress is $[\tau_d] = 400$ kgf/cm^2.

Solution. Upon the statical effect of load Q on the rod the maximum shearing stresses in the latter are

$$\tau_{max}=\frac{Q\rho}{W_p}\cong\frac{Q\rho}{0.2d^3}=\frac{3\times 8}{0.2\times 6^3}=\frac{5}{9} \text{ kgf/cm}^2$$

The dynamic factor is

$$k_d=\frac{[\tau_d]}{\tau_{max}}=\frac{400\times 9}{5}=720$$

Since the moment of inertia of the mass of load Q about the axis of rotation is

$$I_m=\frac{Q}{g}\rho^2=\frac{3\times 64}{981}\cong 0.2 \text{ kgf-cm-sec}^2$$

the moment of inertia of the mass of the rod is

$$I_{m_0}=\frac{\pi}{32}\times\frac{\gamma}{g}d^4l\cong 0.1\,\frac{8\times 10^{-3}}{981}\times 6^4\times 40\cong 0.042 \text{ kgf-cm-sec}^2$$

the mass reduction factor of the rod (see Example 127) is $k_m=\frac{1}{3}$, the kinetic energy of the rotary motion of the load is

$$T=\frac{I_m\omega_0^2}{2}\cong 0.1\omega_0^2 \text{ kgf-cm}$$

and the elastic strain energy of the rod in torsion upon the statical effect of the load is

$$U=\frac{Q^2\rho^2l}{2GI_p}\cong\frac{9\times 64\times 40}{2\times 8\times 10^5\times 0.1\times 6^4}=\frac{1}{9\times 10^3} \text{ kgf-cm}$$

then the dynamic factor according to formula (243) is

$$k_d=1+\sqrt{1+\frac{T}{U}\times\frac{1}{1+k_m\frac{I_{m_0}}{I_m}}}$$

$$=1+\sqrt{1+\frac{0.1\times\omega_0^2\times 9\times 10^3}{1+\frac{0.042}{3\times 0.2}}}\cong 1+\sqrt{1+841\omega_0^2}$$

Comparing the obtained dynamic factors we have

$719=\sqrt{1+841\omega_0^2}$ from which

$$\omega_0=\frac{\pi n}{30}\cong\sqrt{\frac{719^2}{841}}\cong 25 \text{ and } n=\frac{30}{\pi}\times 25\cong 240 \text{ rpm}$$

Neglecting the mass of the rod, we obtain

$$k_d=720=\sqrt{\frac{T}{U}}=30\omega_0$$

Therefore

$$\omega_0=\frac{\pi n}{30}=\frac{720}{30}=24 \text{ and } n=\frac{30\times 24}{\pi}\cong 230 \text{ rpm}$$

Thus, if the mass of the rod is neglected, the permissible speed of rotation of the incident load is reduced by $\frac{240-230}{240}\times 100\cong 4\%$.

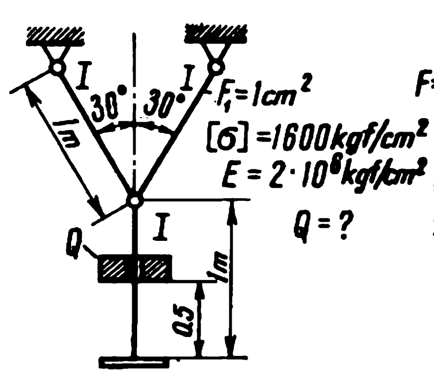
Problems 984 through 1003. Determine the quantities indicated in the problems for elastic systems subject to impacts.

The dead mass of the elements of the system can be neglected. In problems which are to be solved in the general form the dynamic factor is to be found from the approximate formula

$$k_d = \sqrt{\frac{v_0^2}{g\delta}}$$

In the frame (girder) systems and in curved beams take only the bending strains into consideration.

984



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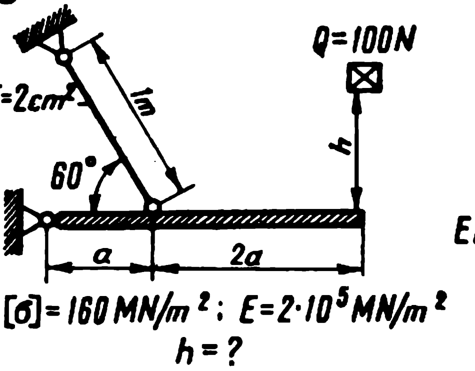

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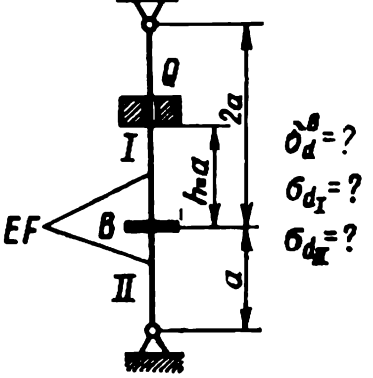

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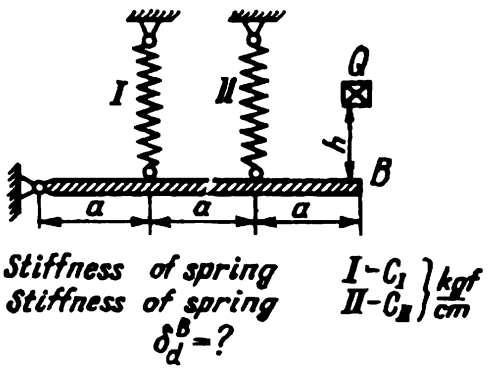

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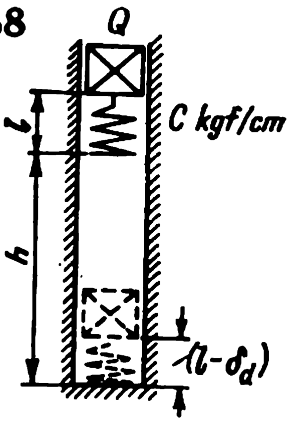

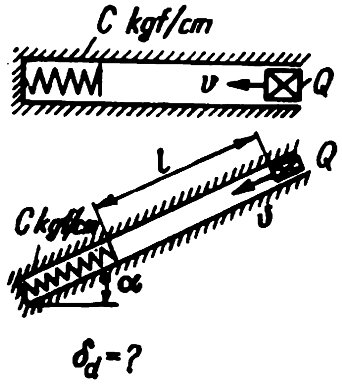

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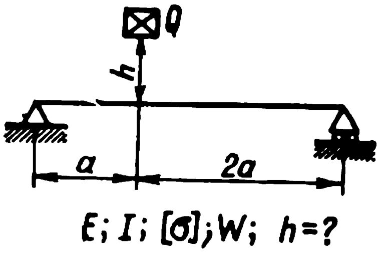

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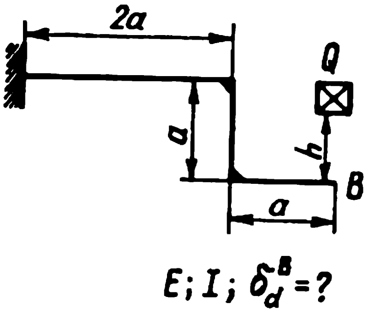

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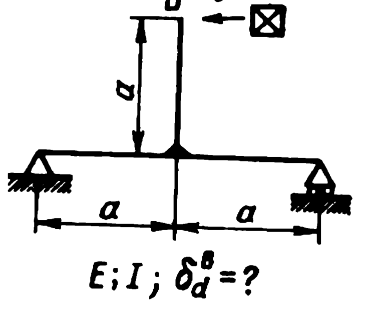

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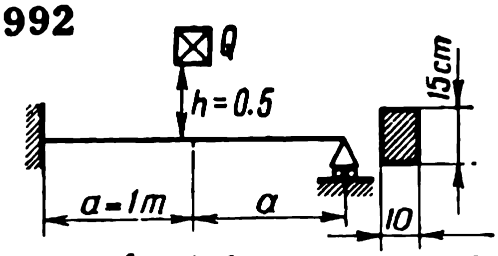
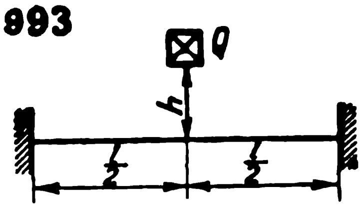


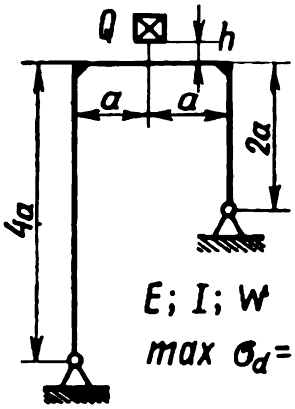
$E=2\cdot 10^6\,kgf/cm^2$; $[\sigma]=1600\,kgf/cm^2$; $Q=?$

993



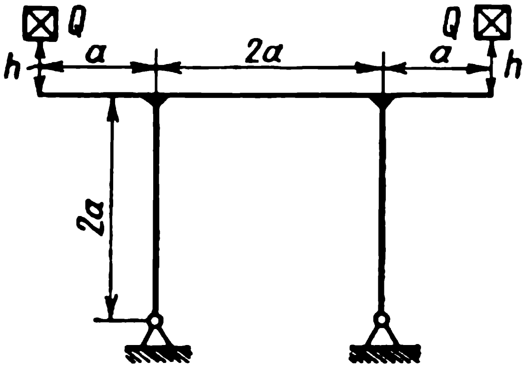
E; I; W; $[\sigma]$ $h=?$

994



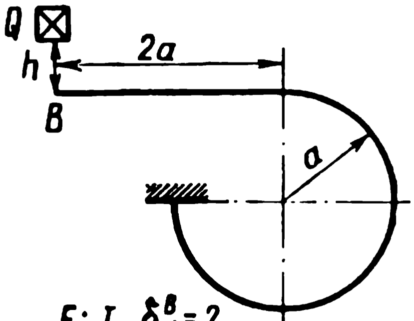
E; I; W
$max\ \sigma_d=?$

995



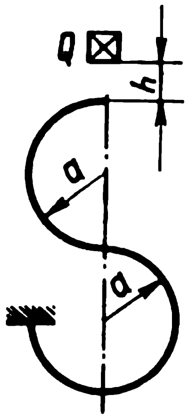
E; I; W $max\ \sigma_d=?$

996



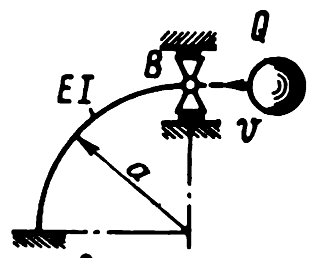
E; I $\delta_d^B=?$

997



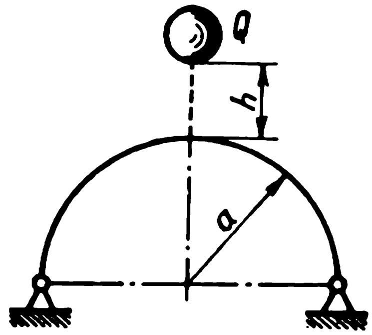
E; I; W $max\ \sigma_d=?$

998



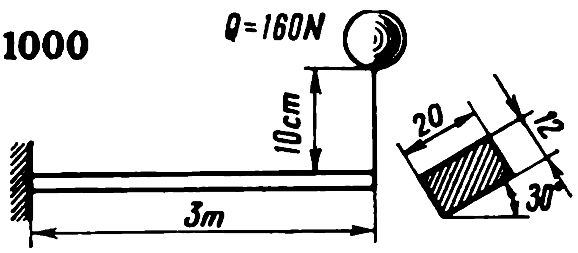
$\delta_B=?$

999



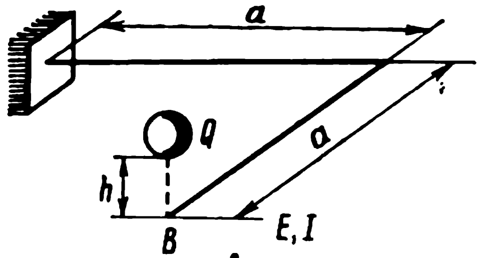
E; I; W $max\ \sigma_d=?$

1000



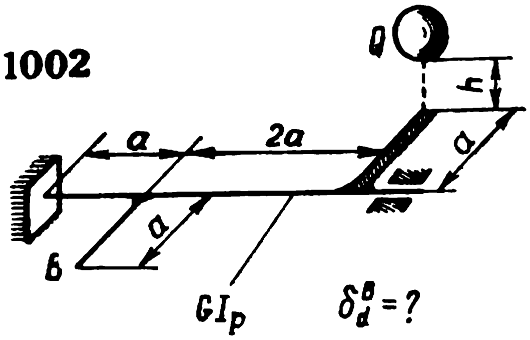
$max=\sigma_d$? $E=10^4\,MN/m^2$

1001



$\delta_d^B=?$

1002



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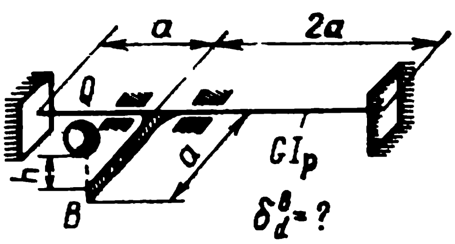
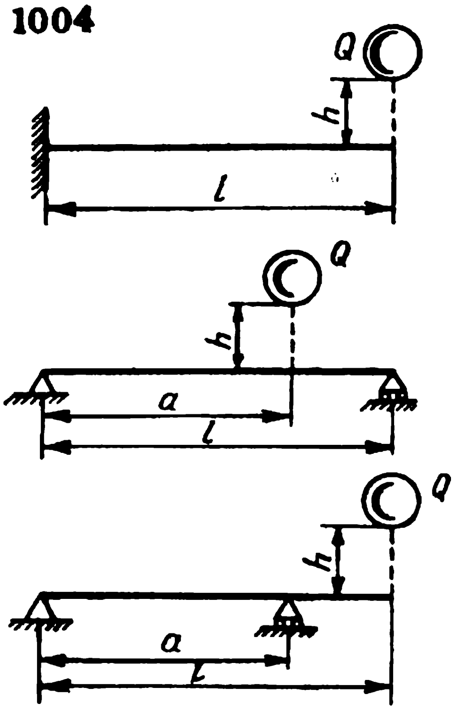


Problem 1004. Prove that for the given beams the maximum dynamic stresses (neglecting the mass of the beams) do not depend on the method of support and on quantity a.

Assume that $h \gg \delta$.

Problem 1005. Find a at which max σ_d is a minimum. The mass of the beams is to be neglected; assume that $h \gg \delta$.

1004



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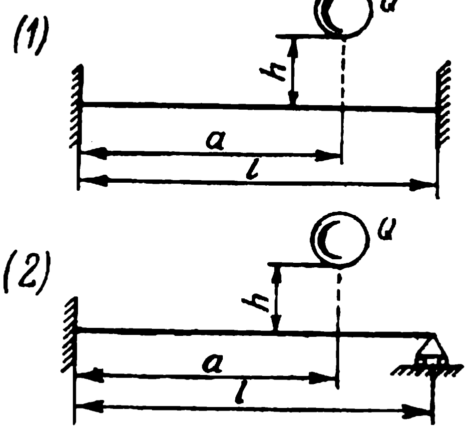
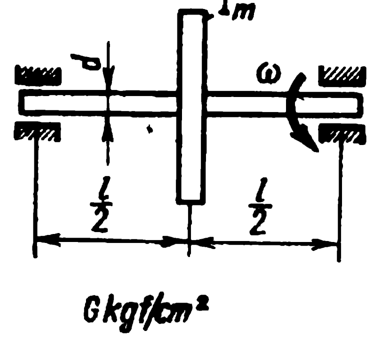


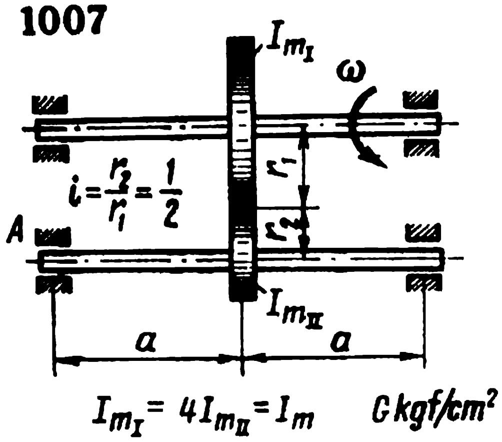
Problem 1006. Find max τ_d for a shaft turning at an angular velocity ω and carrying a flywheel with a moment of inertia of mass I_m in case of abrupt seizure of the end:

(a) in one bearing; (b) in both bearings simultaneously.

1006



1007

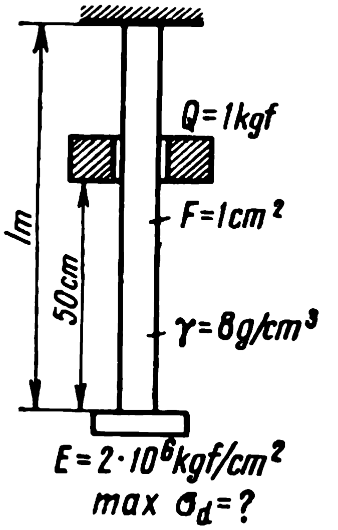


Problem 1007. Determine max τ_d in the shaft of the transmission upon abrupt braking of the shaft end in bearing A.

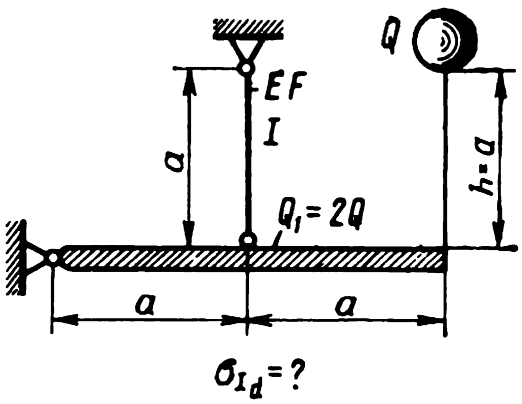
Problems 1008 through 1011. Determine the quantities indicated in the problems for systems subject to impacts.

Take into consideration the dead weight of the elements for which it is specified. In the calculations take into account the remarks made for Problems 984 through 1003.

1008



1009



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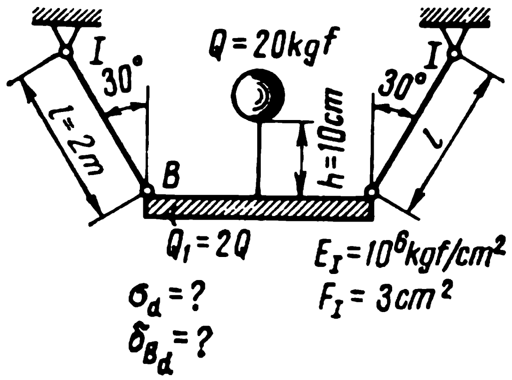
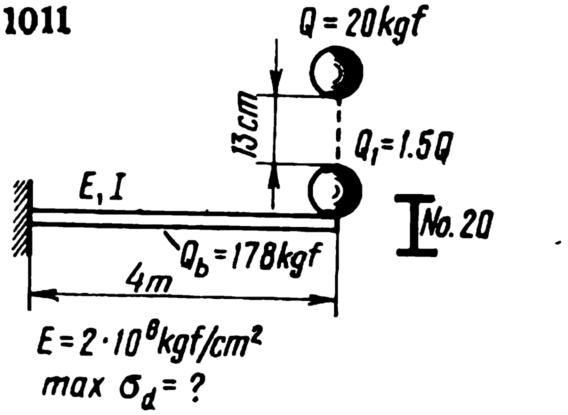
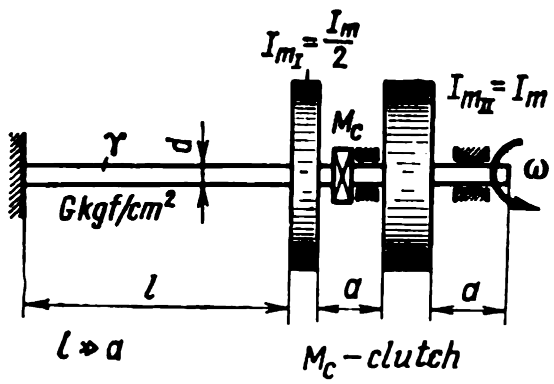


1011

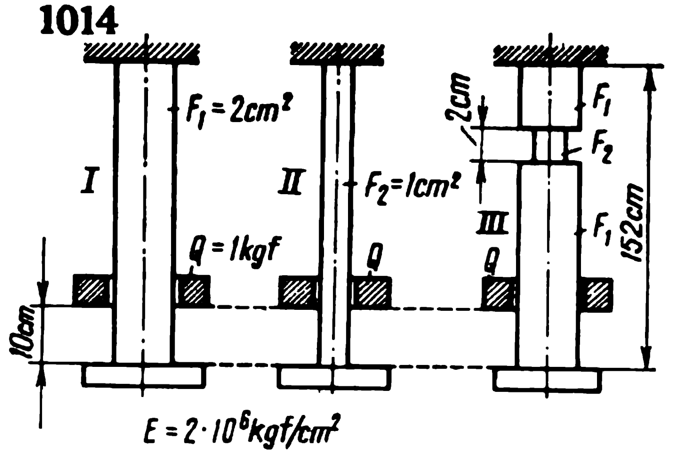


Problem 1012. Solve Problem 993, taking the mass of the beam (Q_0) into consideration.

1013



1014



Problem 1013. Determine max τ_d in the shaft due to a torsional impact caused by the engagement of the clutch M_c. The clutch engages the rotating system of the right-hand part of the shaft to the stationary left-hand part.

Problem 1014. Determine and compare σ_{d_I}, $\sigma_{d_{II}}$, and $\sigma_{d_{III}}$ in rods *I*, *II* and *III* due to the impact of the same load Q dropped from the same height $h = 10$ cm.

CHAPTER 15. ALTERNATING STRESSES

This chapter deals with stresses that vary periodically over a long period.

A set of the consecutive magnitudes of alternating stresses occurring during one period in the process of their variation is called a *stress cycle.*

In the case of systematic action of alternating stresses on a body, cracks may develop at the places of maximum stress concentration leading to brittle failure. The process of the formation and development of cracks in the material due to the action of alternating stresses is called *fatigue of the material.*

The resistance of materials to alternating stresses is called the *fatigue strength*. The maximum periodically varying stress which the material is able to resist for an unlimited period of time is called the *fatigue* (*endurance*) *limit.*

Commonly a conventional finite fatigue limit is determined, based on a finite number of stress alternations, for example, (5 to 10) $\times$ 10^6 cycles for ferrous metals, (50 to 100) $\times$ 10^6 cycles for nonferrous metals, etc.

A cycle is characterized by its asymmetry factor r, which is the algebraic ratio

$$r = \frac{p_{min}}{p_{max}} \tag{250}$$

in which p_{max} (σ_{max} and τ_{max}) and p_{min} (σ_{min} and τ_{min}) are the maximum and minimum stresses in the cycle.

The quantity

$$p_m = \frac{p_{max} + p_{min}}{2} \tag{251}$$

is the mean stress of the cycle and

$$p_a = \frac{p_{max} - p_{min}}{2} \tag{252}$$

the amplitude of the cycle.

The fatigue limit is

$$p_r = p^r_{max} = p^r_m + p^r_a \tag{253}$$

in which p^r_{max}, p^r_m, p^r_a are the maximum stress, mean stress and the amplitude of the cycle for the material at the fatigue limit.

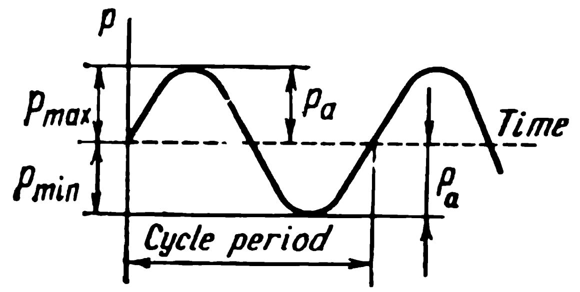


Fig. 239

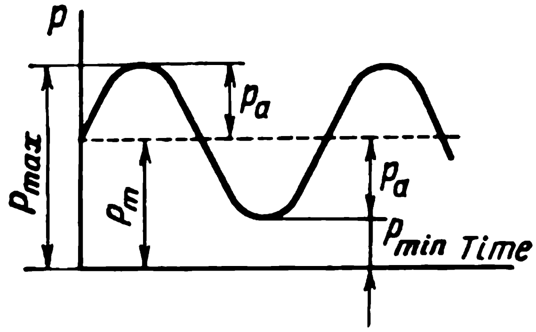


Fig. 240

In the case of a symmetrical cycle (Fig. 239)

$$p_{max} = -p_{min}; \quad p_m = 0; \quad r = -1; \quad p_r = p_{-1}$$

In the case of a constant-sign positive asymmetrical cycle (Fig. 240)

$$p_{max} > 0; \quad p_{min} > 0; \quad 0 < r < 1$$

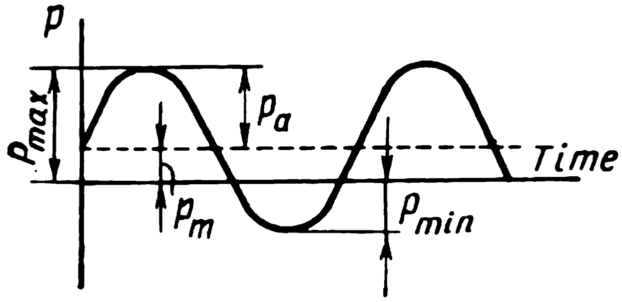


Fig. 241

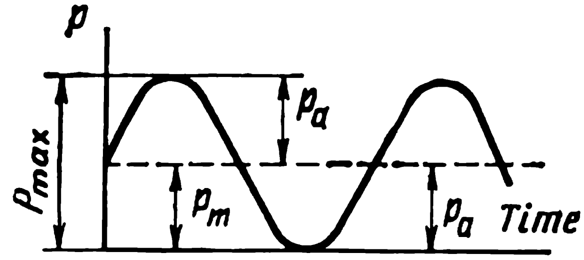


Fig. 242

In the case of an alternating asymmetrical cycle (Fig. 241)

$$p_{max} > 0; \quad p_{min} < 0; \quad -1 < r < 0$$

In the case of a positive pulsating (starting from zero) cycle (Fig. 242)

$$p_{max} > 0; \; p_{min} = 0; \; p_m = p_a = \frac{p_{max}}{2}; \; r = 0; \; p_r = p_0$$

15.1.

Basic Factors Affecting the Fatigue Strength *

CHARACTER OF THE CYCLE AND THE KIND OF DEFORMATION (STRAIN)

In the case of a symmetric cycle, the fatigue limit of a material (all other conditions being equal) is at its lowest value. The fatigue limit in the case of a symmetric cycle p_{-1} (σ_{-1} or τ_{-1}) is determined by a

* The scope of this section is restricted to information required for solving the problems contained in this manual.

curve (Fig. 243) which is plotted on the basis of experimental data, proceeding as follows. We draw a tangent to the right-hand end of the curve which approaches a horizontal line. The ordinate intersected by this tangent is the conventional fatigue limit.

The following approximate relations have been established between fatigue limits in symmetrical bending σ_{-1}, symmetrical axial tension-compression σ_{-1p}, symmetrical torsion τ_{-1} and the ultimate strength σ_u:

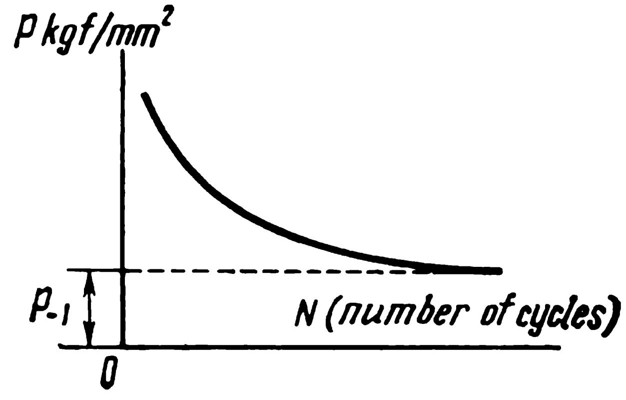


Fig. 243

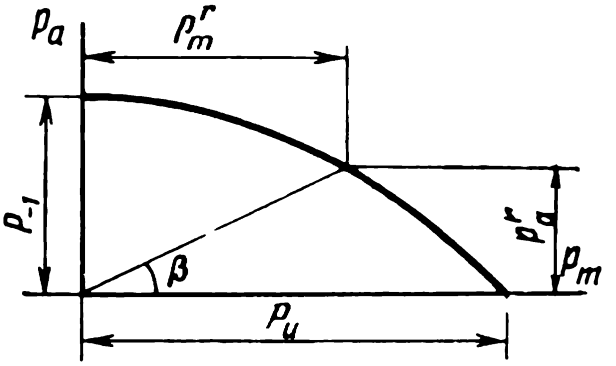


Fig. 244

for steel

$$\sigma_{-1} \cong (0.4 \text{ to } 0.6)\ \sigma_u; \quad \sigma_{-1p} \cong (0.7 \text{ to } 0.8)\ \sigma_1;$$

$$\tau_{-1} \cong (0.4 \text{ to } 0.7)\ \sigma_{-1}$$

for cast iron

$$\sigma_{-1} \cong (0.4 \text{ to } 0.5)\ \sigma_u; \quad \tau_{-1} \cong (0.7 \text{ to } 0.9)\ \sigma_{-1}$$

for nonferrous metals

$$\sigma_{-1} \cong (0.25 \text{ to } 0.5)\ \sigma_u$$

In an asymmetrical cycle the fatigue limit can be determined from an experimental curve of limiting amplitudes plotted in the coordinates p_m vs p_a (Fig. 244).

In the graph p_u denotes the ultimate strength of the material for the given kind of strain.

At the given asymmetry factor of the cycle $r = \frac{p_{min}}{p_{max}}$, the value of $\tan\beta = \frac{p_a}{p_m} = \frac{1-r}{1+r}$ and of angle β should be determined.

Then a straight line is drawn from the origin of coordinates at an angle β to the axis p_m. Adding the coordinates p_m^r and p_a^r of the point of intersection of the straight line and the curve, we find the fatigue limit

$$p_r = p_m^r + p_a^r$$

Example 138. Suppose steel, grade 40 (USSR Std), in the normalized state is to be tested. The recorded data obtained during the fatigue test in bending (symmetric cycle) give:

σ, MN/m²	300;	290;	280;	270;	260;	250;	240;	235;	230;	230
$N\frac{1}{10^6}$	0.52;	0.71;	1.21;	2.32;	3.44;	4.82;	5.85;	8.51;	>10;	>10

the last two specimens did not fail.

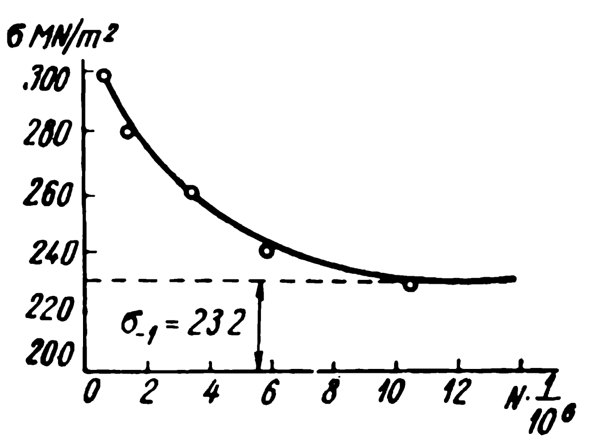


Fig. 245

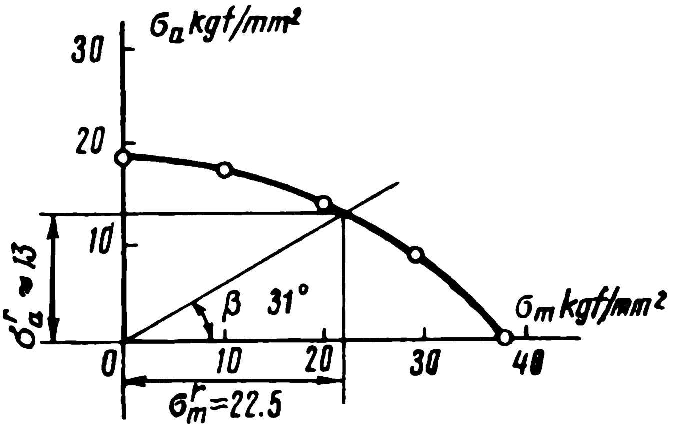


Fig. 246

Determine σ_{-1}.

Solution. Using the experimental data we first plot the endurance curve in the coordinates σ, MN/m² vs $\frac{N}{10^6}$ (N = number of cycles) (Fig. 245).

Then we draw a tangent to the right-hand end portion of the curve and determine the approximate fatigue (endurance) limit $\sigma_{-1} \cong 232$ MN/m² on the axis of ordinates.

Example 139. The experimental data for steel, grade Ст. 3, are: $\sigma_u = 38.8$ kgf/mm², $\sigma_{-1} = 18.5$ kgf/mm² and the limiting amplitudes σ_a^r of the cycle at the given mean stresses σ_m^r are

σ_m^r,	kgf/mm²	10	20	30
σ_a^r,	kgf/mm²	17.5	14	8.5

Determine $\sigma_{0.25}$.

Solution. Using the experimental data, we plot the curve of the limiting amplitudes in the coordinates σ_m vs σ_a (Fig. 246).

Since the fatigue limit is to be determined for an asymmetry factor of the cycle equal to $r = 0.25$, then

$$\tan\beta = \frac{1-0.25}{1+0.25} = 0.6 \text{ and } \beta \cong 31^\circ$$

Next we draw a straight line from the origin of coordinates at an angle $\beta = 31^\circ$ to the axis σ_m. The coordinates of the point of inter-

section of this straight line with the curve of the limiting amplitudes of stress are

$$\sigma_m^{0.25} \cong 22.5 \text{ kgf/mm}^2 \text{ and } \sigma_a^{0.25} \cong 13 \text{ kgf/mm}^2$$

The fatigue limit will then be

$$\sigma_{0.25} = \sigma_m^{0.25} + \sigma_a^{0.25} \cong 22.5 + 13 = 35.5 \text{ kgf/mm}^2$$

STRESS CONCENTRATION

Stress concentration is a local increase of stresses near the stress raiser (i.e. sharp local changes in the shape of the body in the form of holes, necks, notches, fillets, etc.).

The ratio of the maximum local stress p_l (σ_l or τ_l) to the nominal stress p (σ or τ) in the section weakened by the stress raiser (without taking the concentration into account), established in statical loading and based on the assumption of an ideal homogeneity, isotropy and elasticity of the material, is called the *theoretical stress concentration factor*:

$$\alpha = \frac{p_l}{p}; \quad \left(\alpha_\sigma = \frac{\sigma_l}{\sigma} \text{ and } \alpha_\tau = \frac{\tau_l}{\tau}\right) \tag{254}$$

Factor $\alpha > 1$ represents only the effect of the geometry of the stress raiser on the maximum local stress.

For alternating stresses the concept of *effective stress concentration factor* is introduced:

$$\alpha_{eff} = \frac{p_{-1}}{p_{-1}^c} \left(\alpha_{eff}^\sigma = \frac{\sigma_{-1}}{\sigma_{-1}^c} \text{ and } \alpha_{eff}^\tau = \frac{\tau_{-1}}{\tau_{-1}^c}\right) \tag{255}$$

which is the ratio of the fatigue limit p_{-1} of a smooth specimen in a symmetrical cycle to the fatigue limit p_{-1}^c of a similar specimen but with the stress raiser.

Factor α_{eff} represents the effect of both the geometry of the stress raiser and the material of the specimen on the fatigue limit.

The ratio

$$q = \frac{\alpha_{eff} - 1}{\alpha - 1} \tag{256}$$

is called the *sensitivity factor of the material to stress concentration*. It varies in the range $0 \leqslant q \leqslant 1$.

For cast iron $q = 0$, for structural steels $q = 0.6$ to 0.8 (the lower values refer to medium strength steels and the higher ones, to high-strength steels); for steels with

$$\sigma_u \geqslant 130 \text{ kgf/mm}^2; \quad q \cong 1$$

(Figure 1 of Appendix 5 is a graph of approximate values of q for steels, depending on σ_u and α_σ without taking the size of the body into account.)

Since q also depends on the shape and size of the body, values of α_{eff} established in full-scale experiments are more reliable for practical applications.

SIZE EFFECT FACTOR

The *size effect* is the influence of the absolute physical size of a body on the fatigue (endurance) limit.

The reduction of the fatique limit with an increase in the absolute physical size of a body is assessed by the *size effect factor* $\varepsilon_{se} < 1$ which is the ratio of the fatigue limit of the specimen of the given diameter D to the fatigue limit of a standard specimen of diameter d:

$$\varepsilon_{se} = \frac{(p_{-1})_D}{(p_{-1})_d} \tag{257}$$

(Figure 2 of Appendix 5 is a graph of the values ε_{se} versus D for carbon and alloyed steels with various surface finishes. This graph can be used for the approximate determination of ε_{se} both in bending and in torsion.)

If the effective stress concentration factor has been taken from a graph which takes the size effect factor into account, no correction need be made for the dimensions of the body.

SURFACE FINISH

The effect of the state of the surface quality of a body on the fatigue limit is taken into account by the *surface quality factor* $\varepsilon_{sf} < 1$ which is a ratio of the fatigue limit of a specimen with the given state of surface quality (p_{-1}) to the fatigue limit of a similar specimen with a polished surface p_{-1}.

$$\varepsilon_{sf} = \frac{(p_{-1})}{p_{-1}} \tag{258}$$

(Figure 3 of Appendix 5 is a graph of the values of ε_{sf} versus σ_u for various states of surface finish of the specimen.)

The strengthening effect of work hardening of the surface, surface hardening, casehardening, nitriding and other processing factors is taken into account by the coefficient β, taken from reference books and used as a multiplier for the coefficient ε_{sf}.

15.2.

Strength Analysis For a Uniaxial Stressed State and Pure Shear (Torsion)

It is assumed that the nature of the stressed state at the point being examined is not changed with a change in load and the cycles of stress alternation remain similar (r = const).

In a *symmetrical cycle* of alternating stresses the factor of safety is established on the basis of the fatigue limit of the part. The effect of the basic factors (stress concentration, size effect factor, and surface finish) on the endurance of a part can be taken into account by a single overall factor

$$\varepsilon = \frac{\varepsilon_{se}\beta\varepsilon_{sf}}{\alpha_{eff}} \tag{259}$$

Therefore the factor of safety of a part in a symmetrical cycle of operation can be assessed by using the equation

$$n = \frac{\varepsilon p_{-1}}{p_{max}} \tag{260}$$

in which p_{max} is the maximum (nominal) stress in the part.

The strength condition will be of the form

$$p_{max} \leqslant [p_{-1}] \tag{261}$$

and the allowable stress

$$[p_{-1}] = \frac{\varepsilon p_{-1}}{[n]} \tag{262}$$

in which $[n]$ is the allowable (recommended) factor of safety.

In the case of an *asymmetric cycle* of variable stresses with a given asymmetry factor r (or $\tan\beta = \frac{1-r}{1+r}$) the factor of safety in fatigue can be found by the use of the simplified limiting amplitude diagram plotted on the basis of the fatigue limit p_{-1} in a symmetrical cycle and the ultimate strength p_u in the case of a static tension (Fig. 247).

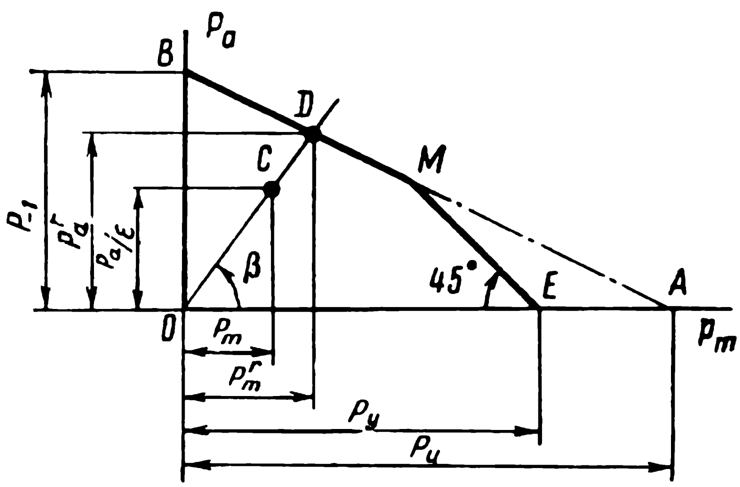


Fig. 247

If the working cycle is characterized on this diagram by point C and the similar limiting cycle, by point D, the factor of safety will be expressed graphically by the ratio $n = \frac{OD}{OC}$ or analytically as

$$n = \frac{p_{-1}}{\frac{p_{-1}}{p_u} p_m + \frac{p_a}{\varepsilon}} \tag{263}$$

In the case of alternating stresses of small amplitude, it may turn out that the limiting state of a ductile material will be determined

not by fatigue but by the yield (line EM), and then the factor of safety is to be determined from the yield point

$$n_y = \frac{p_y}{p_{max}} = \frac{p_y}{p_m + p_a} \tag{264}$$

In checking the strength it is advisable to compare the factors of safety calculated by formulas (263) and (264) and to take the lower value. The accepted factor of safety must be not less than the allowable value. From formula (263), equating $n = [n]$, $p_m = [p_m]$ and $p_a = [p_a]$, the allowable stress in an asymmetric cycle $[p_r]$ is

$$[p_r] = \frac{2[p_{-1}][p]}{(1-r)[p] + (1+r)[p_{-1}]\dfrac{p_y}{p_u}} \tag{265}$$

in which $[p]$ ($[\sigma]$ or $[\tau]$) is the permissible stress in static loading.

The strength condition then becomes

$$p_{max} = p_m + p_a \leqslant [p_r] \tag{266}$$

For a still greater simplification of calculations, but with an increased factor of safety, use can be made of the straight-line diagram of allowable stresses proposed by Soderberg (Fig. 248).

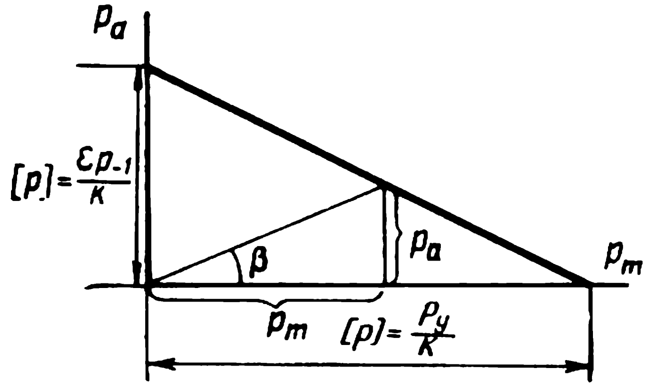


Fig. 248

According to this diagram the factor of safety is

$$n = \frac{p_{-1}}{\dfrac{p_{-1}}{p_y}p_m + \dfrac{p_a}{\varepsilon}} \tag{267}$$

and the allowable stress is

$$[p_r] = \frac{2[p][p_{-1}]}{(1-r)[p] + (1+r)[p_{-1}]} \tag{268}$$

Example 140. A round bar of diameter d=40 mm has a smooth ground surface and is made of steel, grade Ст. 4, for which $\sigma_u =$ $= 480$ MN/m² and $\sigma_{-1} = 200$ MN/m².

Determine the factor of safety n for the bar when it is subjected to conditions of alternating bending in a symmetrical cycle, if $M_{max} =$ $= -M_{min} = 640$ N-m.

Solution. The maximum and minimum stresses in the cycle are

$$\sigma_{max} = -\sigma_{min} = \frac{M_{max}}{W} \cong \frac{640}{0.1 \times 4^3 \times 10^{-6}} = 10^8 \text{ N/m}^2 = 100 \text{ MN/m}^2$$

From the graph of Fig. 2 (Appendix 5) we find that for a round bar 40 mm in diameter, made of carbon steel and having a smooth surface the size effect factor $\varepsilon_{se} = 0.86$.

According to formula (260) the factor of safety is

$$n=\frac{\varepsilon_{se}\sigma_{-1}}{\sigma_{max}}\cong\frac{0.86\times 200}{100}\cong 1.7$$

Example 141. A round stepped bar of diameter $D=80$ mm and $d=40$ mm (Fig. 249) is made of steel grade 40X, for which $\sigma_u = 1000$ MN/m^2 and $\sigma_{-1p}=250$ MN/m^2. At the fillet $\frac{\rho}{d}=0.2$. The bar surface is well polished.

Determine the maximum alternating axial force P_{max}, varying in a symmetrical cycle, at which the factor of safety $[n]=1.8$.

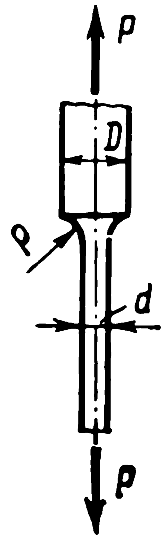


Fig. 249

Solution. From the graph of Fig. 4 (Appendix 5) at $\frac{\rho}{d}=0.2$ for steel with $\sigma_u=1000$ MN/m^2 by linear interpolation we find the effective stress concentration factor $\alpha_{eff}=1.7$. For a well polished part $\varepsilon_{sf}=1$.

The permissible stress is

$$[\sigma_{-1p}]=\frac{\sigma_{-1p}}{[n]\alpha_{eff}}=\frac{250}{1.8\times 1.7}\cong 82\ \text{MN/m}^2$$

The minimum cross-sectional area of the bar is

$$F=\frac{\pi d^2}{4}=\pi\times 4\cong 12.6\ \text{cm}^2$$

The maximum alternating axial force is

$$P_{max}=[\sigma_{-1p}]\,F=82\times 10^6\times 12.6\times 10^{-4}=103{,}000\ \text{N}=103\ \text{KN}$$

Example 142. A round shaft of diameter $d=50$ mm is made of carbon steel with $\sigma_u=60$ kgf/mm^2 and has a well polished surface. It has a round through transverse hole of diameter $a=10$ mm and is subjected to alternating torsion in a symmetrical cycle with $\max M_t = -\min M_t = 90$ kgf-m (Fig. 250).

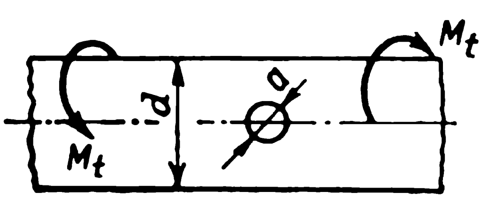


Fig. 250

Determine the factor of safety of the shaft.

Solution. Let us assume approximately that $\sigma_{-1}=0.4\sigma_u$ and

$$\tau_{-1}=0.5\sigma_{-1}=0.2\times 60=12\ \text{kgf/mm}^2$$

From the graph of Fig. 5 (Appendix 5) for steel with $\sigma_u=60$ kgf/mm^2, the effective stress concentration factor, taking into account the absolute physical size of the shaft, $\alpha_{eff}=1.77$.

The section modulus in torsion of a round shaft weakened by the hole with the ratio $\frac{a}{d} = 0.2$, in accordance with reference data, is

$$W_t = \frac{\pi d^3}{16} 0.84 \cong 20.6 \text{ cm}^3$$

Due to the action of the torque max $M_t = 90$ kgf-m

$$\tau_{\max} = \frac{\max M_t}{W_t} = \frac{90 \times 10^2}{20.6} \cong 437 \text{ kgf/cm}^2$$

The required factor of safety is

$$n = \frac{\tau_{-1}}{\alpha_{eff}\tau_{\max}} = \frac{12 \times 10^2}{1.77 \times 437} \cong 1.55$$

Example 143. Determine the allowable stress in alternating bending in a cycle characterized by $r = -0.6$ for a part of alloyed structural steel with $\sigma_u = 100$ kgf/mm² and $\sigma_m = 80$ kgf/mm², the factor of safety being $[n] = 2$. The part is of round cross section $d = 40$ mm in diameter and has a stress raiser for which the theoretical stress concentration factor is $\alpha_\sigma = 1.6$. The strengthening factor due to surface work hardening is $\beta = 1.4$.

Solution. We assume that $\sigma_{-1} = 0.4\sigma_u = 0.4 \times 100 = 40$ kgf/cm². Assuming that the sensitivity factor of the material to stress concentration is equal to $q = 0.8$, we calculate the effective stress concentration factor by formula (256).

$$\alpha_{eff} = 1 + q(\alpha_\sigma - 1) = 1 + 0.8\ (1.6 - 1) = 1.48$$

The size effect factor ε_{se} can be found by curve *5* for alloyed steel with moderate stress concentration from the graph of Fig. 2 (Appendix 5), i.e. $\varepsilon_{se} \cong 0.65$.

The allowable stress for a constant load is

$$[\sigma] = \frac{\sigma_y}{[n]} = \frac{80}{2} = 40 \text{ kgf/mm}^2$$

According to formula (259) the overall factor for the effect of various factors on fatigue in a symmetrical cycle is

$$\varepsilon = \frac{\varepsilon_{se}\beta\varepsilon_{sf}}{\alpha_{eff}} \cong \frac{0.65 \times 1.4 \times 1}{1.48} \cong 0.615$$

According to formula (262) the allowable stress in a symmetrical cycle is

$$[\sigma_{-1}] = \frac{\varepsilon\sigma_{-1}}{[n]} \cong \frac{0.615 \times 40}{2} = 12.3 \text{ kgf/mm}^2$$

According to formula (265) the allowable stress in an asymmetrical cycle (asymmetry factor $r = -0.6$) is

$$[\sigma_{-0.6}] = \frac{2[\sigma][\sigma_{-1}]}{(1-r)[\sigma]+(1+r)[\sigma_{-1}]\frac{\sigma_y}{\sigma_u}}$$

$$= \frac{2\times 40\times 12.3}{(1+0.6)\,40+(1-0.6)\,12.3\times 0.8} \cong 14.5 \text{ kgf/mm}^2$$

Example 144. Check the strength of a stepped round shaft of diameters $D = 60$ mm and $d = 30$ mm (Fig. 251) made of carbon steel, grade 45, with $\sigma_u = 70$ kgf/mm², $\tau_y = 22$ kgf/mm², $\tau_{-1} = 16$ kgf/mm², and a factor of safety $[n] = 1.6$. At the fillet $\frac{\rho}{d} = 0.1$ and the strengthening factor due to shot blasting is $\beta = 1.1$. The shaft is subjected to alternating torsion at max $M_t = 48$ kgf-m, min $M_t = -24$ kgf-m.

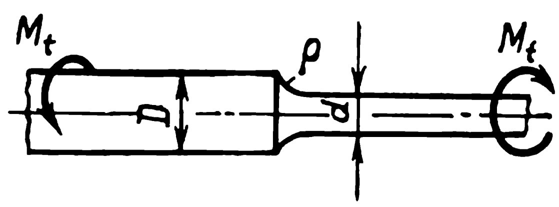


Fig. 251

Solution. The allowable shearing stress in constant torsion

$$[\tau] = \frac{\tau_y}{[n]} = \frac{22}{1.6} \cong 13.8 \text{ kgf/mm}^2$$

From the graph of Fig. 6 (Appendix 5) by linear interpolation at $\frac{\rho}{d} = 0.1$ for steel with $\sigma_u = 70$ kgf/mm² we find the effective stress concentration factor to be $\alpha_{eff}^{\tau} \cong 1.28$.

The overall factor for the effect of all the factors on fatigue is

$$\varepsilon = \frac{\beta}{\alpha_{eff}^{\tau}} \cong \frac{1.1}{1.28} \cong 0.86$$

The allowable stress in a symmetrical cycle is

$$[\tau_{-1}] = \frac{\varepsilon\tau_{-1}}{[n]} \cong \frac{0.86\times 16}{1.6} = 8.6 \text{ kgf/mm}^2$$

The asymmetry factor of the given cycle is

$$r = \frac{\min M_t}{\max M_t} = -\frac{24}{48} = -0.5$$

The allowable stress is found by formula (268) at $r = -0.5$. Thus

$$[\tau_{-0.5}] = \frac{2[\tau][\tau_{-1}]}{(1-r)[\tau]+(1+r)[\tau_{-1}]}$$

$$\cong \frac{2\times 13.8\times 8.6}{(1+0.5)\,13.8+(1-0.5)\times 8.6} \cong 9.5 \text{ kgf/mm}^2$$

$$\cong 950 \text{kgf/cm}^2$$

The maximum working stress due to the action of max M_t is

$$\tau_{\max} = \frac{\max M_t}{W_p} = \frac{48 \times 10^2}{\frac{\pi \times 3^3}{16}} \cong 906 \text{ kgf/cm}^2$$

Thus the shaft is of the required strength.

15.3.

Strength Analysis For a Complex Stressed State

We shall consider only a planar stressed state shown in Fig. 252 for a ductile material.

The strength condition is written in accordance with either the third or fourth strength theory in the following elliptic form:

$$\frac{\sigma_{\max}^2}{[\sigma_r]^2} + \frac{\tau_{\max}^2}{[\tau_r]^2} \leqslant 1 \qquad (269)$$

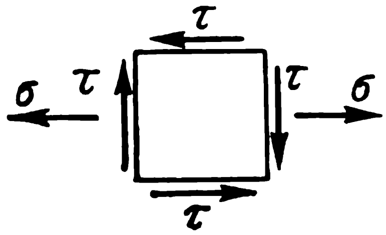


Fig. 252

Here $\sigma_{\max} = \sigma_m + \sigma_a$; $\tau_{\max} = \tau_m + \tau_a$; and $[\sigma_r]$ and $[\tau_r]$ are the allowable normal and shearing stresses found by formula (265) or (268) in accordance with the characteristics of the cycles of variation of normal and shearing stresses.

The overall factor of safety n of a part, subject to the combined action of variables σ and τ, is determined from the equation

$$n = \frac{n_\sigma n_\tau}{\sqrt{n_\sigma^2 + n_\tau^2}} \qquad (270)$$

in which n_σ and n_τ are partial factors of safety due to the action on the part of either variables σ or variables τ alone. These partial factors of safety are found by formulas (263) or (267).

Example 145. A steel shaft (grade Ст. 5) is subjected to periodically and synchronously varying bending moments $M_{\max} = 400$ N-m, $M_{\min} = -160$ N-m and torques $\max M_t = 640$ N-m, $\min M_t = 320$ N-m. The diameters of the stepped shaft are $D = 50$ mm and $d = 40$ mm, and the fillet radius $\rho = 0.25$ cm. For steel, grade Ст. 5, $\sigma_u = 540$ MN/m², $\sigma_y = 280$ MN/m², $\tau_u = 350$ MN/m², $\tau_y = 180$ MN/m², $\sigma_{-1} = 240$ MN/m² and $\tau_{-1} = 140$ MN/m².

Determine the actual factor of safety n for the shaft under these conditions.

Solution. By linear interpolation we find from the graphs of Figs. 6 and 7 (Appendix 5) the stress concentration factors at $\frac{\rho}{d} = \frac{0.25}{4} = 0.0625$ for $\sigma_u = 540$ MN/m².

$$\alpha^{\sigma}_{eff_0} \cong 1.81 \text{ and } \alpha^{\tau}_{eff_0} \cong 1.42$$

Since in this case $\frac{D}{d} = \frac{5}{4} = 1.25 < 2$, corrections should be made in $\alpha^{\sigma}_{eff_0}$ and $\alpha^{\tau}_{eff_0}$.

We find the necessary correction factors from the graph of Fig. 8 (Appendix 5). They are $\xi_{\sigma} = 0.83$ and $\xi_{\tau} = 0.79$.

The final values of $\alpha^{\sigma}_{eff_0}$ and $\alpha^{\tau}_{eff_0}$ can be found by the formula of Fig. 8 (Appendix 5):

$$\alpha^{\sigma}_{eff} = 1 + \xi_{\sigma}(\alpha^{\sigma}_{eff_0} - 1) \cong 1 + 0.83\,(1.81 - 1) \cong 1.67$$

and

$$\alpha^{\tau}_{eff} = 1 + \xi_{\tau}(\alpha^{\tau}_{eff_0} - 1) \cong 1 + 0.79\,(1.42 - 1) \cong 1.33$$

According to curve *3* of the graph given in Fig. 3 (Appendix 5) we find the surface quality factor for $\sigma_u = 540$ MN/m²; it is $\varepsilon_{sf} = 0.88$.

The overall factors for the effect of all the factors on fatigue are

$$\varepsilon_{\sigma} = \frac{\varepsilon_{sf}}{\alpha^{\sigma}_{eff}} = \frac{0.88}{1.67} \cong 0.53 \text{ and } \varepsilon_{\tau} = \frac{\varepsilon_{sf}}{\alpha^{\tau}_{eff}} = \frac{0.88}{1.33} \cong 0.66$$

The maximum and minimum normal and shearing stresses in alternating bending and torsion have the following values

$$\sigma_{max} = \frac{M_{max}}{W} = \frac{M_{max}}{\frac{\pi d^3}{32}} = \frac{400}{6.28 \times 10^{-6}} = 64 \times 10^6 \text{ N/m}^2 = 64 \text{ MN/m}^2;$$

$$\sigma_{min} = \frac{M_{min}}{W} = \frac{-160}{6.28 \times 10^{-6}} = -25.5 \times 10^6 \text{ N/m}^2 = -25.5 \text{ MN/m}^2;$$

$$\tau_{max} = \frac{\max M_t}{W_p} = \frac{\max M_t}{\frac{\pi d^3}{16}} = \frac{640}{12.56 \times 10^{-6}} = 51 \times 10^6 \text{ N/m}^2 = 51 \text{ MN/m}^2;$$

$$\tau_{min} = \frac{\min M_t}{W_p} = \frac{320}{12.56 \times 10^{-6}} = 25.5 \times 10^6 \text{ N/m}^2 = 25.5 \text{ MN/m}^2$$

The mean stresses and amplitudes of the cycles of normal and shearing stresses equal

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} = \frac{64 - 25.5}{2} \cong 19 \text{ MN/m}^2;$$

$$\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{64 + 25.5}{2} \cong 44.7 \text{ MN/m}^2;$$

$$\tau_m = \frac{\tau_{max} + \tau_{min}}{2} = \frac{51 + 25.5}{2} \cong 38.2 \text{ MN/m}^2;$$

$$\tau_a = \frac{\tau_{max} - \tau_{min}}{2} = \frac{51 - 25.5}{2} \cong 12.7 \text{ MN/m}^2$$

The factors of safety based on the normal and shearing stresses can be found by formula (263):

$$n_\sigma = \frac{\sigma_{-1}}{\frac{\sigma_{-1}}{\sigma_u}\sigma_m + \frac{\sigma_a}{\varepsilon_\sigma}} = \frac{240}{\frac{240}{540}\times 19 + \frac{44.7}{0.53}} \cong 2.59$$

$$n_\tau = \frac{\tau_{-1}}{\frac{\tau_{-1}}{\tau_u}\tau_m + \frac{\tau_a}{\varepsilon_\tau}} = \frac{140}{\frac{140}{350}\times 38.2 + \frac{12.7}{0.66}} \cong 4.06$$

The overall factor of safety is established by formula (270)

$$n = \frac{n_\sigma n_\tau}{\sqrt{n_\sigma^2 + n_\tau^2}} = \frac{2.59\times 4.06}{\sqrt{2.59^2 + 4.06^2}} \cong 2.18$$

Example 146. The part illustrated in Fig. 253 is made of steel, grade Ст. 6, for which $\sigma_u = 60$ kgf/mm², $\sigma_y = 32$ kgf/mm², $\tau_y = 22$ kgf/mm², $\sigma_{-1} = 25$ kgf/mm² and $\tau_{-1} = 15$ kgf/mm²; $D = 80$ mm, $d = 40$ mm, $\rho = 0.2$ cm, $l = 40$ cm, $a = 10$ cm, $P_0 = \text{const} = 16$ kgf; P varies in a symmetrical cycle from P_{max} to $P_{min} = -P_{max}$.

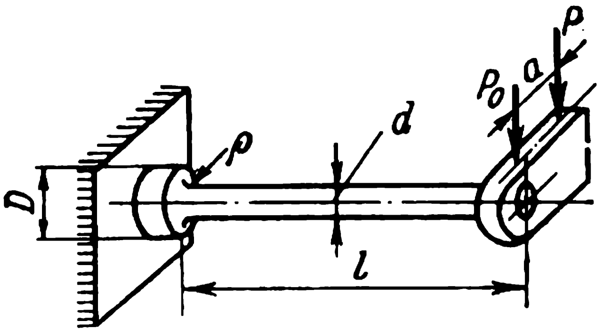


Fig. 253

Determine the allowable value of P_{max}, if the factor of safety stipulated for the part is $[n] = 2$. The surface of the part is roughly turned in a lathe.

Solution. From the graphs of Figs. 6 and 7 (Appendix 5) at

$$\frac{\rho}{d} = \frac{0.2}{4} = 0.05, \quad \frac{D}{d} = 2 \text{ for } \sigma_u = 60 \text{ kgf/mm}^2$$

we find the effective stress concentration factors for the fillet

$$\alpha_{eff}^\sigma \cong 2.07 \text{ and } \alpha_{eff}^\tau \cong 1.56$$

According to curve *4* of Fig. 3 (Appendix 5) for $\sigma_u = 60$ kgf/mm² the surface quality factor is $\varepsilon_{sf} \cong 0.8$. The overall factors for the effect of all the factors on the fatigue are

$$\varepsilon_\sigma = \frac{\varepsilon_{sf}}{\alpha_{eff}^\sigma} = \frac{0.8}{2.07} \cong 0.386 \text{ and } \varepsilon_\tau = \frac{0.8}{1.56} \cong 0.512$$

The maximum and minimum normal stresses in the fillet due to the alternating bending in an asymmetric cycle are

$$\sigma_{max} = \frac{P_0 + P_{max}}{W} l \cong \frac{16 + P_{max}}{0.1\times 64} 40 = 100 + 62.5 P_{max} \text{ kgf/cm}^2$$

$$\sigma_{min} = \frac{P_0 - P_{max}}{W} l \cong \frac{16 - P_{max}}{0.1\times 64} 40 = 100 - 62.5 P_{max} \text{ kgf/cm}^2$$

The maximum and minimum shearing stresses in the fillet due to the alternating torsion in a symmetrical cycle become

$$\tau_{max} = \frac{P_{max}}{W_p} a \cong \frac{P_{max}}{0.2 \times 64} 10 = 0.732 P_{max} \text{ kgf/cm}^2$$

$$\tau_{min} = -\frac{P_{max}}{W_p} a \cong -0.732 P_{max} \text{ kgf/cm}^2$$

The mean stresses and amplitudes of the cycles of the normal and shearing stresses equal

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} = 100 \text{ kgf/cm}^2; \quad \sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2}$$

$$= 62.5 P_{max} \text{ kgf/cm}^2;$$

$$\tau_m = 0; \quad \tau_a = 0.732 P_{max} \text{ kgf/cm}^2$$

According to formula (267) we find the factors of safety based on the normal and shearing stresses

$$n_\sigma = \frac{25}{\frac{25}{32} \times 1 + \frac{0.625}{0.386} P_{max}} = \frac{25}{0.732 + 1.62 P_{max}};$$

$$n_\tau = \frac{\varepsilon_\tau \tau_{-1}}{\tau_a} = \frac{0.512 \times 15}{0.0732 P_{max}} \cong \frac{105}{P_{max}}$$

Since the overall factor of safety according to formula (270) is

$$n = 2 = \frac{n_\sigma n_\tau}{\sqrt{n_\sigma^2 + n_\tau^2}} = \frac{25 \times 105}{P_{max}(0.732 + 1.62 P_{max})}$$

$$\times \frac{1}{\sqrt{\frac{25^2}{(0.732 + 1.62 P_{max})^2} + \frac{105^2}{P_{max}^2}}}$$

$$= \frac{25 \times 105}{\sqrt{25^2 P_{max}^2 + 105^2 (0.732 + 1.62 P_{max})^2}}$$

then

$$25^2 P_{max}^2 + 105^2 (0.732 + 1.62 P_{max})^2 = \frac{25^2 \times 105^2}{4}$$

or

$$P_{max}^2 + 0.885 P_{max} - 58.1 = 0$$

From which the required maximum allowable force equals

$$P_{max} \cong -0.442 \pm \sqrt{0.442^2 + 58.1} \cong -0.442 \pm 7.63 \text{ kgf}$$

Since only one root of the equation satisfies the problem, $P_{max} \cong$ $\cong 7.2$ kgf.

Problem 1015. Using the lower limits of the empirical dependences, determine the approximate value of the fatigue limit for steel speci-

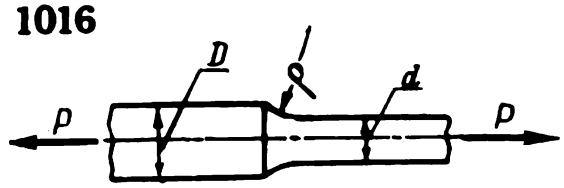
mens with stress raisers in symmetrical cycles of alternating torsion, if $\sigma_u = 960$ MN/m², $\alpha_\tau = 1.6$ and $q = 0.9$.

Problem 1016. A stepped round steel bar of diameters $d = 36$ mm and $D = 72$ mm is subjected to alternating tension-compression at $P_{max} = -P_{min} = 5$ tnf.

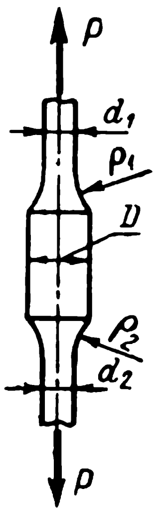
Determine the minimum allowable fillet radius ρ, if $\sigma_u = 40$ kgf/mm², $\sigma_{-1p} = 12$ kgf/mm² and the factor of safety should be $[n] = 2$.

Problem 1017. A stepped round bar of diameters $D = 60$ mm, $d_1 = 30$ mm and $d_2 = 32$ mm is made of steel, grade 20X, for which $\sigma_u = 80$ kgf/mm² and $\sigma_{-1p} = 24$ kgf/mm². At the fillets $\frac{\rho_1}{d_1} = 0.3$ and $\frac{\rho_2}{d_2} = 0.1$. The surface of the bar is well polished.

1016



1017

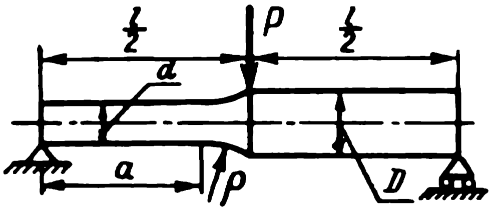


Determine the actual factor of safety n if the bar is immersed in fresh water and subjected to an alternating axial force $P_{max} = -P_{min} = 3$ tnf in a symmetrical cycle.

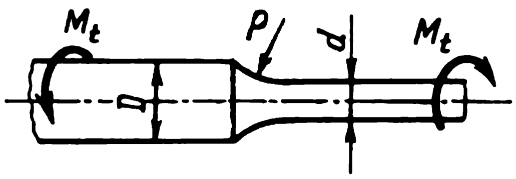
Problem 1018. Determine the maximum bending force P_{max} at which a stepped rotating steel shaft can operate as illustrated in the figure if the factor of safety is $[n] = 1.8$.

Let $\sigma_u = 1200$ MN/m², $\sigma_{-1} = 360$ MN/m², $D = 70$ mm, $d = 50$ mm, $\frac{\rho}{d} = 0.15$, $l = 60$ cm and $a = 40$ cm. The surface of the shaft has been ground.

1018



1019



Problem 1019. A stepped shaft of diameters $D = 60$ mm, $d = 50$ mm and with $\frac{\rho}{d} = 0.05$ is to operate in fresh water in an alternating symmetrical cycle of torsion with the factor of safety $[n] = 1.5$. The surface of the shaft has been ground. Determine which shaft will be stronger: one of steel, grade Ст. 5, for which $\sigma_u = 52$ kgf/mm² and $\tau_{-1} = 14$ kgf/mm² or one of steel, grade 40X, for which $\sigma_u = 100$ kgf/mm² and $\tau_{-1} = 22$ kgf/mm².

Problem 1020. Determine $\sigma_{\pm 0.6}$ for steel, grade 30XM, with the experimentally obtained values $\sigma_u = 90$ kgf/mm², $\sigma_{-1p} = 36$ kgf/mm² and the limiting amplitudes σ_a^r for the mean stresses σ_m^r equal to

σ_m^r, kgf/mm²	20	40	60
σ_a^r, kgf/mm²	32	26	16

Problem 1021. Determine $\sigma_{-0.5}$ for steel, for which the curve of the limiting amplitudes (in the cycles being considered $\sigma_m > 0$) is approximately defined by the equation $\sigma_a^r = 20\left[1 - \left(\frac{\sigma_m^r}{40}\right)^2\right]$.

Problem 1022. Determine the factor of safety for a steel part subject to alternating stresses of tension-compression cycles, if $\sigma_u = 90$ kgf/mm², $\sigma_y = 70$ kgf/mm², $\sigma_{-1p} = 30$ kgf/mm², $\sigma_{max} = 16$ kgf/mm², $\sigma_{min} = -6$ kgf/mm², $\alpha_{eff} = 1.5$; $\varepsilon_{se} = 0.7$; $\varepsilon_{sf} = 0.9$ and $\beta = 1.3$.

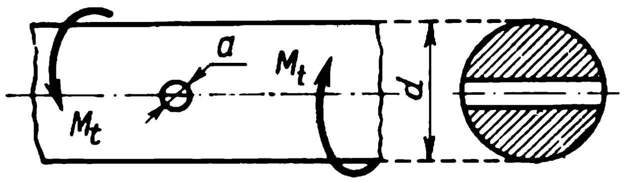
Problem 1023. A steel part is to operate in an alternating bending cycle with the stress amplitude $\sigma_a = 200$ MN/m² and factor of safety $[n] = 2$. What should the asymmetry factor r of the cycle be if $\sigma_u = 1100$ MN/m², $\sigma_y = 900$ MN/m², $\sigma_{-1} = 480$ MN/m²; $\alpha_\sigma = 1.2$; $q = 0.9$, $\varepsilon_{se} = 0.8$; $\varepsilon_{sf} = 1$ and $\beta = 1.4$.

Problem 1024. Determine the minimum allowable stress for a steel part in an alternating tension-compression cycle for which the asymmetry factor is $r = -0.4$ and factor of safety is $[n] = 1.4$. The part is made of carbon steel for which $\sigma_u = 50$ kgf/mm² and $\sigma_y = 24$ kgf/mm². The part is of round cross section of diameter $d = 30$ mm and has a stress raiser for which $\alpha_\sigma = 2$. The strengthening factor due to work hardening of the surface is $\beta = 1.2$.

Problem 1025*. A round steel shaft of diameter $d = 40$ mm has a through transverse round hole of diameter $a = 6$ mm and is subjected to a pulsating cycle ($r = 0$) with the factor of safety $[n] = 1.8$.

Determine the maximum alternating torque max M_t, if the shaft is made of steel, grade 30XГСА, for which $\sigma_u = 110$ kgf/mm², $\tau_y = 51$ kgf/mm² and $\tau_{-1} = 23$ kgf/mm².

1025



Problem 1026*. Determine the actual factor of safety n for a part in alternating tension and torsion; the part is made of steel, grade 45,

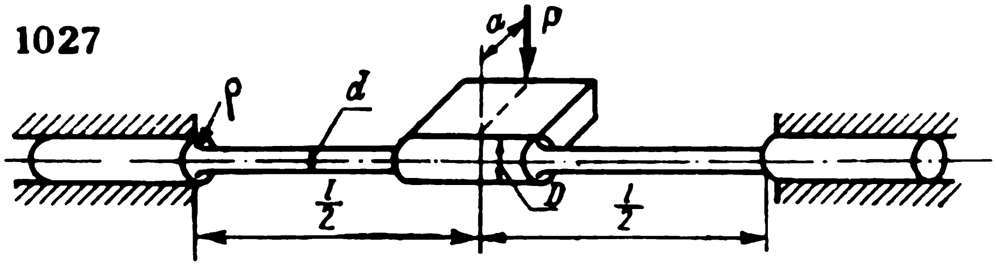
* In solving problems marked with an asterisk, make use of Soderberg's straight-line diagram.

for which $\sigma_u = 60$ kgf/mm², $\sigma_y = 32$ kgf/mm², $\sigma_{-1p} = 20$ kgf/mm², $\tau_y = 22$ kgf/mm², $\tau_{-1} = 16$ kgf/mm², $\alpha_{eff}^{\sigma} = 1.7$, $\alpha_{eff}^{\tau} = 1.4$; $\varepsilon_{se} = 0.7$, $\varepsilon_{sf} = 1$, $\beta = 1$, $\sigma_{max} = 1000$ kgf/cm², $\sigma_{min} = 200$ kgf/cm², $\tau_{max} = 400$ kgf/cm² and $\tau_{min} = -100$ kgf/cm².

Problem 1027*. A cylindrical steel rod of round cross section of diameter $d = 30$ mm is fixed at the ends. The lug at the middle of the rod is subjected to a symmetrically varying force P.

Determine the maximum force P_{max}, if the rod is made of steel, grade 40XH for which $\sigma_u = 900$ MN/m², $\sigma_{-1} = 400$ MN/m² and $\tau_{-1} = 240$ MN/m². The diameter of the rod in the middle is $D = 60$ mm, $\frac{\rho}{d} = 0.1$, $l = 32$ cm, $a = 10$ cm, $\beta = 1.2$ and $[n] = 1.6$.

1027



Problem 1028*. A stepped round steel shaft of diameters $D = 60$ mm and $d = 50$ mm, and a fillet radius $\rho = 5$ mm is subjected to alternating bending and torsion.

Determine the actual factor of safety n of the shaft, if, in the dangerous cross section at the fillet, $M_{max} = 3$ kN-m and $M_{min} = 1.5$ kN-m in bending; and max $M_t = 2$ kN-m, min $M_t = -0.5$ kN-m in torque.

The shaft is made of steel, grade 40XH, for which $\sigma_u = 900$ MN/m²; $\sigma_y = 750$ MN/m²; $\sigma_{-1} = 400$ MN/m²; $\tau_y = 390$ MN/m² and $\tau_{-1} = 240$ MN/m².

The strengthening effect of processing factors is assessed by the factor $\beta = 1.3$.

* In solving problems marked with an asterisk, make use of Soderberg's straight-line diagram.

APPENDICES

Appendix 1

Data on Torsion of Non-Circular Sections

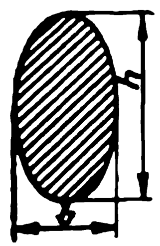
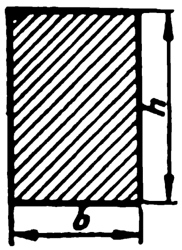
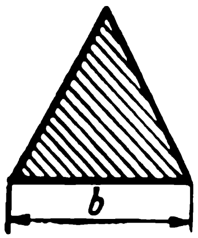
Cross section	Moment of inertia of section in torsion I_t, cm^4	Section modulus in torsion W_t, cm^3	Points with maximum shear stress $\tau_{max} = \frac{M_t}{W_t}$	Notes
	$I_t = \frac{\pi}{16} \cdot \frac{m^3}{m^2+1} b^4 = \frac{16F^4}{\pi^3 bh (b^2+h^2)}$	$W_t = \frac{\pi b^3}{16} m = \frac{\pi b^2 h}{16}$	At end of minor semiaxis $\tau_{max} = \frac{M_t}{W_t}$ At end of major semiaxis $\tau = \frac{\tau_{max}}{m}$	$\frac{h}{b} = m > 1$ F = area of section
	$I_t = \alpha b^4$	$W_t = \beta b^3$	In middle of longer sides $\tau_{max} = \frac{M_t}{W_t}$ In middle of shorter sides $\tau = \gamma \tau_{max}$ At corners stress equals zero	Table of factors α, β and γ
Equilateral triangle 	$I_t = \frac{b^4}{46.19} \approx 0.02165\, b^4$	$W_t = 0.05\, b^3$	$\tau_{max} = \frac{20\, M_t}{b^3}$ (at middle of sides) Shear stress equals zero at vertices	

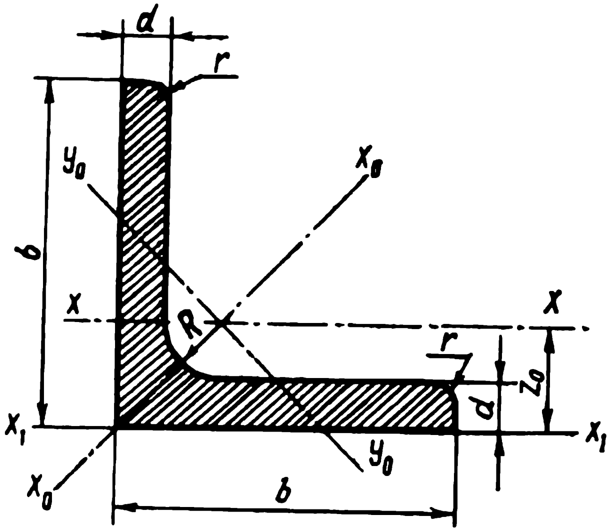
Table of factors α, β and γ

$m = \frac{h}{b}$	α	β	γ
1.0	0.140	0.208	1.0
1.5	0.294	0.346	0.859
2.0	0.457	0.493	0.795
3.0	0.790	0.801	0.753
4.0	1.123	1.127	0.745

Appendix 2
Table 1

ROLLED STEEL SECTIONS

Rolled Steel Equal Angles. USSR Std GOST 8509-57



No. of Section	Dimensions, mm				Area of Section, cm²	Weight per Metre, kgf	Data for Axes							z_0, cm
							$x - x$		$x_0 - x_0$		$y_0 - y_0$		$x_1 - x_1$	
	b	d	R	r			I_x, cm⁴	i_x, cm	$I_{x_0\,max}$, cm⁴	$i_{x_0\,max}$, cm	$I_{y_0\,min}$, cm⁴	$i_{y_0\,min}$, cm	I_{x_1}, cm⁴	
2	20	3	3.5	1.2	1.13	0.89	0.40	0.59	0.63	0.75	0.17	0.39	0.81	0.60
		4			1.46	1.15	0.50	0.58	0.78	0.73	0.22	0.38	1.09	0.64
2.5	25	3	3.5	1.2	1.43	1.12	0.81	0.75	1.29	0.95	0.34	0.49	1.57	0.73
		4			1.86	1.46	1.03	0.74	1.62	0.93	0.44	0.48	2.11	0.76
2.8	28	3	4	1.3	1.62	1.27	1.16	0.85	1.84	1.07	0.48	0.55	2.20	0.80

Continued

No. of Section	Dimensions, mm				Area of Section, cm^2	Weight per Metre, kgf	Data for Axes							z_0, cm
							$x - x$		$x_0 - x_0$		$y_0 - y_0$		$x_1 - x_1$	
	b	d	R	r			I_x, cm^4	i_x, cm	$I_{x_0\,max}$, cm^4	$i_{x_0\,max}$, cm	$I_{y_0\,min}$, cm^4	$i_{y_0\,min}$, cm	I_{x_1}, cm^4	
3.2	32	3	4.5	1.5	1.86	1.46	1.77	0.97	2.80	1.23	0.74	0.63	3.26	0.89
		4			2.43	1.91	2.26	0.96	3.58	1.21	0.94	0.62	4.39	0.94
3.6	36	3	4.5	1.5	2.10	1.65	2.56	1.10	4.06	1.39	1.06	0.71	4.64	0.99
		4			2.75	2.16	3.29	1.09	5.21	1.38	1.36	0.70	6.24	1.04
4	40	3	5	1.7	2.35	1.85	3.55	1.23	5.63	1.55	1.47	0.79	6.35	1.09
		4			3.08	2.42	4.58	1.22	7.26	1.53	1.90	0.78	8.53	1.13
4.5	45	3	5	1.7	2.65	2.08	5.13	1.39	8.13	1.75	2.12	0.89	9.04	1.21
		4			3.48	2.73	6.63	1.38	10.5	1.74	2.74	0.89	12.1	1.26
		5			4.29	3.37	8.03	1.37	12.7	1.72	3.33	0.88	15.3	1.30
5	50	3	5.5	1.8	2.96	2.32	7.11	1.55	11.3	1.95	2.95	1.00	12.4	1.33
		4			3.89	3.05	9.21	1.54	14.6	1.94	3.80	0.99	16.6	1.38
		5			4.80	3.77	11.2	1.53	17.8	1.92	4.63	0.98	20.9	1.42
5.6	56	3.5	6	2	3.86	3.03	11.6	1.73	18.4	2.18	4.80	1.12	20.3	1.50
		4			4.38	3.44	13.1	1.73	20.8	2.18	5.41	1.11	23.3	1.52

		5			5.41	4.25	16.0	1.72	25.4	2.16	6.59	1.10	29.2	1.57
6.3	63	4	7	2.3	4.96	3.90	18.9	1.95	29.9	2.45	7.81	1.25	33.1	1.69
		5			6.13	4.81	23.1	1.94	36.6	2.44	9.52	1.25	41.5	1.74
		6			7.28	5.72	27.1	1.93	42.9	2.43	11.2	1.24	50.0	1.78
7	70	4.5	8.0	2.7	6.20	4.87	29.0	2.16	46.0	2.72	12.0	1.39	51.0	1.88
		5			6.86	5.38	31.9	2.16	50.7	2.72	13.2	1.39	56.7	1.90
		6			8.15	6.39	37.6	2.15	59.6	2.71	15.5	1.38	68.4	1.94
		7			9.42	7.39	43.0	2.14	68.2	2.69	17.8	1.37	80.1	1.99
		8			10.7	8.37	48.2	2.13	76.4	2.68	20.0	1.37	91.9	2.02
7.5	75	5	9	3	7.39	5.80	39.5	2.31	62.6	2.91	16.4	1.49	69.6	2.02
		6			8.78	6.89	46.6	2.30	73.9	2.90	19.3	1.48	83.9	2.06
		7			10.1	7.96	53.3	2.29	84.6	2.89	22.1	1.48	98.3	2.10
		8			11.5	9.02	59.8	2.28	94.9	2.87	24.8	1.47	113	2.15
		9			12.8	10.1	66.1	2.27	105	2.86	27.5	1.46	127	2.18
8	80	5.5	9	3	8.63	6.78	52.7	2.47	83.6	3.11	21.8	1.59	93.2	2.17
		6			9.38	7.36	57.0	2.47	90.4	3.11	23.5	1.58	102	2.19
		7			10.8	8.51	65.3	2.45	104	3.09	27.0	1.58	119	2.23
		8			12.3	9.65	73.4	2.44	116	3.08	30.3	1.57	137	2.27
9	90	6	10	3.3	10.6	8.33	82.1	2.78	130	3.50	34.0	1.79	145	2.43
		7			12.3	9.64	94.3	2.77	150	3.49	38.9	1.78	169	2.47
		8			13.9	10.9	106	2.76	168	3.48	43.8	1.77	194	2.51
		9			15.6	12.2	118	2.75	186	3.46	48.6	1.77	219	2.55

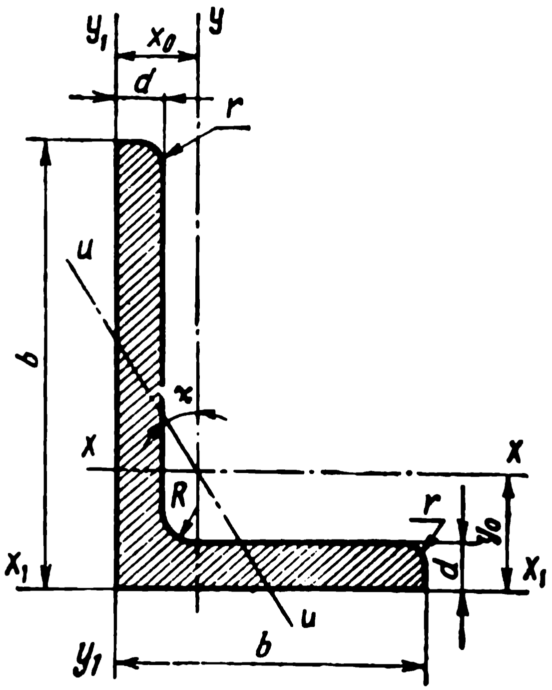
Continued

No. of Section	Dimensions, mm				Area of Section, cm^2	Weight per Metre, kgf	Data for Axes							z_0, cm
							$x - x$		$x_0 - x_0$		$y_0 - y_0$		$x_1 - x_1$	
	b	d	R	r			I_x, cm^4	i_x, cm	$I_{x_0\,max}$, cm^4	$i_{x_0\,max}$, cm	$I_{y_0\,min}$, cm^4	$i_{y_0\,min}$, cm	I_{x_1}, cm^4	
10	100	6.5	12	4	12.8	10.1	122	3.09	193	3.88	50.7	1.99	214	2.68
		7			13.8	10.8	131	3.08	207	3.88	54.2	1.98	231	2.71
		8			15.6	12.2	147	3.07	233	3.87	60.9	1.98	265	2.75
		10			19.2	15.1	179	3.05	284	3.84	74.1	1.96	333	2.83
		12			22.8	17.9	209	3.03	331	3.81	86.9	1.95	402	2.91
		14			26.3	20.6	237	3.00	375	3.78	99.3	1.94	472	2.99
		16			29.7	23.3	264	2.98	416	3.74	112	1.94	542	3.06
11	110	7	12	4	15.2	11.9	176	3.40	279	4.29	72.7	2.19	308	2.96
		8			17.2	13.5	198	3.39	315	4.28	81.8	2.18	353	3.00
12.5	125	8	14	4.6	19.7	15.5	294	3.87	467	4.87	122	2.49	516	3.36
		9			22.0	17.3	327	3.86	520	4.86	135	2.48	582	3.40
		10			24.3	19.1	360	3.85	571	4.84	149	2.47	649	3.45
		12			28.9	22.7	422	3.82	670	4.82	174	2.46	782	3.53
		14			33.4	26.2	482	3.80	764	4.78	200	2.45	916	3.61
		16			37.8	29.6	539	3.78	853	4.75	224	2.44	1051	3.68
14	140	9	14	4.6	24.7	19.4	466	4.34	739	5.47	192	2.79	818	3.78
		10			27.3	21.5	512	4.33	814	5.46	211	2.78	911	3.82
		12			32.5	25.5	602	4.31	957	5.43	248	2.76	1097	3.90

16	160	10	16	5.3	31.4	24.7	774	4.96	1 229	6.25	319	3.19	1 356	4.30
		11			34.4	27.0	844	4.95	1 341	6.24	348	3.18	1 494	4.35
		12			37.4	29.4	913	4.94	1 450	6.23	376	3.17	1 633	4.39
		14			43.3	34.0	1046	4.92	1 662	6.20	431	3.16	1 911	4.47
		16			49.1	38.5	1175	4.89	1 866	6.17	485	3.14	2 191	4.55
		18			54.8	43.0	1299	4.87	2 061	6.13	537	3.13	2 472	4.63
		20			60.4	47.4	1419	4.85	2 248	6.10	589	3.12	2 756	4.70
18	180	11	16	5.3	38.8	30.5	1216	5.60	1 933	7.06	500	3.59	2 128	4.85
		12			42.2	33.1	1317	5.59	2 093	7.04	540	3.58	2 324	4.89
20	200	12	18	6	47.1	37.0	1823	6.22	2 896	7.84	749	3.99	3 182	5.37
		13			50.9	39.9	1961	6.21	3 116	7.83	805	3.98	3 452	5.42
		14			54.6	42.8	2097	6.20	3 333	7.81	861	3.97	3 722	5.46
		16			62.0	48.7	2363	6.17	3 755	7.78	970	3.96	4 264	5.54
		20			76.5	60.1	2871	6.12	4 560	7.72	1182	3.93	5 355	5.70
		25			94.3	74.0	3466	6.06	5 494	7.63	1438	3.91	6 733	5.89
		30			115.5	87.6	4020	6.00	6 351	7.55	1688	3.89	8 130	6.07
22	220	14	21	7	60.4	47.4	2814	6.83	4 470	8.60	1159	4.38	4 941	5.93
		16			68.6	53.8	3175	6.81	5 045	8.58	1306	4.36	5 661	6.02
25	250	16	24	8	78.4	61.5	4717	7.76	7 492	9.78	1942	4.98	8 286	6.75
		18			87.7	68.9	5247	7.73	8 337	9.75	2158	4.96	9 342	6.83
		20			97.0	76.1	5765	7.71	9 160	9.72	2370	4.94	10 401	6.91
		22			106.1	83.3	6270	7.69	9 961	9.69	2579	4.93	11 464	7.00
		25			119.7	94.0	7006	7.65	11 125	9.64	2887	4.91	13 064	7.11
		28			133.1	104.5	7717	7.61	12 244	9.59	3190	4.89	14 674	7.23
		30			142.0	111.4	8177	7.59	12 965	9.56	3389	4.89	15 753	7.31

Table 2

Rolled Steel Unequal Angles. USSR Std GOST 8510-57



No. of Section	Dimensions, mm					Area of Section, cm²	Weight per Metre, kgf	Data for Axes											
								$x-x$		$y-y$		x_1-x_1		y_1-y_1		$u-u$			
	B	b	d	R	r			I_x, cm⁴	i_x, cm	I_y, cm⁴	i_y, cm	I_{x_1}, cm⁴	Distance from Centroid, y_0, cm	$I_{y_1 min}$, cm⁴	Distance from Centroid, x_0, cm	$I_{u min}$, cm⁴	$i_{u min}$, cm	Slope of Axis, $\tan\alpha$	
2.5/1.6	25	16	3	3.5	1.2	1.16	0.91	0.70	0.78	0.22	0.44	1.56	0.86	0.43	0.42	0.13	0.34	0.392	
3.2/2	32	20	3	3.5	1.2	1.49	1.17	1.52	1.01	0.46	0.55	3.26	1.08	0.82	0.49	0.28	0.43	0.382	
			4			1.94	1.52	1.93	1.00	0.57	0.54	4.38	1.12	1.12	0.53	0.35	0.43	0.374	

4/2.5	40	25	3	4.0	1.3	1.89	1.48	3.06	1.27	0.93	0.70	6.37	1.32	1.58	0.59	0.56	0.54	0.385
			4			2.47	1.94	3.93	1.26	1.18	0.69	8.53	1.37	2.15	0.63	0.71	0.54	0.381
4.5/2.8	45	28	3	5.0	1.7	2.14	1.68	4.41	1.43	1.32	0.79	9.02	1.47	2.20	0.64	0.79	0.61	0.382
			4			2.80	2.20	5.68	1.42	1.69	0.78	12.1	1.51	2.98	0.68	1.02	0.60	0.379
5/3.2	50	32	3	5.5	1.8	2.42	1.90	6.17	1.60	1.99	0.91	12.4	1.60	3.26	0.72	1.18	0.70	0.403
			4			3.17	2.49	7.98	1.59	2.56	0.90	16.6	1.65	4.42	0.76	1.52	0.69	0.401
5.6/3.6	56	36	3.5	6.0	2.0	3.16	2.48	10.1	1.79	3.30	1.02	20.3	1.80	5.43	0.82	1.95	0.79	0.407
			4			3.58	2.81	11.4	1.78	3.70	1.02	23.2	1.82	6.25	0.84	2.19	0.78	0.406
			5			4.41	3.46	13.8	1.77	4.48	1.01	29.2	1.86	7.91	0.88	2.66	0.78	0.404
6.3/4.0	63	40	4	7.0	2.3	4.04	3.17	16.3	2.01	5.16	1.13	33.0	2.03	8.51	0.91	3.07	0.87	0.397
			5			4.98	3.91	19.9	2.00	6.26	1.12	41.4	2.08	10.8	0.95	3.73	0.86	0.396
			6			5.90	4.63	23.3	1.99	7.28	1.11	49.9	2.12	13.1	0.99	4.36	0.86	0.393
			8			7.68	6.03	29.6	1.96	9.15	1.09	66.9	2.20	17.9	1.07	5.58	0.85	0.386
7/4.5	70	45	4.5	7.5	2.5	5.07	3.98	25.3	2.23	8.25	1.28	51	2.25	13.6	1.03	4.88	0.98	0.407
			5			5.59	4.39	27.8	2.23	9.05	1.27	56.7	2.28	15.2	1.05	5.34	0.98	0.406
7.5/5	75	50	5	8	2.7	6.11	4.79	34.8	2.39	12.5	1.43	69.7	2.39	20.8	1.17	7.24	1.09	0.436
			6			7.25	5.69	40.9	2.38	14.6	1.42	83.9	2.44	25.2	1.21	8.48	1.08	0.435
			8			9.47	7.43	52.4	2.35	18.5	1.40	112	2.52	34.2	1.29	10.9	1.07	0.430
8/5	80	50	5	8	2.7	6.36	4.99	41.6	2.56	12.7	1.41	84.6	2.6	20.8	1.13	7.58	1.09	0.387
			6			7.55	5.92	49.0	2.55	14.8	1.40	102	2.65	25.2	1.17	8.88	1.08	0.386
9/5.6	90	56	5.5	9	3	7.86	6.17	65.3	2.88	19.7	1.58	132	2.92	32.2	1.26	11.8	1.22	0.384
			6			8.54	6.70	70.6	2.88	21.2	1.58	145	2.95	35.2	1.28	12.7	1.22	0.384
			8			11.18	8.77	90.9	2.85	27.1	1.56	194	3.04	47.8	1.36	16.3	1.21	0.380

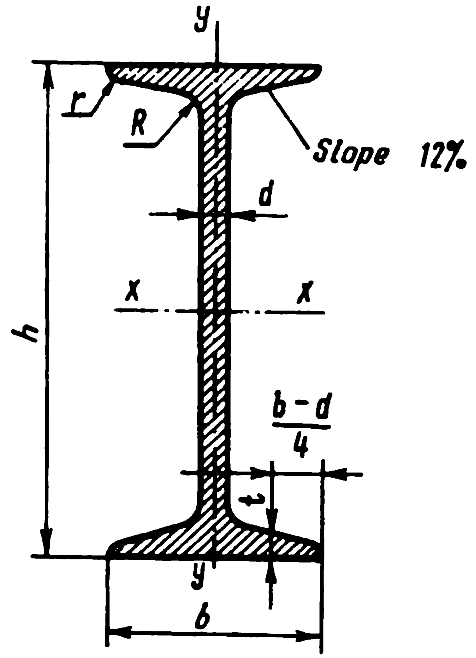
Continued

No. of Section	Dimensions, mm					Area of Section, cm²	Weight per Metre, kgf	Data for Axes										
								$x-x$		$y-y$		x_1-x_1		y_1-y_1		$u-u$		Slope of Axis, $\tan\alpha$
	B	b	d	R	r			I_x, cm⁴	i_x, cm	I_y, cm⁴	i_y, cm	I_{x_1}, cm⁴	Distance from Centroid, y_0, cm	$I_{y_1\,min}$, cm⁴	Distance from Centroid, x_0, cm	$I_{u\,min}$, cm⁴	$i_{u\,min}$, cm	
10/6.3	100	63	6	10	3.3	9.59	7.53	98.3	3.2	30.6	1.79	198	3.23	49.9	1.42	18.2	1.38	0.393
			7			11.1	8.70	113	3.19	35.0	1.78	232	3.28	58.7	1.46	20.8	1.37	0.392
			8			12.6	9.87	127	3.18	39.2	1.77	266	3.32	67.6	1.50	23.4	1.36	0.391
			10			15.5	12.1	154	3.15	47.1	1.75	333	3.40	85.8	1.58	28.3	1.35	0.387
11/7	110	70	6.5	10	3.3	11.4	8.98	142	3.53	45.6	2	286	3.55	74.3	1.58	26.9	1.53	0.402
			7			12.3	9.64	152	3.52	48.7	1.99	309	3.57	80.3	1.6	28.8	1.53	0.402
			8			13.9	10.9	172	3.51	54.6	1.98	353	3.61	92.3	1.64	32.3	1.52	0.400
12.5/8	125	80	7	11	3.7	14.1	11	227	4.01	73.7	2.29	452	4.01	119	1.8	43.4	1.76	0.407
			8			16	12.5	256	4	83.0	2.28	518	4.05	137	1.84	48.8	1.75	0.406
			10			19.7	15.5	312	3.98	100	2.26	649	4.14	173	1.92	59.3	1.74	0.404
			12			23.4	18.3	365	3.95	117	2.24	781	4.22	210	2	69.5	1.72	0.400
14/9	140	90	8	12	4	18	14.1	364	4.49	120	2.58	727	4.49	194	2.03	70.3	1.98	0.411
			10			22.2	17.5	444	4.47	146	2.56	911	4.58	245	2.12	85.5	1.96	0.409

16/10	160	100	9	13	4.3	22.9	18.3	606	5.15	186	2.85	1 221	5.19	300	2.23	110	2.2	0.391
			10			25.3	19.8	667	5.13	204	2.84	1 359	5.23	335	2.28	121	2.19	0.390
			12			30	23.6	784	5.11	239	2.82	1 634	5.32	405	2.36	142	2.18	0.388
			14			34.7	27.3	897	5.08	272	2.8	1 910	5.40	477	2.43	162	2.16	0.385
18/11	180	110	10	14	4.7	28	22.2	952	5.8	276	3.12	1 933	5.88	444	2.44	165	2.42	0.375
			12			33.7	26.4	1123	5.77	324	3.1	2 324	5.97	537	2.52	194	2.40	0.374
20/12.5	200	125	11	14	4.7	34.9	27.4	1449	6.45	446	3.58	2 920	6.5	718	2.79	264	2.75	0.392
			12			37.9	29.7	1568	6.43	482	3.57	3 189	6.54	786	2.83	285	2.74	0.392
			14			43.9	34.4	1801	6.41	551	3.54	3 726	6.52	922	2.91	327	2.73	0.390
			16			49.8	39.1	2026	6.38	617	3.52	4 264	6.71	1061	2.99	367	2.72	0.388
25/16	250	160	12	18	6	48.3	37.9	3147	8.07	1032	4.62	6 212	7.97	1634	3.53	604	3.54	0.410
			16			63.6	49.9	4091	8.02	1333	4.58	8 308	8.14	2200	3.69	781	3.50	0.408
			18			71.1	55.8	4545	7.99	1475	4.56	9 358	8.23	2487	3.77	866	3.49	0.407
			20			78.5	61.7	4987	7.97	1613	4.53	10 410	8.31	2776	3.85	949	3.48	0.405

Table 3

Rolled Steel I-Beams. USSR Std GOST 8239-56

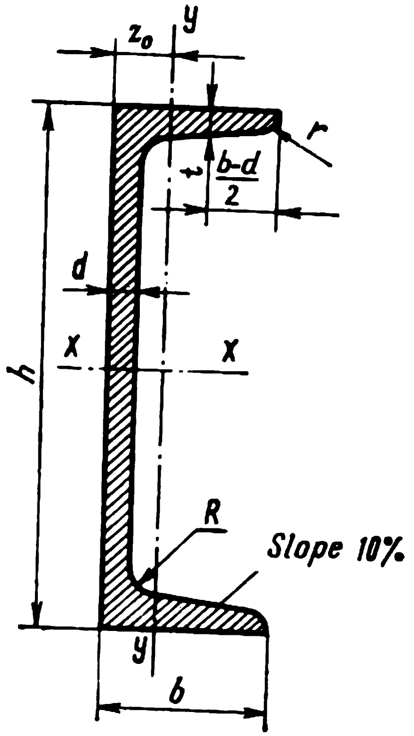


No. of Section	Weight per Metre, kgf	Dimensions, mm						Area of Section, cm^2	Data for Axes						
									x — x				y — y		
		h	b	d	t	R	r		I_x, cm^4	W_x, cm^3	i_x, cm	S_x, cm^3	I_y, cm^4	W_y, cm^3	i_y, cm
10	9.46	100	55	4.5	7.2	7	2.5	12.0	198	39.7	4.06	23.0	17.9	6.49	1.22
12	11.5	120	64	4.8	7.3	7.5	3	14.7	350	58.4	4.88	33.7	27.9	8.72	1.38
14	13.7	140	73	4.9	7.5	8	3	17.4	572	81.7	5.73	46.8	41.9	11.5	1.55
16	15.9	160	81	5.0	7.8	8.5	3.5	20.2	873	109	6.57	62.3	58.6	14.5	1.70
18	18.4	180	90	5.1	8.1	9	3.5	23.4	1 290	143	7.42	81.4	82.6	18.4	1.88

18a	19.9	180	100	5.1	8.3	9	3.5	25.4	1 430	159	7.51	89.8	114	22.8	2.12
20	21.0	200	100	5.2	8.4	9.5	4	26.8	1 840	184	8.28	104	115	23.1	2.07
20a	22.7	200	110	5.2	8.6	9.5	4	28.9	2 030	203	8.37	114	155	28.2	2.32
22	24.0	220	110	5.4	8.7	10	4	30.6	2 550	232	9.13	131	157	28.6	2.27
22a	25.8	220	120	5.4	8.9	10	4	32.8	2 790	254	9.22	143	206	34.3	2.50
24	27.3	240	115	5.6	9.5	10.5	4	34.8	3 460	289	9.97	163	198	34.5	2.37
24a	29.4	240	125	5.6	9.8	10.5	4	37.5	3 800	317	10.1	178	260	41.6	2.63
27	31.5	270	125	6.0	9.8	11	4.5	40.2	5 010	371	11.2	210	260	41.5	2.54
27a	33.9	270	135	6.0	10.2	11	4.5	43.2	5 500	407	11.3	229	337	50.0	2.80
30	36.5	300	135	6.5	10.2	12	5	46.5	7 080	472	12.3	268	337	49.9	2.69
30a	39.2	300	145	6.5	10.7	12	5	49.9	7 780	518	12.5	292	436	60.1	2.95
33	42.2	330	140	7.0	11.2	13	5	53.8	9 840	597	13.5	339	419	59.9	2.79
36	48.6	360	145	7.5	12.3	14	6	61.9	13 380	743	14.7	423	516	71.1	2.89
40	56.1	400	155	8.0	13.0	15	6	71.4	18 930	947	16.3	540	666	85.9	3.05
45	65.2	450	160	8.6	14.2	16	7	83.0	27 450	1220	18.2	699	807	101	3.12
50	76.8	500	170	9.5	15.2	17	7	97.8	39 290	1570	20.0	905	1040	122	3.26
55	89.8	550	180	10.3	16.5	18	7	114	55 150	2000	22.0	1150	1350	150	3.44
60	104	600	190	11.1	17.8	20	8	132	75 450	2510	23.9	1450	1720	181	3.60
65	120	650	200	12.0	19.2	22	9	153	101 400	3120	25.8	1800	2170	217	3.77
70	138	700	210	13.0	20.8	24	10	176	134 600	3840	27.7	2230	2730	260	3.94
70a	158	700	210	15.0	24.0	24	10	202	152 700	4360	27.5	2550	3240	309	4.01
70b	184	700	210	17.5	28.2	24	10	234	175 370	5010	27.4	2940	3910	373	4.09

Table 4

Rolled Steel Channels. USSR Std GOST 8240-56



No. of Section	Weight per Metre, kgf	Dimensions, mm						Area of Section, cm^2	Data for Axes							z_0
									$x - x$				$y - y$			
		h	b	d	t	R	r		I_x, cm^4	W_x, cm^3	i_x, cm	S_x, cm^3	I_y, cm^4	W_y, cm^3	i_y, cm	
5	4.84	50	32	4.4	7.0	6	2.5	6.16	22.8	9.10	1.92	5.59	5.61	2.75	0.954	1.16
6.5	5.90	65	36	4.4	7.2	6	2.5	7.51	48.6	15.0	2.54	9.00	8.70	3.68	1.08	1.24
8	7.05	80	40	4.5	7.4	6.5	2.5	8.98	89.4	22.4	3.16	13.3	12.8	4.75	1.19	1.31

10	8.59	100	46	4.5	7.6	7	3	10.9	174	34.8	3.99	20.4	20.4	6.46	1.37	1.44
12	10.4	120	52	4.8	7.8	7.5	3	13.3	304	50.6	4.78	29.6	31.2	8.52	1.53	1.54
14	12.3	140	58	4.9	8.1	8	3	15.6	491	70.2	5.60	40.8	45.4	11.0	1.70	1.67
14a	13.3	140	62	4.9	8.7	8	3	17.0	545	77.8	5.66	45.1	57.5	13.3	1.84	1.87
16	14.2	160	64	5.0	8.4	8.5	3.5	18.1	747	93.4	6.42	54.1	63.3	13.8	1.87	1.80
16a	15.3	160	68	5.0	9.0	8.5	3.5	19.5	823	103	6.49	59.4	78.8	16.4	2.01	2.00
18	16.3	180	70	5.1	8.7	9	3.5	20.7	1090	121	7.24	69.8	86.0	17.0	2.04	1.94
18a	17.4	180	74	5.1	9.3	9	3.5	22.2	1190	132	7.32	76.1	105	20.0	2.18	2.13
20	18.4	200	76	5.2	9.0	9.5	4	23.4	1520	152	8.07	87.8	113	20.5	2.20	2.07
20a	19.8	200	80	5.2	9.7	9.5	4	25.2	1670	167	8.15	95.9	139	24.2	2.35	2.28
22	21.0	220	82	5.4	9.5	10	4	26.7	2110	192	8.89	110	151	25.1	2.37	2.21
22a	22.6	220	87	5.4	10.2	10	4	28.8	2330	212	8.99	121	187	30.0	2.55	2.46
24	24.0	240	90	5.6	10.0	10.5	4	30.6	2900	242	9.73	139	208	31.6	2.60	2.42
24a	25.8	240	95	5.6	10.7	10.5	4	32.9	3180	265	9.84	151	254	37.2	2.78	2.67
27	27.7	270	95	6.0	10.5	11	4.5	35.2	4160	308	10.9	178	262	37.3	2.73	2.47
30	31.8	300	100	6.5	11.0	12	5	40.5	5810	387	12.0	224	327	43.6	2.84	2.52
33	36.5	330	105	7.0	11.7	13	5	46.5	7980	484	13.1	281	410	51.8	2.97	2.59
36	41.9	360	110	7.5	12.6	14	6	53.4	10820	601	14.2	350	513	61.7	3.10	2.68
40	48.3	400	115	8.0	13.5	15	6	61.5	15220	761	15.7	444	642	73.4	3.23	2.75

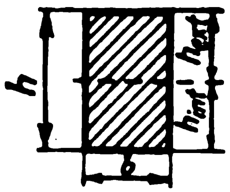
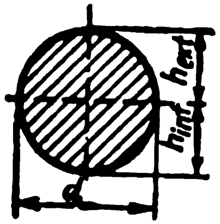
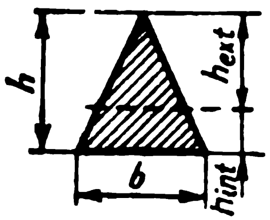
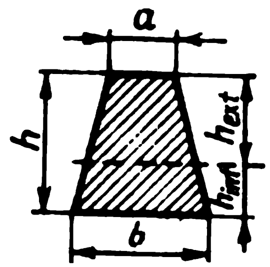
Appendix 3

Factors for Buckling

Slenderness ratio, λ	Allowable-Stress Reduction Factor φ for				
	Steel, grade 4, 3, 2 and OC	Steel, grade 5	Steel, grade СПК	Cast iron	Wood
0	1.00	1.00	1.00	1.00	1.00
10	0.99	0.98	0.97	0.97	0.99
20	0.96	0.95	0.95	0.91	0.97
30	0.94	0.92	0.91	0.81	0.93
40	0.92	0.89	0.87	0.69	0.87
50	0.89	0.86	0.83	0.57	0.80
60	0.86	0.82	0.79	0.44	0.71
70	0.81	0.76	0.72	0.34	0.60
80	0.75	0.70	0.65	0.26	0.48
90	0.69	0.62	0.55	0.20	0.38
100	0.60	0.51	0.43	0.16	0.31
110	0.52	0.43	0.35	—	0.25
120	0.45	0.36	0.30	—	0.22
130	0.40	0.33	0.26	—	0.18
140	0.36	0.29	0.23	—	0.16
150	0.32	0.26	0.21	—	0.14
160	0.29	0.24	0.19	—	0.12
170	0.26	0.21	0.17	—	0.11
180	0.23	0.19	0.15	—	0.10
190	0.21	0.17	0.14	—	0.09
200	0.19	0.16	0.13	—	0.08

Appendix 4

Equations for determining the section moduli of the external W_{ext} and internal W_{int} fibres, approximate values of the displacement e of the neutral axis and factors α_{ext} and α_{int} for formulas used in finding the normal stresses of the external and internal fibres of a curved beam for certain cross sections.

Cross section	W_{int}^{ext}	e	α_{ext}	α_{int}
1 	$\frac{bh^2}{6}$ $\frac{bh^2}{6}$	$\frac{h^2}{12}$	$\frac{1+\frac{h}{6\rho}}{1+\frac{h}{2\rho}}$	$\frac{1-\frac{h}{6\rho}}{1-\frac{h}{2\rho}}$
2 	$\frac{\pi d^3}{32}$ $\frac{\pi d^3}{32}$	$\frac{d^2}{16\rho}$	$\frac{1+\frac{d}{8\rho}}{1+\frac{d}{2\rho}}$	$\frac{1-\frac{d}{8\rho}}{1-\frac{d}{2\rho}}$
3 	$\frac{bh^2}{24}$ $\frac{bh^2}{24}$	$\frac{h^2}{18\rho}$	$\frac{1+\frac{h}{12\rho}}{1+\frac{2h}{3\rho}}$	$\frac{1-\frac{h}{6\rho}}{1-\frac{1}{3}\frac{h}{\rho}}$
4  $\frac{a}{b}=n$	$\frac{n^2+4n+1}{2n+2}\cdot\frac{bh^2}{12}$ $\frac{n^2+4n+1}{2n+1}\cdot\frac{bh^2}{12}$	$\frac{n^2+4n+1}{(n+1)^2}\cdot\frac{h^2}{18\rho}$	$\frac{1+\frac{n^2+4n+1}{(n+1)(n+2)}\cdot\frac{h}{6\rho}}{1+\frac{n+2}{n+1}\cdot\frac{h}{3\rho}}$	$\frac{1-\frac{n^2+4n+1}{(n+1)(2n+1)}\cdot\frac{h}{6\rho}}{1-\frac{2n+1}{n+1}\cdot\frac{h}{3\rho}}$

Appendix 5

Data for Fatigue Strength Analyses

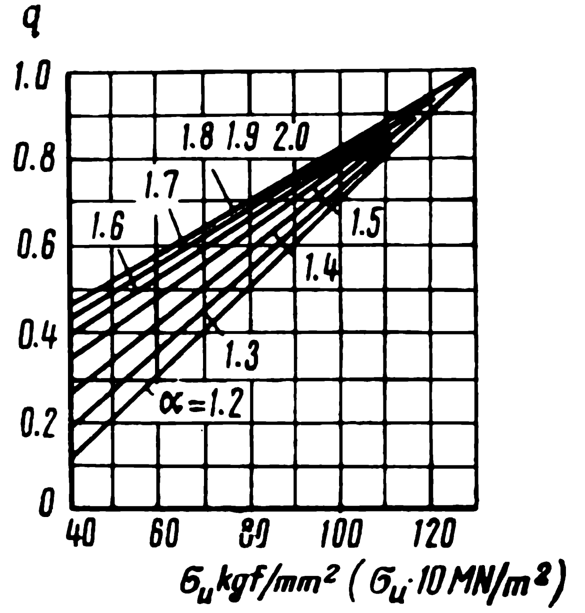


Fig. 1. Graph of the sensitivity of factors of materials to stress concentration for steel

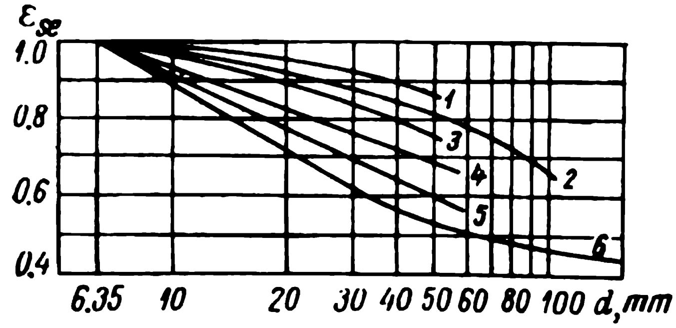


Fig. 2. Graph of size effect factors in bending:

1—carbon steel, smoothly polished; 2—carbon steel, smoothly ground; 3—alloy steel, smoothly polished; 4—alloy steel, smoothly ground; carbon steel with stress concentration; 5—alloy steel with moderate stress concentration ($\alpha^{\sigma}_{eff}<2$); 6—structural steel ($\sigma_u<65$ kgf/mm² or $\sigma_u<650$ MN/m²), shaft with press-fitted part made of un-pressworked steel; at $d<60$ mm—alloy steel with intense stress concentration

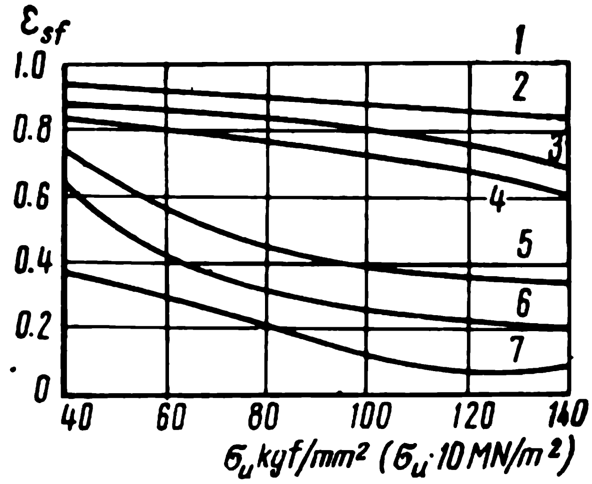


Fig. 3. Graph of surface quality factors for steel:

1—well polished; 2—roughly polished; 3—fine ground or fine turned; 4—rough ground or rough turned; 5—test in fresh water with stress concentration; 6—test in fresh water with no stress concentration; 7—test in sea water with no stress concentration

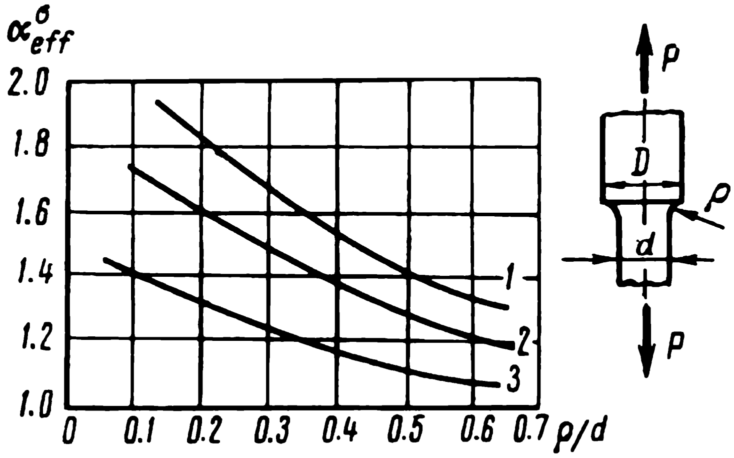


Fig. 4. Graph of effective stress concentration factors for stepped shafts in tension (compression) with the ratio $\frac{D}{d}=2$ at $d = 30$ to 50 mm:

1—for steel with $\sigma_u=120$ kgf/mm² or $\sigma_u=1200$ MN/m²; 2—for steel with $\sigma_u=80$ kgf/mm² or $\sigma_u=800$ MN/m²; 3—for steel with $\sigma_u=40$ kgf/mm² or $\sigma_u=400$ MN/m²

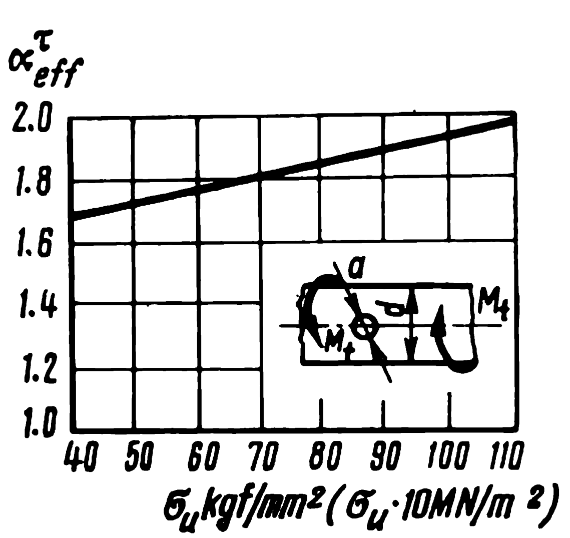


Fig. 5. Graph of effective stress concentration factors for shafts with a transverse hole, subject to torsion:

$\frac{a}{d} = 0.05$ to 0.25; $\tau = \frac{M_t}{W_{t\cdot net}}$ at $d = 30$ to 50 mm

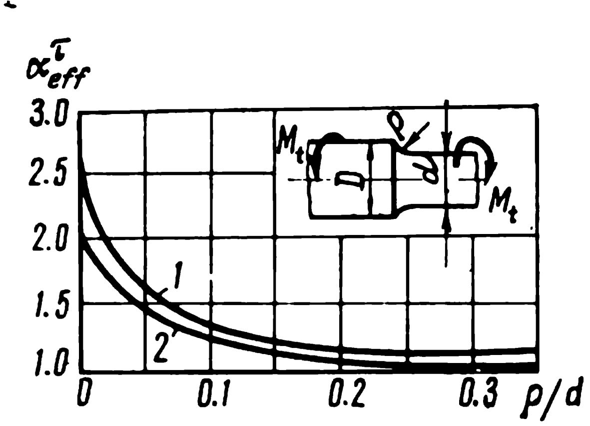


Fig. 6. Graph of effective stress concentration factors for stepped shafts in torsion with the ratio $\frac{D}{d} = 2$ at $d = 30$ to 50 mm:

1—for steel with σ_u=120 kgf/mm² (1200 MN/m²); 2—for steel with σ_u=50 kgf/mm² (500 MN/m²)

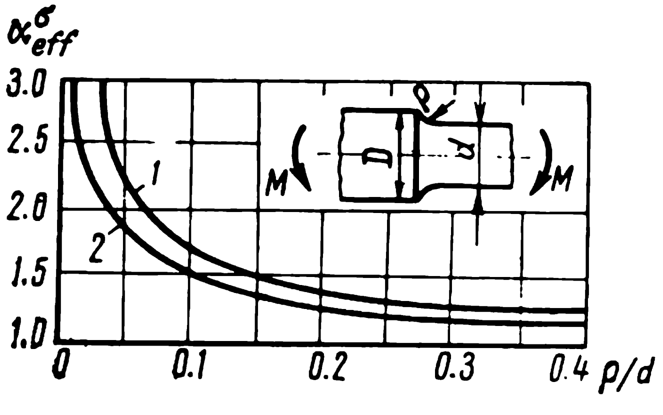


Fig. 7. Graph of effective stress concentration factors for stepped shafts in bending with the ratio $\frac{D}{d} = 2$ at $d = 30$ to 50 mm:

1—for steel with σ_u=120 kgf/mm² (1200 MN/m²); 2—for steel with σ_u=50 kgf/mm² (500 MN/m²)

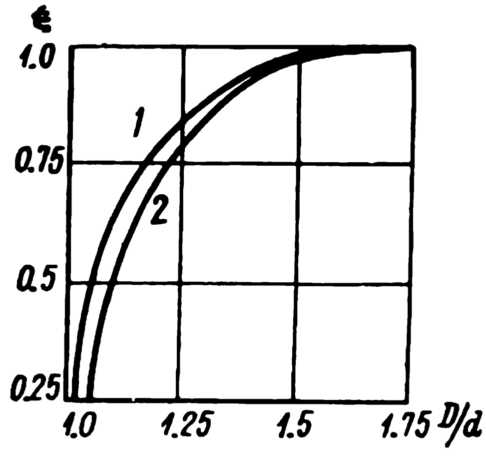


Fig. 8. Graph of correction factors for the ratio $\frac{D}{d} < 2$ (Figs. 6 and 7):

1—bending; 2—torsion

If $\frac{D}{d} < 2$, then α_{eff} should be determined by the formula

$$\alpha_{eff} = 1 + \xi(\alpha_{eff_0} - 1)$$

where α_{eff_0} is the factor found in the curves of Figs. 6 and 7.

ANSWERS TO PROBLEMS

9. -0.15 mm; 2.75 J. **10.** (a) 0.5 mm: 3.33 J; (b) 0; 0.833 J. **11.** 1 mm; 120 kgf-cm. **12.** ~ 0.531 mm; ~ 56.8 kgf-cm. **13.** ~ 0.833 mm; ~ 76.7 kgf-cm. **14.** ~ 1.067 mm; 282.7 kgf-cm. **15.** ~ 0.87 mm; ~ 4.35 J; **16.** ~ 1.36 mm; ~ 27.2 J. **17.** $\frac{4\mu}{E}\cdot\frac{P}{\pi d^2}$. **18.** $\frac{Ea}{P}\ \Delta b$. **19.** 0.09 mm²; 0.003 mm. **20.** $\frac{4}{3}\times$ $\times\frac{1-2\mu}{E}\cdot\frac{P}{\pi d^2}$. **21.** $3\ \frac{1-2\mu}{E} Pl$. **22.** $2\ \frac{1-2\mu}{E} Pl$. **23.** $\frac{2}{3}\cdot\frac{1-2\mu}{E} Pl$. **24.** 3750 kgf. **25.** $\delta_y=\frac{Pa}{8EF}(1+3\sqrt{3})$; $\delta_x=\frac{Pa}{8EF}(3-\sqrt{3})$; $\sigma_I=\frac{\sqrt{3}}{2}\times$ $\times\frac{P}{F}$; $\sigma_{II}=\frac{P}{2F}$. **26.** $\delta_y\cong 1.36$ mm; $\delta_x\cong 0.53$ mm; ~ 440; ~ 470 kgf/cm². **27.** $\frac{(2P+\sqrt{2}\,qa)a}{EF}$; $\frac{P+\sqrt{2}\,qa}{\sqrt{2}\,F}$; $\frac{P}{\sqrt{2}\,F}$. **28.** $\frac{Pa}{EF\cos^2\beta}(\sin^2\beta+2)$; $\frac{P}{F}\times$ $\times\tan\beta$; $\frac{P}{F\cos\beta}$. **29.** $\delta_x=\frac{\sqrt{2}}{2}\cdot\frac{Pa}{EF}$; $\delta_y=\frac{Pa}{2EF}(2+\sqrt{2})$; $\frac{P}{\sqrt{2}\,F}$; $\frac{P}{F}$. **30.** $\delta_x=\frac{3}{2}\cdot\frac{Pa}{EF_1}$; $\delta_y=\frac{2Pa}{EF_2}$; $\frac{P}{F_1}$; $\frac{3}{2}\cdot\frac{P}{F_1}$. **31.** $\frac{(2P+qa)\,a}{4EF}$; $\frac{2P+qa}{4F}$. **32.** $\delta_x=\delta_y=\frac{Pa}{EF}$; $\frac{P}{F}$. **33.** $\frac{Pa}{E_2F_2}$; $\frac{P}{2F_2}$. **34.** $\frac{10Pa}{E_1F_1}$; $\frac{2P}{F_1}$; $\frac{4P}{F_1}$. **35.** $\delta_x\cong 5.4$ mm; $\delta_y\cong 1.8$ mm; ~ 900; ~ 1800; ~ 1200 kgf/cm². **36.** 8.5 mm; 1000; 500 kgf/cm². **37.** 3.9 mm; 30; 150 MN/m². **38.** $\delta_y=0.7$ mm; $\delta_x=0.35$ mm; 80; 40; 20 MN/m². **39.** $\delta_y=2.125$ mm; $\delta_x=0.08$ mm; 100; 150; 75 MN/m². **40.** ~ 5.6 mm; ~ 142; 164 MN/m². **41.** $\frac{P}{\sqrt{3}\,[\sigma]}$; $\frac{P}{2\sqrt{3}\,[\sigma]}$; $\frac{P}{2[\sigma]}$. **42.** $\frac{P}{[\sigma]}$. **43.** $\frac{3P}{2[\sigma]}$; $\frac{P}{[\sigma]}$. **44.** $\frac{7qa}{2[\sigma]}$; $\frac{7\sqrt{2}\,qa}{4[\sigma]}$. **45.** 5 cm². **46.** 5.33 cm². **47.** 4.9 cm²; 8 cm². **48.** $F_1=F_2=\frac{7P}{8[\sigma]}$; $F_3=\frac{7}{4}\cdot\frac{P}{[\sigma]}$. **49.** 9 kN. **50.** 64 kN. **51.** 176.5 kgf. **52.** 2.07 m. **53.** 894 kgf/cm². **54.** $d=\sqrt{\frac{P}{\pi([\sigma]-\gamma l)}}$; $\Delta a=$ $=\frac{l}{4E}\left(\frac{2P}{F}+\gamma l\right)$. **55.** $\Delta a=\frac{a}{E}\left[\frac{P}{F}+\gamma\left(l-\frac{a}{2}\right)\right]$. **56.** 3 m²; 617 m³; ~ 0.4 cm; **57.** 2500 cm²; 2620 cm²; 19.6 tnf. **58.** 600; 800; -800 kgf/cm². **59.** -277; 300; 262; kgf/cm². **60.** 130; -20; -120; 5 kgf/cm². **61.** $\max\sigma_I=$ $=\frac{22}{7}\gamma a$; $\min\sigma_{III}=-\frac{31}{14}\gamma a$. **62.** ~ 1040; ~ 1560 kgf/cm². **63.** ~ 1164; ~ 875 kgf/cm². **64.** ~ 61; ~ 87 kgf/cm². **65.** ~ 70.7; ~ 141.4 MN/m². **66.** 120 MN/m². **67.** ~ 10; ~ 7.1 MN/m². **68.** 25; 1000 kgf/cm². **69.** ~ 390; ~ 312 kgf/cm². **70.** ~ 903; ~ 903; ~ 452 kgf/cm². **71.** ~ 33 mm. **72.** ~ 35.7 cm. **73.** $\frac{3}{4}\cdot\frac{P}{[\sigma]}$. **74.** ~ 739; ~ 261 kgf/cm². **75.** 300; 200 kgf/cm². **76.** ~ 1091; ~ 545 kgf/cm². **77.** $\frac{\Delta E}{5a}\sin\beta$; $\frac{2\Delta E}{5a}\sin\beta$. **78.** ~ 16.4; ~ 15.4; ~ 16.4 MN/m².

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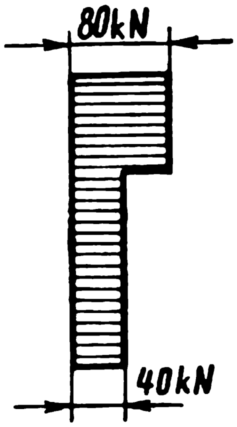

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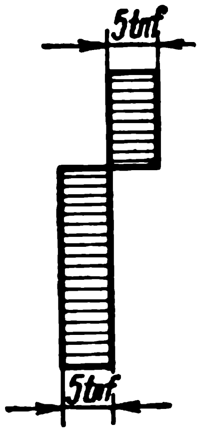

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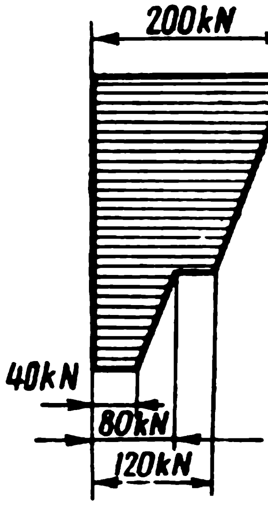

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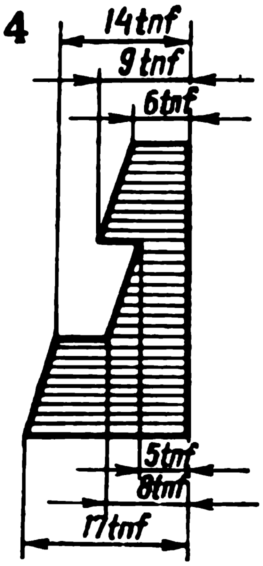

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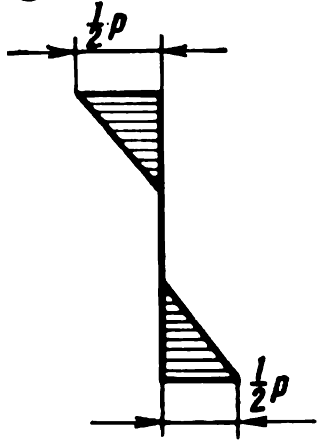

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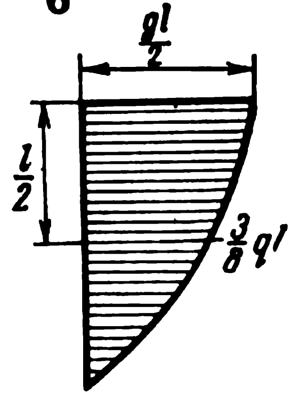

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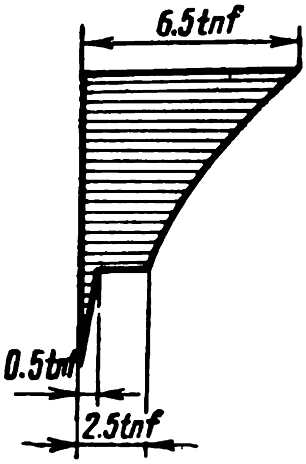

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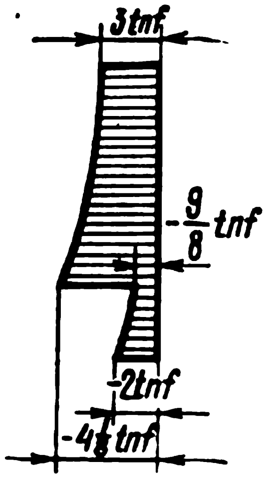

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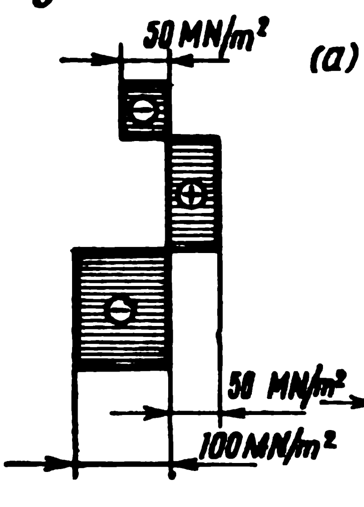

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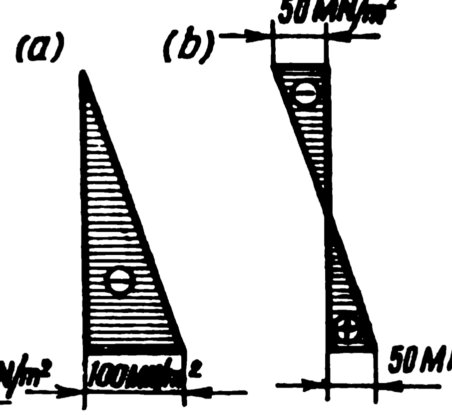

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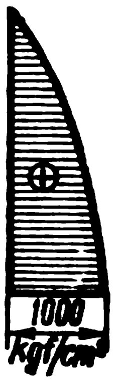

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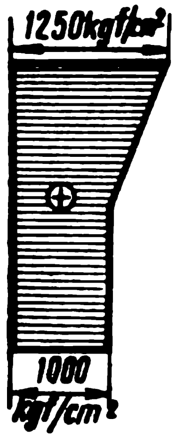

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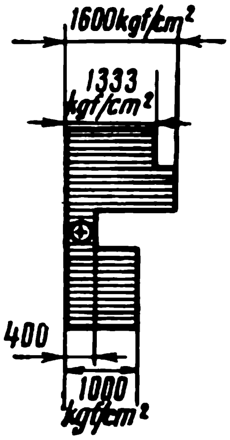

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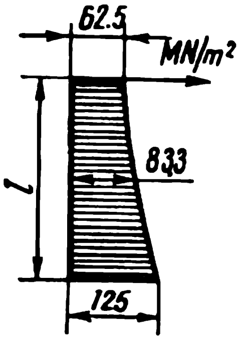

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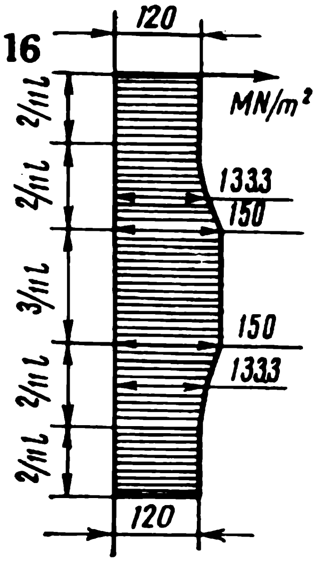
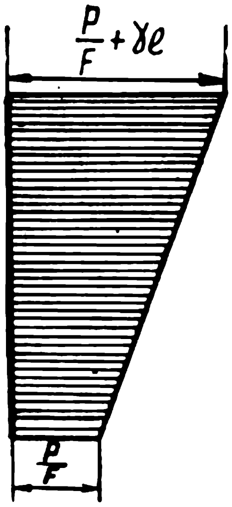


79. 20; 40; 60 MN/m^2. **80.** ~ 764; ~ 917; ~ 1070 kgf/cm^2. **81.** ~ 87; ~ 302 kgf/cm^2. **82.** ~ 172; ~ 98; 1000 kgf/cm^2. **83.** ~ 596; ~ 43 kgf/cm^2. **84.** 35 MN/m^2. **85.** ~ 513; ~ 726; 1025 kgf/cm^2. **86.** ~ 6; ~ 10.4; ~ 10.4 MN/m^2. **87.** ~ 106 MN/m^2. **88.** 800; 600; 1400 kgf/cm^2; 120; 200; 160 kgf/cm^2. **89.** 800; 450 kgf/cm^2; 144; 192 kgf/cm^2. **90.** 500; 1500 kgf/cm^2; 576; 192 kgf/cm^2. **91.** 400; 450; 550 kgf/cm^2; 288; 432; 144 kgf/cm^2. **92.** 50; 37.5; 75; 62.5 MN/m^2;

55



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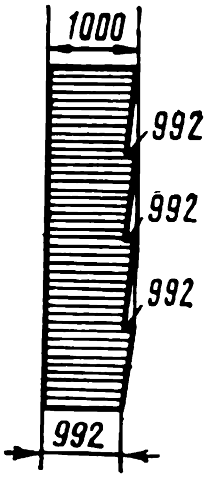

48; 36; 12; 6 MN/m^2. **93.** 130; 10; 50 MN/m^2; 16; 32; 16 MN/m^2. **94.** ~ 19.1; 27.5 tnf; ~ 671; 322; 168 kgf/cm^2. **95.** $P = P' \cong 14.3$ tnf; $\sigma_{I_t} \cong 147$ kgf/cm^2; $\sigma_{II_t} = \sigma_{III_t} \cong 93$ kgf/cm^2. **96.** ~ 9.06 tnf; ~ 11.53 tnf; ~ 220; ~ 294; 280 kgf/cm^2. **97.** ~ 7.14 tnf; 8.74 tnf; ~ 93; 93; 54 kgf/cm^2. **98.** 5.2 tnf; 6.4 tnf; 72 kgf/cm^2. **99.** ~ 63 kN; ~ 82 kN. **100.** 2.05 tnf; 2.8 tnf; ~ 300; 225 kgf/cm^2. **101.** ~ 28.3 kN; 32 kN; ~ 16.3; ~ 8.1; 5.4 MN/m^2. **102.** 1000 kgf/cm^2; $q = 100$ kgf/cm. **103.** $\sigma_I = \frac{2E}{D}\delta \tan\alpha$; $\frac{2E}{(D+b_1)}\delta\tan\alpha$; $q \cong \frac{4a\delta\tan\alpha}{D^2}(b_1E_1 + b_2E_2)$. **104.** (a) 217; 290 kgf/cm^2; 17.4 kgf/cm^2; (b) 369°. **105.** 60 kgf/cm^2. **106.** $E\left(\alpha\Delta t - \frac{\Delta}{D}\right)$. **107.** $\sigma_I = \sigma_{III} = 1333$ kgf/cm^2.

Problem No.	τ_1	τ_2	τ_3	σ_α	τ_α	σ_α	τ_α	σ_α	τ_α	p_0	σ_0	τ_0
				on areas parallel to								
				axis 1		axis 2		axis 3				
	kgf/cm²											
108	0	±600	±600	0	0	300	520	900	520	694	400	566
109	±600	±600	0	—300	+520	−900	520	0	0	694	−400	566
110	±200	±600	±400	300	173	300	520	1000	346	730	533	500
111	±500	±750	±250	−250	+433	−625	649	375	216	645	—167	624
112	±100	±300	±200	−450	87	—450	260	—100	173	416	−333	249
113	±400	±400	0	600	346	200	346	800	0	654	533	377
114	0	±500	±500	−1000	0	−750	433	−250	433	817	−667	471
115	±100	±300	±200	350	87	350	260	700	173	529	467	249
116	±400	±700	±300	0	346	—250	606	650	260	589	133	573
117	±200	±500	±300	−300	173	−350	433	250	260	432	−133	411
118	±100	±300	±200	−850	87	—850	260	−500	173	775	−733	249
119	±450	±450	0	375	390	−75	390	600	0	519	300	424

Problem No.	τ_1	τ_2	τ_3	σ_α	τ_α	σ_α	τ_α	σ_α	τ_α	p_0	σ_0	τ_0
				on areas parallel to								
				axis 1		axis 2		axis 3				
	MN/m²											
120	0	±50	±50	−80	0	—55	43.3	−5	43.3	66.5	−46.7	47.1
121	0	0	0	50	0	50	0	50	0	50	50	0
122	0	0	0	−70	0	−70	0	−70	0	70	—70	0

Problem No.	ε_1	ε_2	ε_3	$\frac{\Delta V}{V}$	U_0	U_0^{dist}	U_0^{vol}
	%				kgf-cm/cm³		
108′	0.060	−0.018	—0.018	0.024	0.36	0.312	0.048
109′	0.018	0.018	−0.06	—0.024	0.36	0.312	0.048
110′	0.054	0.002	−0.024	0.032	0.328	0.243	0.085
111′	0.04	0.0075	−0.0575	—0.010	0.388	0.379	0.009
112′	0.015	—0.011	−0.024	—0.020	0.094	0.061	0.033
113′	0.028	0.028	—0.024	0.032	0.224	0.139	0.085
114′	0.03	—0.035	—0.035	—0.04	0.350	0.217	0.133
115′	0.031	0.005	−0.008	0.028	0.126	0.061	0.065
116′	0.046	0.007	−0.045	0.008	0.326	0.321	0.005
117′	0.032	—0.007	—0.033	—0.008	0.170	0.165	0.005
118′	0.007	—0.019	−0.032	—0.044	0.222	0.061	0.161
119′	0.0255	0.0255	−0.033	0.018	0.202	0.175	0.025

Problem No.	ε_1	ε_2	ε_3	$\frac{\Delta V}{V}$	U_0	U_0^{dist}	U_0^{vol}
	%				J/m³		
120′	0.034	—0.031	—0.031	—0.028	28.2×10^3	21.7×10^3	6.5×10^3
121′	0.01	0.01	0.01	0.03	7.5×10^3	0	7.5×10^3
122′	—0.014	—0.014	—0.014	—0.042	14.7×10^3	0	14.7×10^3

Problem No.	σ_{eq_I}	$\sigma_{eq_{II}}$	$\sigma_{eq_{III}}$	$\sigma_{eq_{IV}}$	σ_{eq_V}
	kgf/cm²				
108″	1200	1200	1200	1200	1200
109″	—	360	1200	1200	600
110″	1200	1080	1200	1060	1200
111″	500	800	1500	1320	1000
112″	—	300	600	530	300
113″	800	560	800	800	800
114″	—	600	1000	1000	500
115″	800	620	600	530	700
116″	800	920	1400	1220	1100
117″	400	640	1000	870	700
118″	—	140	600	530	100
119″	600	510	900	900	750

Problem No.	σ_{eq_I}	$\sigma_{eq_{II}}$	$\sigma_{eq_{III}}$	$\sigma_{eq_{IV}}$	σ_{eq_V}
	MN/m²				
120″	20	68	100	100	60
121″	50	20	0	0	25
122″	—	—	0	0	0

123. (a) 6.6×10^{-5} mm²/kgf; 1.51×10^4 kgf/mm².
(b) 9.6×10^{-5} mm²/kgf; 1.04×10^4 kgf/mm².
(c) 14.6×10^{-5} mm²/kgf; 0.685×10^4 kgf/mm².

124. $0;\ -\mu p;\ -p;\ -\frac{1-\mu^2}{E}pa;\ \frac{\mu(1+\mu)}{E}pb.$ **125.** $0;\ -\frac{4\mu_2 pE_1F_1}{(E_2ab+4E_1F_1)};$ $-p;\ -\frac{(1-2\mu)\,p\,(4\mu_2E_1F_1+E_2ab+4E_1F_1)}{E_2(E_2ab+4E_1F_1)}$. **126.** $\sigma_1=\sigma_2=\frac{-\mu p}{(1-\mu)+\frac{d}{(d_1-d)}\frac{E}{E_1}};$

$\sigma_3=-p$. 127. $\sigma_1=\sigma_2=\frac{-\mu p}{1-\mu}$; $\sigma_3=-p$; $\Delta a=\frac{(\mu+2\mu^2-1)\,pa}{E(1-\mu)}$; $U=$ $=\frac{(1+\mu)(1-2\mu)\,p^2}{2E(1-\mu)}$. 128. $\sigma_1=\sigma_2=1000$ kgf/cm²; $\sigma_3=0$; $\tau_1=\tau_2=500$ kgf/cm²; $\tau_3=0$; $\Delta D=0.35$ mm. 129. 2.5 mm. 130. $\frac{pD}{2[\sigma]\cos\alpha}$; $\frac{pD^2\tan\alpha}{8[\sigma]}$. 131. 15 mm; 11.1 cm². 132. $\sigma_1^{sph}=\sigma_2^{sph}=75$ kgf/cm²; $\sigma_3^{sph}=0$ (for the inside surface $\sigma_3^{sph}=$ $=-0.3$ kgf/cm²); $\sigma_1^{cyl}=100$ kgf/cm²; $\sigma_2^{cyl}=83$ kgf/cm²; $\sigma_3^{cyl}=0$ (for the inside surface $\sigma_3^{cyl}=-0.2$ kgf/cm²); $\sigma_b=67$ kgf/cm². 133. $\frac{\gamma y\tan\alpha}{2\delta\cos\alpha}\left(h-\frac{2}{3}y\right)$; $\frac{\gamma y(h-y)\tan\alpha}{\delta\cos\alpha}$; $\frac{\gamma y(h-y)\tan\alpha}{\delta\cos\alpha}$ for $0<y\leqslant\frac{3}{4}h$. 134. 69, 3; 40 MN/m². 135. 0; ± 80 MN/m². 136. 50; 70; 20 MN/m²; 0.625×10^{-3}; 0.785×10^{-3}; 0.25×10^{-3}. 137. 20; 100; 80; 84.9 MN/m²; 0.25×10^{-3}; 1.25×10^{-3}; 10^{-3}; 1.061×10^{-3}. 138. 500; 0; -500 kgf/cm²; 0.625×10^{-3}; 0.27×10^{-3}. 139. (a) 12.5 kgf-cm; (b) 0.86 kgf-cm. 140. $\sigma_1=-\sigma_3=\frac{2M_t}{\pi D_m^2\delta}$; $\sigma_2=0$; $[M_t]\cong$ $\cong 157$ kgf-m; $(M_t)_y\cong 236$ kgf-m. 141. $d_0\cong 1.59$ cm; $d\cong 1.35$ cm; $\delta\cong 0.74$ cm; $b\cong 1.35$ cm. 142. $d\cong 2.8$ cm; $a=3$ cm; $h=6$ cm; $b=3.75$ cm; $c=1$ cm; $f=1.25$ cm; $k=1.25$ cm. 143. $d=4$ cm; $d_1=5.6$ cm; $d_2\cong 6.9$ cm; $t_1\cong 2$ cm; $t_2\cong 1.4$ cm. 144. $d\cong 3.1$ cm; $d_1\cong 4.37$ cm; $a\cong 1{,}06$ cm; $d_2\cong 4.88$ cm. 145. $b\cong 4.87$ cm; $a\cong 5.97$ cm; $\delta\cong 1.99$ cm; $h\cong 7.97$ cm; $c\cong 2.66$ cm; $l\cong 2.99$ cm. 146. $a\cong 3.16$ cm; $d\cong 1.06$ cm; $c\cong 1.19$ cm; $b\cong 9.46$ cm²; $l\cong 11.56$ cm. 147. $a\cong 3.16$ cm; $c\cong 1.06$ cm; $b_1\cong 3.87$ cm; $b\cong 5$ cm; $d\cong$ $\cong 0.87$ cm. 148. $d\cong 3.57$ cm; $\delta=2.1$ cm; $b\cong 21.4$ cm; $l\cong 13.1$ cm. 149. $d\cong$ $\cong 3.25$ cm; $\delta_1\cong 1.92$ cm; $\delta_2\cong 1.44$ cm; $b\cong 9.75$ cm. 150. $z_c=1.5a$; $y_c=4a$. 151. $z_c=2a$; $y_c=3.9a$. 152. $z_c=2a$; $y_c=3.71a$. 153. $z_c=0.5a$; $y_c=$ $=\frac{\left(\pi-\alpha-\frac{2}{3}\sin\alpha\right)}{\pi-\alpha}a$. 154. $z_c=0.5a$; $y_c=0.6b$. 155. $z_c=17.79$ cm; $y_c=$ $=11.88$ cm. 156. $\frac{4}{15}b^4$. 157. $\frac{7}{96}a^4$; $\frac{5}{96}a^4$. 158. $I_x=I_y=7869$ cm⁴; $I_{xy}=0$. 159. $11.9a^4$; $12.4a^4$. 160. $5.407a^4$; $5.382a^4$. 161. $0.726R^4$; $0.678R^4$. 162. $5.25a^4$; $1.57a^4$. 163. 5791 cm⁴; 11591 cm⁴. 168. $\frac{48M}{\pi d^3}$; $\frac{208M^2 l}{G\pi d^4}$. 169. $\frac{4.06M}{b^3}$; $\frac{11M^2 l}{Gb^4}$. 170. $\frac{80N}{\pi\omega d^3}$ N/m²; $\frac{1552N^2 l}{\pi\omega^2 Gd^4}$ J. 171. $\frac{15.3ml}{d^3}$; $\frac{262m^2l^3}{Gd^4}$. 172. $\frac{16M}{\pi d^3}$; $7.85\,\frac{M^2 l}{Gd^4}$. 173. $\frac{10ml}{d^3}$; $18.85\,\frac{m^2l^3}{Gd^4}$. 174. ~ 4.07 cm. 175. ~ 4 cm. 176. $D\cong 6.84$ cm. $d\cong 3.42$ cm. 177. ~ 3.64 cm. 178. $d_1\cong 2.94$ cm; $d_2\cong 4$ cm; $\varphi\cong 9.2°$. 179. $d_1=d_3\cong 2.94$ cm; $d_2\cong 3.72$ cm; $\varphi\cong 1.05\times10^{-2}$ rad. 180. $a\cong 2.88$ cm; $d\cong 2.94$ cm; $\varphi\cong 0.41°$. 181. $d_1\cong 8$ cm; $d_2\cong 5.9$ cm; $\varphi\cong 2.06°$. 182. $d\cong 2.5$ cm; $\varphi\cong 0.98°$. 183. $d\cong 2.5$ cm; $\varphi\cong 6.7'$. 184. 490 r.p.m.; 126 kgf/cm². 185. ~ 5 hp; 127 kgf/cm². 186. ~ 4.48 kgf-m; ~ 105 kgf/cm². 187. ~ 354 kgf/cm²; 2.38°. 188. ~ 70 kgf-m; $\sim 0.18°$. 189. ~ 363 N-m; ~ 77.2 MN/m²; $\sim 3\times10^{-2}$ rad.

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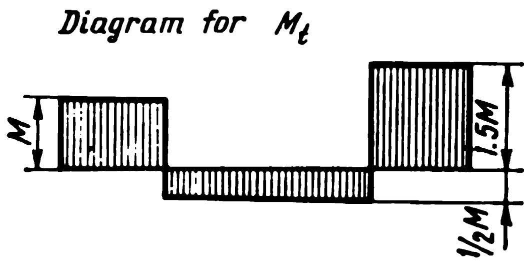

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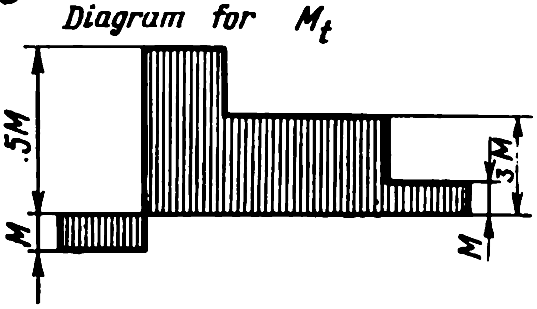

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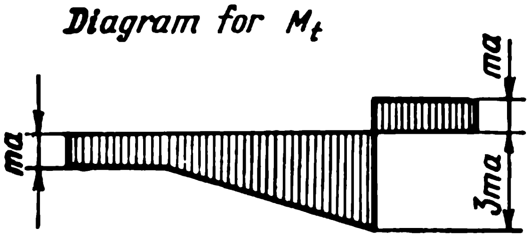

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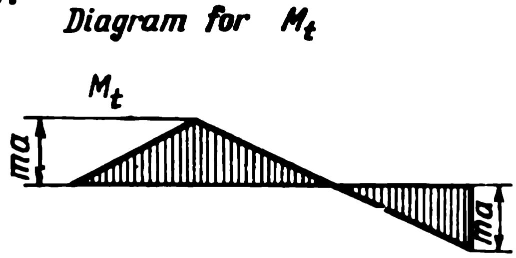

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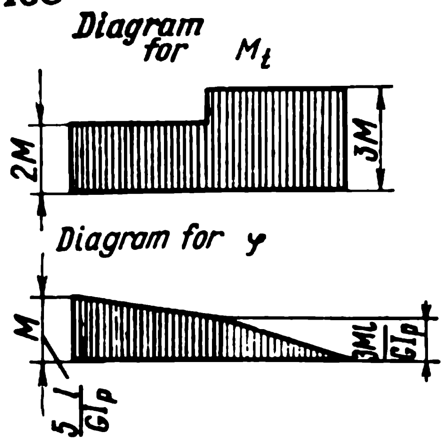

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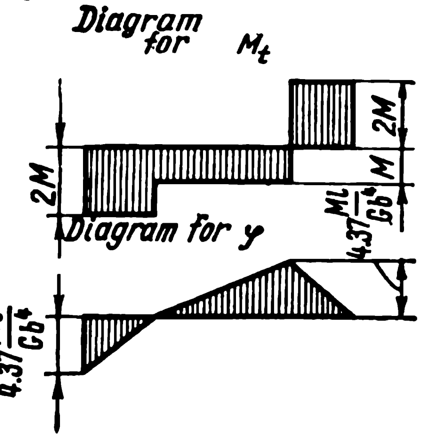

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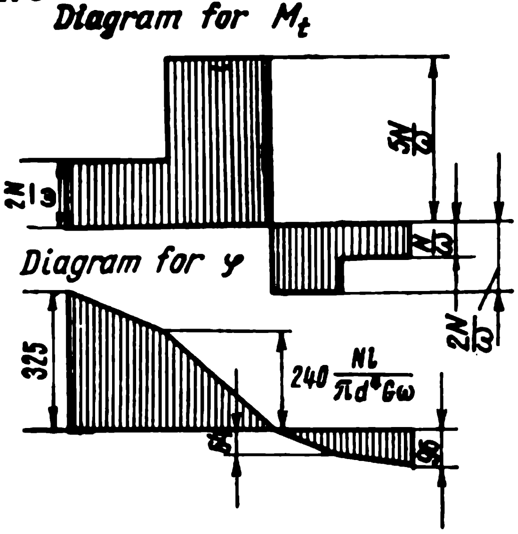

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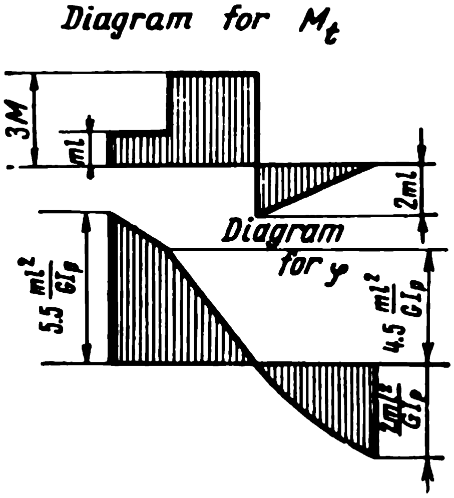
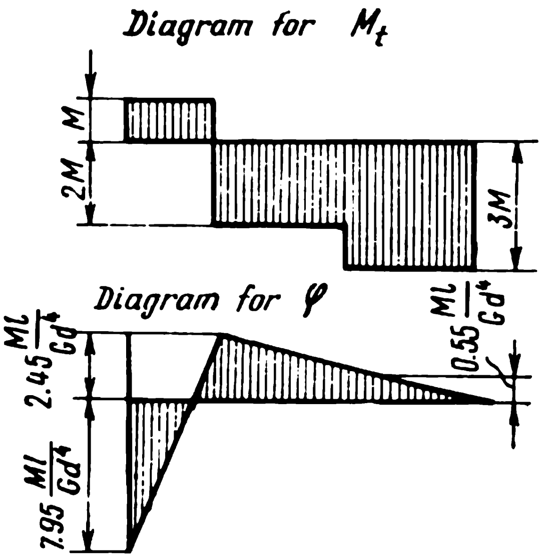


190. 81.25 kgf-m; ~ 318 kgf/cm^2; $\sim 1.64°$. **191.** ~ 24.6 kW; 1.41×10^{-2} rad. **192.** ~ 194 r.p.m.; $\sim 1.18°$. **193.** ~ 448 kgf/cm^2; $\sim 2°$. **194.** 400 kgf/cm^2. **195.** 7; 19; $\frac{1}{3}$; $\frac{2}{3}$. **196.** ~ 35 kgf/cm^2; $\sim 2°$. **197.** ~ 51.3 N-m; ~ 27.5 MN/m^2. **198.** 78.5 kgf; 5 mm (upward); 30 mm (downward). **199.** ~ 425; ~ 123. **200.**

172



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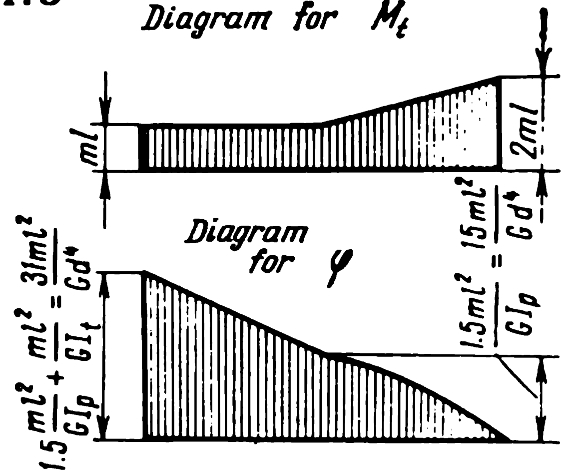
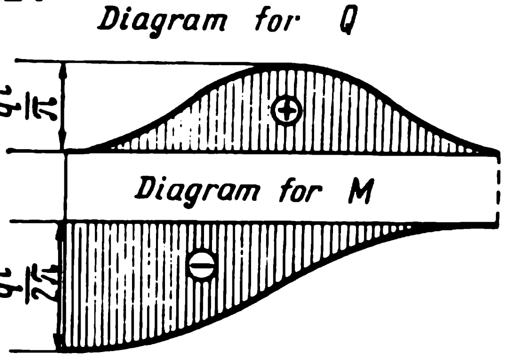


~ 0.98; ~ 1.22. **201.** -693 kgf/cm^2; ~ 400 kgf/cm^2. **202.** $d=\sqrt[3]{\frac{32M}{3\pi[\tau]}}$; $\varphi_A^\circ = \frac{180a[\tau]}{\pi G}\sqrt[3]{\frac{3\pi[\tau]}{32M}}$. **203.** ~ 3.94 cm; $\sim 0.48°$. **204.** ~ 1.97 cm; 0. **205.** ~ 2.17 kN-m; 1.48×10^{-2} rad; ~ 16.1 J. **206.** ~ 126 kgf-m; $\sim 0.215°$; ~ 23.5 kgf-cm. **207.** ~ 573 kgf-cm^2; $\sim 0.41°$. **208.** ~ 780 kgf/cm^2; $\sim 1.19°$.

224



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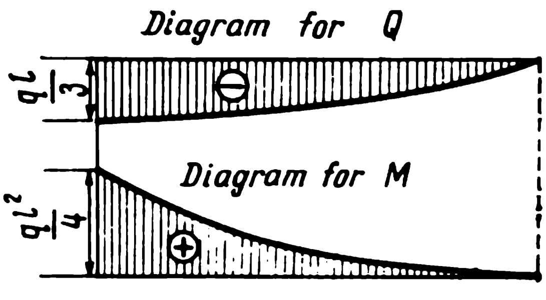

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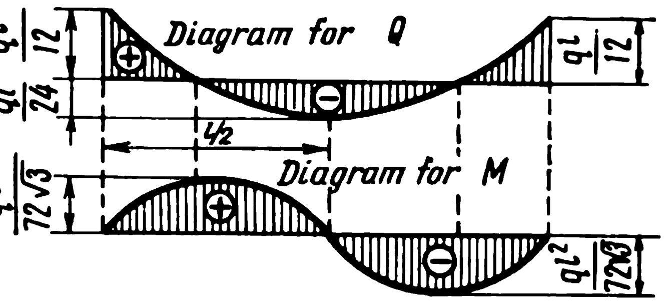
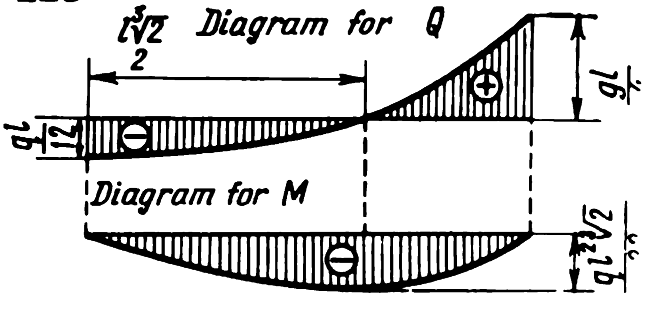


228



Problem No.	Q_{max}	M_{max}
223	$\frac{ql}{4}$	$\frac{ql^2}{6}$
224*	$\frac{ql}{\pi}$	$\frac{ql^2}{2\pi}$
225*	$\frac{ql}{3}$	$\frac{ql^2}{4}$
227*	$\frac{ql}{12}$	$\frac{ql^2}{12\sqrt{3}}$
228*	$\frac{ql}{4}$	$\frac{\sqrt[3]{2}\,ql^2}{32}$
229*	$\frac{ql}{\pi}$	$\frac{ql^2}{\pi^2}$
230*	$\frac{ql}{2\pi}$	$\frac{ql^2}{4\pi^2}$
242	$\frac{qb(l+a)}{2l}$	$\frac{qb^2(l+a)^2}{8l^2}$
243	4 tn f	4 tn f·m
244	$\frac{3}{2}qa$	$\frac{17}{16}qa^2$

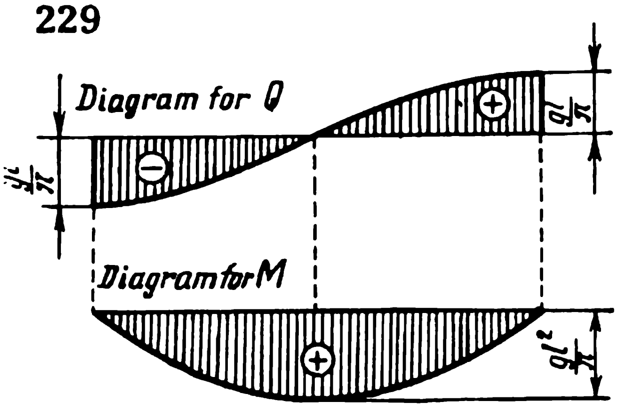
Problem No.	Q_{max}	M_{max}
245	$\frac{7}{6}qa$	$\frac{5\sqrt{5}}{9}qa^2$
247	$\frac{3}{4}qa$	$\frac{qa^2}{4}$
248	qa	qa^2
249	$2qa$	qa^2
250	$2P$	$\frac{Pl}{3}$
252	$0.462ql$	$0.0535ql^2$
	(at $a=0.21l$)	
253	qa	$\frac{3}{2}qa^2$
254	$\frac{3}{2}P$	$\frac{Pl}{6}$
255	$\frac{ql}{2}$	$\frac{ql^2}{16}$
	$\left(\text{at } a=\frac{\sqrt{2}}{4}l\right)$	
256	$\frac{5}{4}qa$	qa^2

Problem No.	Q_{max}	M_{max}
258	$2.25qa$	$3.531qa^2$
259*	$0.21ql$	$0.0128ql^2$
	(at $a=0.275l$)	
262*	$2qa$	$1.167qa^2$
263	$\frac{3}{4}qa$	$\frac{qa^2}{2}$
264	$\frac{qa}{2}$	$\frac{qa^2}{8}$
265	4 tn f	3 tn f·m
266	qa	qa^2
267	P	Pa
268	ql	$\frac{ql^2}{2}$
269*	P	Pa
270*	P	Pa

Note. For the problems marked with an asterisk diagrams representing Q and M are given.

209. ~ 33.3 MN/m^2; $\sim 6.2\times10^{-3}$ rad. **210.** $b\cong 1.28a$. **211.** 270 kgf/cm^2 **212.** ~ 390 N-m; $\sim 2.81\times10^{-3}$ rad. **213.** 12.7 mm. ~ 0. **214.** ~ 3.17 cm; $\sim 5.8\times10^{-3}$ rad. **215.** $\tau_{\max_i}=M\dfrac{d_i}{2\sum\limits_{i=1}^{n} I_{p_i}}$. **216.** ~ 306; ~ 318 kgf/cm^2. **217.** ~ 750 kgf/cm^2; ~ 37 kgf/cm^2; $\sim 1.14°$. **218.** ~ 1000 kgf/cm^2; ~ 100 kgf/cm^2. **219.** $\tau_{\max}\cong 0.052\dfrac{M}{\pi d^3}$; $\sigma_I\cong 3.3\dfrac{M}{\pi d^3}$; $\sigma_{II}\cong 0.45\dfrac{M}{\pi d^3}$; $\varphi_{AB}\cong 0.57\dfrac{Ma}{G\pi d^4}$.

229



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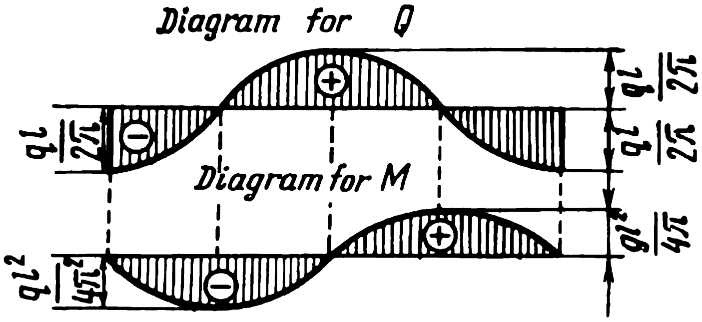
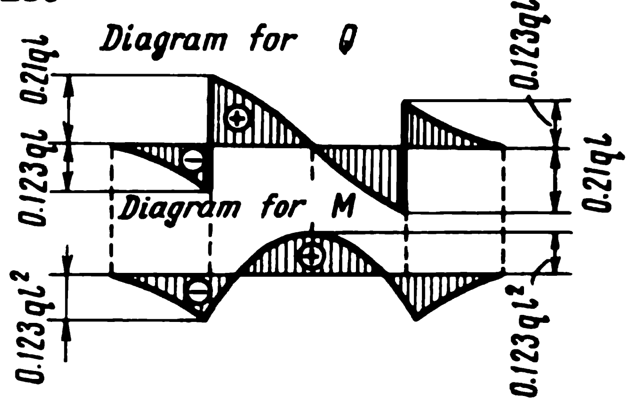


220. ~ 402 kgf/cm^2; ~ 134 kgf/cm^2; $\sim 0.213°$. **221.** ~ 468; ~ 742; ~ 883 kgf/cm^2; $\sim 2.5°$; $\sim 2.13°$. **286.** $0.082H^3$. **287.** $0.0183a^3$. **288.** $\dfrac{13}{60}b^3$. **289.** $\dfrac{a^3}{6}(1-1.5k^2-4k^4)$.

290. 229 cm^3. **291.** $0.0597a^3$. **292.** $0.0706a^3$. **293.** 359.3 cm^3. **294.** 79 cm^3. **295.** $0.0425a^3$. **296.** $0.147a^3$. **297.** $0.74R^3$. **298.** 205 cm^3. **299.** ~ 737 cm^3. **300.** ~ 663 cm^3. **301.** 0.560 cm^3. **302.** 0.470 cm^3. **303.** 1.865 cm^3. **304.** 0.780 cm^3. **305.** 1.478 cm^3. **306.** 2770 cm^3. **307.** 1196 kgf/cm^2. **308.** 91.7 kgf/cm^2. **309.** 865; -432 kgf/cm^2. **310.** $\dfrac{4\sqrt{3}}{\pi d^3}M$. **311.** $\sim 3.83\dfrac{Pa}{d^3}$; 0. **312.** $12\sigma_0$. **313.** $\sim 0.0894\sigma_0$. **314.** $a=5$ cm. **315.** $b=19$ cm. **316.** No. 24a. **317.** $D=29.3$ cm. **318.** $b=13.4$ cm. **319.** $n=4$ bars. **320.** No. 33. **321.** No. 14a. **322.** No. 45. **323.** $a=3.42$ cm. **324.** 16 cm. **325.** No. 50. **326.** $a=1$ cm. **327.** $b=20$ cm. **328.** $h_1=177.7$ cm; $h_2=38.3$ cm. **329.** $a=1$ cm. **330.** $P=1067$ kgf. **331.** $P=1400$ kgf. **332.** $P=$ $=453$ kgf. **333.** $q=330$ kgf/m. **334.** $P\cong 2.14$ tnf. **335.** $P\cong 2.16$ tnf. **336.** $M\cong 1.92$ kN·m. **337.** $M\cong 3.44$ kN-m. **338.** $q=9.36$ kN/m. **339.** $q=$ $=3.75$ kN/m. **340.** $q=38.4$ kgf/cm. **341.** $P=a^2\sigma_0$. **342.** $P=1$ tnf. **352.** $d=8$ cm; $\tau_{\max}=16$ kgf/cm^2. **353.** $h=24.2$ cm; $\tau_{\max}=5.6$ kgf/cm^2. **354.** No. 27a; $\tau_{\max}=$ $=125$ kgf/cm^2. **355.** $l=3$ m; $\tau_{\max}=60.5$ kgf/cm^2. **356.** 2.79 MN/m^2. **357.** $H\cong$ $\cong 8.0$ cm; $\tau_{\max}=1.82$ MN/m^2. **358.** $1.2a$. **359.** ~ 6.65 cm. **360.** $\sim 1.034a$. **361.** $\sim 0.702a$. **362.** $-0.5827r$. **363.** $-0.762a$. **364.** 1745; -417 kgf/cm^2; $\alpha=26°$. **365.** 11.3; -120.3 kgf/cm^2. **366.** 370; -5 kgf/cm^2. **367.** 26; -0.4 kgf/cm^2. **368.** $P=49$ tnf. **369.** No. 24. **370.** No. 40. **371.** $b=15.7$ cm. **372.** No. 55. **373.** No. 50. **374.** 1.59. **375.** 1.4. **376.** 3. **377.** 1.28. **378.** 1.7. **379.** 2. **380.** $b=$ $=20$ mm; $b'=17.5$ mm. **381.** $d=10$ cm; $d'=8.4$ cm. **382.** $q=2.93$ tnf/m; $q'=4.16$ tnf/m. **383.** $P=1.47$ tnf; $P'=1.66$ tnf. **384.** $q=10.7$ kN/m; $q'=$

259



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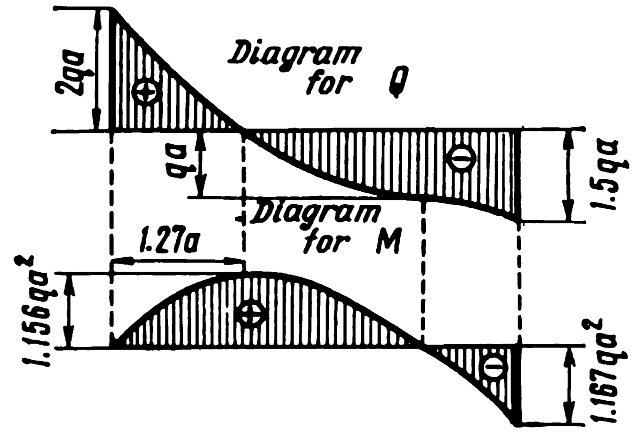

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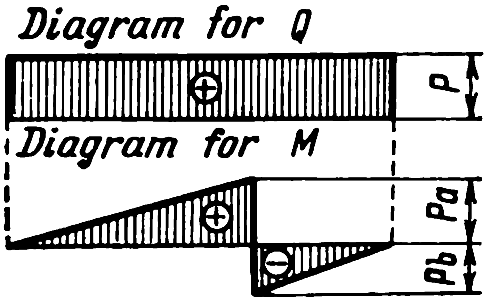

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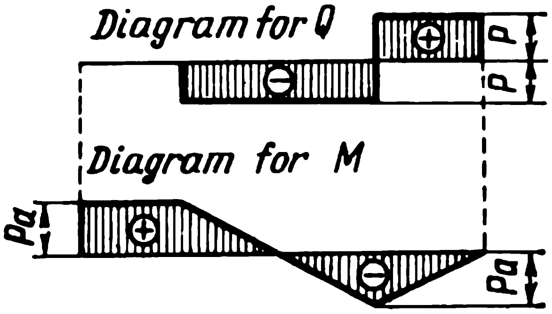

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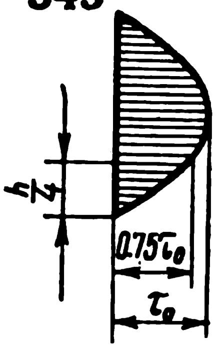

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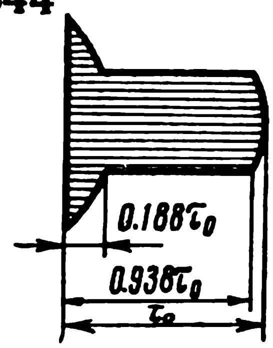

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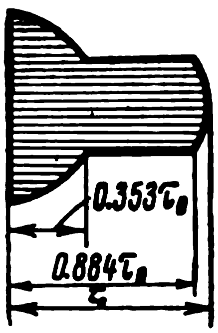

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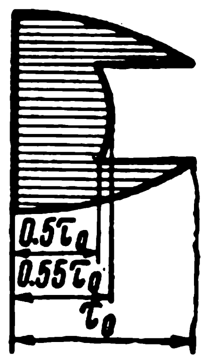

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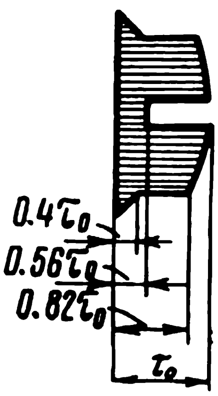

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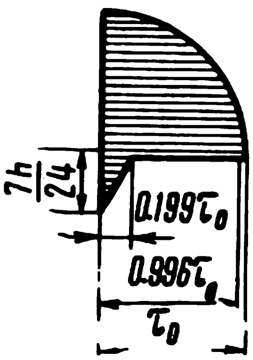

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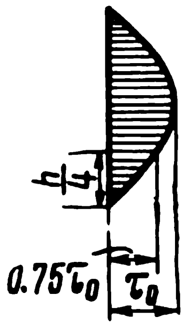

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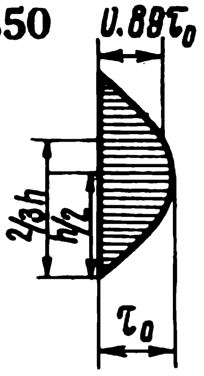

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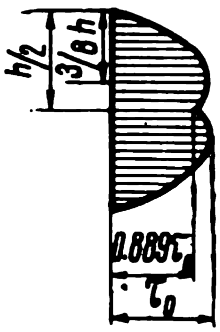


$= 12.2$ kN/m. **385.** $P = 8.05$ kN; $P' = 14.56$ kN. **387.** Less by 0.05%. **388.** 100%; 60%. **389.** 33%; 28%. **390.** 25%; 22%. **391.** 27.3%; 23.8%.

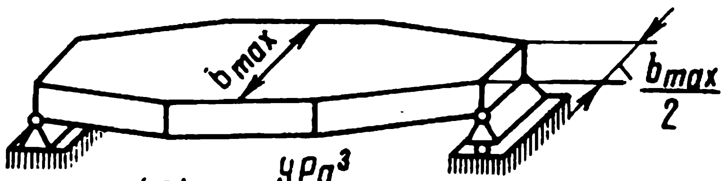
Note. In the answers to Problems 392-443 numerical coefficients at $\frac{Ma^2}{EI}$, $\frac{Pa^3}{EI}$ or $\frac{qa^4}{EI}$ are given for deflections, and those at $\frac{Ma}{EI}$, $\frac{Pa^2}{EI}$ or $\frac{qa^3}{EI}$ for angles of rotation. **392.** $\frac{-23}{8}$; $-\frac{3}{2}$. **393.** $\frac{-19}{24}$; $\frac{-1}{3}$. **394.** $\frac{-25}{4}$; $\frac{-5}{3}$. **395.** $\frac{-16}{3}$; $\frac{-5}{3}$. **396.** $\frac{-59}{24}$; $\frac{-3}{2}$. **397.** -2; $\frac{-13}{12}$. **398.** -2; $\frac{-17}{16}$.

458

$$d = d_{max}\left(\frac{x}{l}\right)^{\frac{2}{3}};\ |f|_{max} = \frac{3ql^4}{8EI_{max}}$$

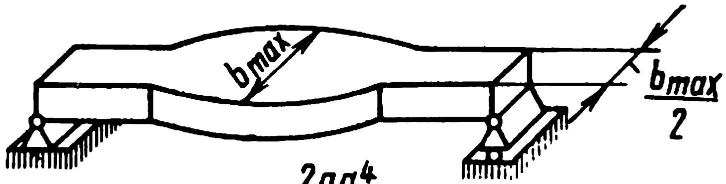
399. $\frac{-85}{72}$; $\frac{-65}{72}$. **400.** $\frac{-5}{8}$; $\frac{-7}{36}$. **401.** $\frac{1}{6}$; $\frac{1}{3}$. **402.** $\frac{-1}{3}$; $\frac{-1}{12}$. **403.** $\frac{-7}{9}$; $\frac{2}{9}$. **404.** $\frac{-5}{12}$; $\frac{1}{6}$. **405.** $\frac{19}{24}$; $\frac{-2}{3}$. **406.** $\frac{19}{90}$; $\frac{101}{360}$. **407.** $\frac{1}{72}$; $\frac{1}{15}$. **408.** $\frac{25}{24}$; $\frac{7}{6}$. **409.** $\frac{-62}{45}$; $\frac{187}{120}$. **410.** $\frac{-11}{6}$; -1. **411.** $\frac{-7}{6}$; $\frac{-1}{2}$. **412.** -7; 2. **413.** $\frac{-5}{12}$; $\frac{-1}{2}$. **414.** $\frac{-11}{36}$; $\frac{1}{9}$. **415.** 0; $-\frac{1}{3}$. **416.** $\frac{-11}{24}$; $\frac{-7}{12}$. **417.** $\frac{1}{12}$; $\frac{-11}{12}$. **418.** $\frac{-2}{3}$; $\frac{1}{3}$. **419.** $\frac{1}{6}$; $-\frac{1}{3}$. **420.** $\frac{-4}{3}$; $\frac{-3}{2}$. **421.** $\frac{-10}{3}$; 3. **422.** $\frac{-7}{12}$; $\frac{-1}{2}$; $\frac{1}{4}$. **423.** $\frac{-5}{12}$; $\frac{-7}{12}$; 0. **424.** $\frac{1}{6}$; $\frac{7}{12}$; $\frac{-3}{4}$. **425.** $\frac{-17}{18}$; $\frac{-1}{2}$; $\frac{-7}{9}$. **426.** $\frac{45}{8}$; 2. **427.** 3; 2. **428.** $\frac{19}{8}$; $\frac{11}{6}$. **429.** $\frac{2}{9}$; $\frac{2}{3}$. **430.** $\frac{9}{4}$. $\frac{27}{8}$. **431.** $\frac{65}{162}$; $\frac{5}{9}$. **432.** $\frac{131}{90}$; $\frac{5}{4}$. **433.** $\frac{733}{648}$; $\frac{10}{9}$. **434.** $\frac{-23}{24}$; $\frac{-71}{24}$. **435.** -2; -11. **436.** $\frac{-7}{3}$. **437.** $\frac{-31}{12}$. **438.** $\frac{-5}{8}$. **439.** $\frac{-1}{12}$; $\frac{-1}{8}$. **440.** $\frac{1}{6}$; $\frac{-5}{6}$. **441.** $\frac{-1}{4}$. **442.** $\frac{-13}{12}$. **443.** $\frac{-11}{12}$. **444.** 1.31 cm. **445.** 3.55 cm. **446.** 1.35 cm. **447.** 1.29 cm. **448.** 1.6 cm. **449.** No. 22. **450.** No. 22 (16). **451.** No. 27. **452.** 12×20 cm^2. **453.** 20 cm. **454.** $\frac{4}{3}$. **455.** $\frac{19}{18}$. **456.** $\frac{144}{7}$. **457.** 4.4. **467.** 160 kgf; 4 cm.

459



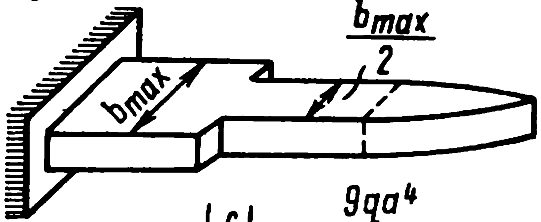
$$|f|_{max}=\frac{9Pa^3}{4EI_{max}}$$

460



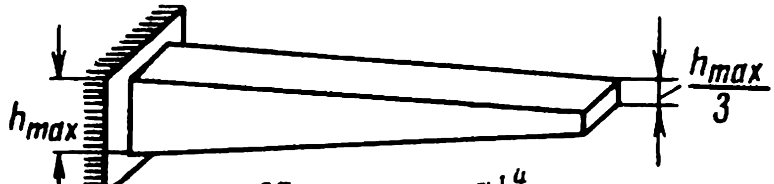
$$|f|_{max}=\frac{2qa^4}{EI_{max}}$$

461



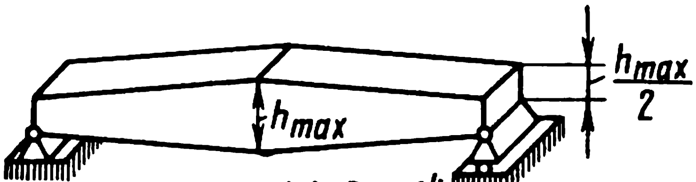
$$|f|_{max}=\frac{9qa^4}{2EI_{max}}$$

462



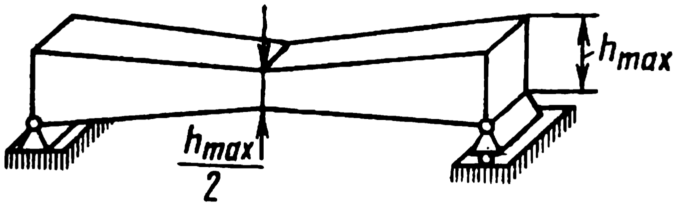
$$|f|_{max}=\frac{27}{32}(2-\ln 3)\cdot\frac{ql^4}{EI_{max}}$$

463



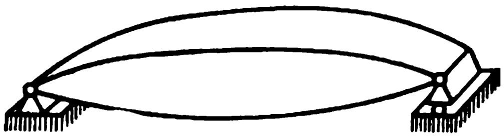
$$|f|_{max}=\frac{1-\ln 2}{4}\cdot\frac{ql^4}{EI_{max}}$$

464



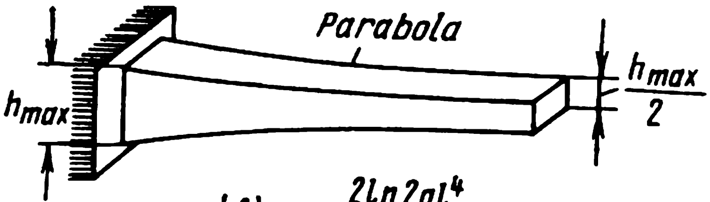
$$|f|_{max}=\frac{2\ln 2-1}{4}\cdot\frac{ql^4}{EI_{max}}$$

465



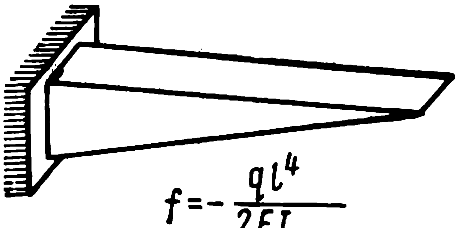
$$h=\frac{2h_{max}}{l}\sqrt{lx-x^2};\quad |f|_{max}=\frac{\pi-2}{64}\cdot\frac{ql^4}{EI_{max}}$$

466



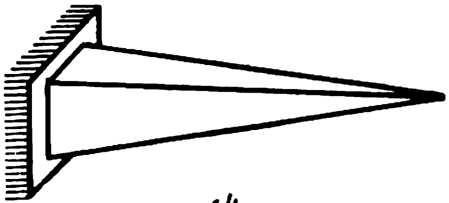
$$|f|_{max}=\frac{2\ln 2\, ql^4}{EI_{max}}$$

468



$$f=-\frac{ql^4}{2EI_{max}}$$

469



$$f=-\frac{ql^4}{6EI_{max}}$$

Shear stresses do not deform the beams. $\theta=\infty$, since the curvature $\frac{1}{\rho}=\frac{M}{EI}$ is great near the free ends of the beams, the approximate equation of the elastic line is not applicable here.

470*. (1) $\frac{3}{4}l$; (2) 2; 3; 10. 471*. (1) $\frac{2}{5}l$; (2) $\frac{81}{77}$; ~ 2.082; ~ 2.912. 472*. (1) $\frac{l}{3}$; (2) ~ 1.10; ~ 2.420; ~ 3.251. 473*. (1) $\frac{l}{4}$; (2) $\frac{4}{3}$; ~ 2.841; ~ 3.271. 474*. (1) $\frac{l}{4}$; (2) $\frac{32}{27}$; 2; $\frac{8}{3}$. 475*. (1) $\frac{l}{2}$; (2) $\frac{32}{27}$; ~ 2.898; 4 and 2. 476. $1.061 \frac{M_{max}}{W_{max}}$ at $x=0.3826l$. 477. $1.294 \frac{M_{max}}{W_{max}}$ at $x=0.2665l$.

Note. In the answers to Problems 478-491 numerical coefficients at $\frac{Ql^3}{EI_{max}}$, $\frac{Pa^3}{EI_{max}}$ or $\frac{M_0a^2}{EI_{max}}$ are given for deflections, and those at $\frac{Pa^2}{EI_{max}}$ or $\frac{M_0a}{EI_{max}}$ for angles of rotation. 478. $\frac{1}{12}$. 479. $\frac{1}{6}$. 480. $\frac{2}{3}(3-4\ln 2)\cong 0.1516$. 481. $\frac{1}{7}$. 482. $\frac{17}{6}$; $\frac{3}{2}$. 483. 10; $\frac{9}{2}$. 484. $\frac{4}{3}$; $\frac{1}{3}$. 485. $\frac{7}{6}$; $\frac{2}{3}$. 486. 3; 2. 487. $\frac{265}{72}$; $\frac{14}{3}$. 488. $\frac{205}{108}$; $\frac{41}{18}$. 489. $\frac{121}{576}$; $\frac{11}{24}$. 490. $\frac{25}{72}$; $\frac{5}{6}$. 491. $\frac{98}{81}$; $\frac{14}{9}$. 492. $|M|_{max}=\frac{7}{32}qa^2$; $|Q|_{max}=\frac{41}{63}qa$. 493. $|M|_{max}=\frac{13}{24}qa^2$; $|Q|_{max}=\frac{49}{72}qa$ 494. $\theta=\frac{ql^3}{240EJ}$. 495. $f_B=\frac{ql^4}{16EI}$. 496. $|M|_{max}=\frac{5}{6}M$; $|Q|_{max}=\frac{1}{6}\frac{M}{a}$. 497. $|M|_{max}=Pa$; $|Q|_{max}=\frac{4}{3}P$. 498. $|M|_{max}=\frac{9}{32}qa^2$; $|Q|_{max}=\frac{57}{64}qa$. 499. $|M|_{max}=Pa$; $|Q|_{max}=P$. 500. $[M]=3.14$ tnf-m. 501. $[M]=2.85$ tnf. 502. $[P]=7.73$ tnf. 503. $[P]=11.6$ tnf. 504. $[q]=7.4$ tnf/m. 505. $[q]=5.2$ tnf/m. 506. No. 12. 507. No. 14. 508. No. 18. 509. No. 10. 510. $\frac{15}{28}M$. 511. $\frac{8}{11}Pa$. 512. $\frac{9}{8}qa^2$. 513. $\frac{3}{4}Pa$. 514. $\frac{3qa^4}{32EJ}$. 515. $\frac{qa^4}{20EI}$. 516. $\frac{7Ma^2}{12EI}$. 517. $\frac{Pa^3}{4EI}$. 518. $\frac{ql^4}{24EI}$. 519. $\frac{Pa^3}{12EI}$. 520. 720 kgf. 521. $\frac{5\sqrt{3}}{12}qa$. 522. 47 kgf. 523. 172 kgf. 524. $\frac{12M}{5a}$ tnf. 525. 2.56; 3.2 kN. 526. $\sigma_1=109$ MN/m². 527. $N_3=\frac{64\Delta EI}{3a^3}$. 528. —7.5; 15 MN/m². 529. 288.5; 43.3 kgf/cm². 530. $B=\frac{3}{20}P$; $M_{max}=\frac{7}{10}Pl$. 531. $B_y=\frac{17}{12}P$; $B_x=\frac{3}{16}P$; $M_{max}=Pa$. 532. $B_x=\frac{3}{13}P$; $M_{max}=\frac{7}{13}Pa$; 533. $B_x=\frac{P}{8}$; $M_{max}=\frac{Pa}{3}$.

* The distance is indicated from the left end of the beam.

534. $N=25$ kgf; $M_{max}=750$ kgf-cm. 535. $N=98$ kgf; $M_{max}=1440$ kgf-cm. 536. $N=\frac{3}{38}qa$; $M_{max}=\frac{11}{152}qa^2$. 537. Joint pressure $N=\frac{3}{4}F$; $M_{max}=\frac{Pa}{2}$. 538. $B_x=\frac{48\Delta EI}{7a^3}$; $B_y=\frac{3}{8}B_x$. 539. $B_x=\frac{3\Delta EI}{8a^3}$. 540. $N=\frac{3(\alpha_{Cu}-\alpha_{St})\Delta t E_{St}E_{Cu}FI}{5a^2E_{Cu}F+3E_{St}I}$. 541. $N=180$ kgf. 542. $B_x=\frac{3a\Delta tEI}{5a^2}$. 543. $B_x=\frac{12\alpha\Delta tEI}{a^2}$. 544. At the intermediate support $M_2=1.5$ tnf-m. 545. At the intermediate supports $M_2=-\frac{31}{12}Pa$; $M_3=\frac{25}{92}Pa$. 546. $\sqrt[3]{\frac{32qa^2}{\pi[\sigma]}}$. 547. $\sqrt[3]{\frac{6Pa}{[\sigma]}}$. 548. No. 16. 549. No. 18a. 550. 6.28 tnf/m. 551. 5.9 tnf/m. 552. ~ 2.6 tnf-m. 553. 1.86 tnf/m. 554. 19.6 kN-m. 555. 24.3 kN. 556. $-\frac{7Pa^3}{6EI}$. 557. $-\frac{13}{768}\cdot\frac{Pl^3}{EI}$. 558. $0.707Pl$; $0.293l$; $0.707l$. 559. 0.662 tnf-m. 560. $-\frac{5}{384}\cdot\frac{ql^4}{EI}$. 561. $\frac{Pl^3}{144EI}$. 562. 2 tnf. 563. 3.65 tnf. 564. $\frac{3\eta W[\sigma]}{a}$. 565. $\eta W[\sigma]$. 566. $\frac{16\eta W[\sigma]}{a^2}$. 567*. $2[\sigma]\left(F+\frac{\sqrt{2}\,\eta W}{a}\right)$. 568. 17.6 cm; 8.8 cm. 569. 14 cm; 7 cm. 570. 9.2 cm; 4.6 cm. 571. 9.2 cm; 4.6 cm. 572. 1.97. 573. 1.62. 574. 3.06. 575. 1.6.
576.

Problem No.	222	223	224	225	226
U	$\frac{q^2l^5}{504EI}$	$\frac{187q^2l^5}{20160EI}$	$\frac{(3+8\pi^2)q^2l^5}{192\pi^4EI}$	$\frac{13q^2l^5}{1620EI}$	$\frac{m^2l^3}{6EI}$
Problem No.	**227**	**228**	**229**	**230**	**231**
U	$\frac{q^2l^5}{60480EI}$	$\frac{q^2l^5}{2592EI}$	$\frac{q^2l^5}{4\pi^4EI}$	$\frac{\pi^4q^2l^5}{64EI}$	0

577. 985 kgf-cm. 578. 210 kgf-cm. 579. 3565 kgf-cm. 580. $\frac{3P^2a^3}{8b^4E}$. 581. $\frac{[\sigma]l^2}{4E}\cdot\sqrt{bq[\sigma]}$. 582. $U_1:U_2:U_3=1:\frac{7}{16}:0.25$. 583. In the beam of equal resistance U is 1.5 times higher. 584. (1) 1.2; (2) $\frac{32}{27}$; (3) $\frac{4}{3}$; (4) 2; (5) 2.4;

* Here F is the area of cross section of a diagonal bar.

(6) 1.8. **585.** (577) $U_\tau=535$ kgf-cm, i.e. 35% from U; (578); $U_\tau=1$ kgf-cm, i.e. 0.5% from U; (579) $U_\tau=314$ kgf-cm, i.e. 11.3% from U. **586.** 1104 kgf/cm^2; $\beta=67°35'$. **587.** 1454 kgf/cm^2; $\beta=59°53'$; $f_v=1.13$ cm; $f_h=0.65$ cm. **588.** 95 kgf/cm^2; $\beta=61°38'$. **589.** 1059 kgf/cm^2; $\beta=84°20'$. **590.** 118.9 kgf/cm^2; $\beta=49°06'$. **591.** 1210 kgf/cm^2; $\beta=74°35'$. **592.** 82.6 kgf/cm^2. $\beta=16°06'$. **593.** 9.13 MN/m^2; $\beta=82°59'$; $f_1=1.030$ cm; $\beta_1=7°11.6'$ with the direction of the force P_1. **594.** $\sim 4.025\frac{Pl}{a^3}$; $\beta=75°$. **595.** 9 MN/m^2; $\beta=14°56'$. **596.** 160 kgf; $\beta=23°58'$; $f=1.334$ cm; $\beta_1=34°56'$ from the vertical. **597.** 9220 kgf; $\beta=73°26'$. **598.** 47.5 kN; $\beta=33°40'$. **599.** 12.9 kN; $\beta=75°37'$; $f=0.377$ cm; $\beta_1=88°11'$ from the vertical. **600.** $44.9\frac{a^3[\sigma]}{l}$; $\beta=26°47'$. **601.** $0.0295\frac{d^3[\sigma]}{a}$; $\beta=15°37'$. **602.** $\sim 12\times 18$ cm^2; $\beta=52°25'$. **603.** No. 27; $\beta\cong 84°52'$. **604.** $\sim 18\times 24$ cm^2; $\beta=36°15'$; $f\cong 1.94$ cm; $\beta_1\cong 41°30'$. **605.** $D\cong 12$ cm; $\beta\cong 70°54'$. **608.** $f_{max}=5.5\frac{Pa^3}{Eb^4}$; $\sim 45°$ with the rectangle sides. **609.** $\sigma_{max}=\frac{27Pl}{16\sqrt{5}a^3}$ at $x=\frac{3}{4}l$ from the free end. **610.** $\sigma_{max}\cong 0.453\frac{ql^2}{a^3}$ at $x\cong 0.317l$ from the support. **611.** $f_{max}\cong 17.8\frac{Pa^3}{Eb^4}$; $\sim 15°42'$.

Note. In the answers to Problems 612-620 the position of the neutral axis is determined by the segments y_0 and z_0 cut off by the neutral axis on the principal axes of inertia of the section.

612. 0; -40 kgf/cm^2; $y_0=\infty$; $z_0=15$ cm. **613.** -3.51; -4.63 kgf/cm^2; ∞; -363 cm. **614.** 25; -50 kgf/cm^2; -6.67 cm; -8 cm. **615.** 65; -286 kgf/cm^2; -12.5 cm; ∞. **616.** $26.5\frac{P}{a^2}$; $-27.5\frac{P}{a^2}$; $\frac{a}{54}$; $-\frac{a}{27}$. **617.** 1285; -1355 kgf/cm^2; 0.29 cm; 1.47 cm. **618.** 17.6; -14.4 MN/m^2; -0.834 cm; 0.625 cm. **619.** $-4.88p$; $-7.37p$; $6.2a$; ∞. **620.** 9.3; -11.3 MN/m^2; -4.67 cm; -1.43 cm. **621.** 650 kgf/cm^2; $z_0=-\frac{d}{24}$. **622.** 44.7 kgf/cm^2; $y_0=\frac{a}{6}$; $z_0=\frac{a}{15}$. **623.** 300 kgf/cm^2; $y_0=8.3$ cm; $z_0=7.8$ cm. **624.** 85.8; -80.8 kgf/cm^2. **625.** $79.3\frac{P}{d^2}$; $-93.5\frac{P}{d^2}$. **626.** 58.4; -63.6 MN/m^2. **627.** 1122; -1034 kgf/cm^2. **628.** $\frac{P}{3ab}$ at $a_x=3a$; $-\frac{4P}{3ab}$ at $a_x=\frac{3}{2}a$. **629.** $\frac{P}{\pi d^2}$ at $d_x=2d$; $-\frac{125P}{27d^2}$ at $d_x=1.2d$. **630.** $a\geqslant 2\sqrt{\frac{P}{[\sigma]}}$. **631.** $a=4$ cm. **632.** No. 22a; $d\cong 1.26$ cm. **633.** No. 27. **634.** $d\cong 17\sqrt{\frac{P}{[\sigma]}}$. **635.** $d\cong 1.5$ cm. **636.** $t_1\cong 4.65t$. **637.** $d=3$ m. **638.** 64 kgf. **639.** 9040 kgf. **640.** 2780 kgf. **641.** 19,550 kgf. **642.** 4180 kgf. **643.** 360 kgf. **644.** 6000 kgf. **645.** 368 kgf. **646.**

$\frac{\pi d^2[\sigma]}{176}$. **647.** 4590 kgf. **648.** 8.68 kN $\leqslant P \leqslant$ 12.15 kN. **649.** $P_1 = 8P$; $\sigma_{min} = -\frac{32P}{3a^2}$.

Note. The answers to Problems 650-655 give the coordinates of angle points of the contour of the core of section in the principal central axes of inertia YZ (in cm).

650. Quadrangle; $0.160a$; 0; $-0.072a$; 0; 0; $\pm 0.166a$. **651.** Quadrangle; $0.697a$; 0; $-0.650a$; 0; 0; $\pm 0.532a$. **652.** Octagon; 0; $\pm 0.107a$; $\pm 0.107a$; 0; $\pm 0.08a$; $\pm 0.08a$. **653.** Hexagon; $0.226a$; 0; $-0.131a$; 0; 0; $\pm 0.107a$; $0.086a$; a; $\pm 0.113a$. **654.** Rectangle; $\frac{a\sqrt{3}}{12}$; $\pm 12a$; $-\frac{a\sqrt{3}}{12}$; $\pm 12a$. **655.** Quadrangle; $\frac{13}{36}b$; 0; $\frac{13}{45}b$; 0; $-\frac{13}{126}b$; $\pm\frac{15}{56}b$. **656.** $51\frac{Pa}{d^3}$. **657.** 2160 kgf/cm^2. **658.** $91.6\frac{Pa}{d^3}$. **659.** $78\frac{q}{a}$. **660.** 1000 kgf/cm^2. **661.** 107 MN/m^2. **662.** $\cong 4.9\frac{Pl}{\pi d_0^3}$ at $x \cong 0.448l$ from the free end. **663.** $19.4\frac{Pa}{d^3}$. **664.** $\sim$ 1040 kgf/cm^2. **665.** $\sigma_{max_I} = 827$ kgf/cm^2; $\sigma_{min_I} = -223$ kgf/cm^2; $\sigma_{max_{II}} = 1755$ kgf/cm^2; $\sigma_{min_{II}} = -35$ kgf/cm^2; $\delta_p = 0.53$ mm. **666.** $d \geqslant 3.7\sqrt[3]{\frac{Pa}{[\sigma]}}$. **667.** $b = 3.35\sqrt[3]{\frac{Pa}{[\sigma]}}$. **668.** $d \cong 40$ mm, **669.** $d \cong 60$ mm. **670.** $d \cong 85$ mm. **671.** $d_1 \cong 22$ mm; $b \cong 22$ mm; $d_2 \cong 36$ mm. **672.** $a = 3\sqrt{\frac{2P}{[\sigma]}}$. **673.** $d \geqslant 2.46\sqrt[3]{\frac{Pa}{[\sigma]}}$. **674.** $d \cong 50$ mm. **675.** $d \cong 38$ mm; $b \cong 22$ mm. **676.** $d_1 \cong 54.5$ mm; $d_2 \cong 64$ mm. **677.** $d \cong 70$ mm. **678.** $d \geqslant 1.84\sqrt[3]{\frac{Pa}{[\sigma]}}$. **679.** $d \cong 24.2$ mm. **680.** $d \cong 80$ mm. **681.** $P \leqslant \frac{[\sigma]\,d^3}{51a}$. **682.** $q \leqslant \frac{0.082\,[\sigma]\,b^3}{a^2}$. **683.** $M \leqslant 0.026\,[\sigma]\,D^3$. **684.** $q \leqslant 1.4$ kN/m. **685.** $P \leqslant 100$ kgf. **686.** $P \leqslant 0.556\,[\sigma]\,\frac{d^3}{l}$. **687.** $M \cong 117$ kgf-cm. **688.** $P \cong 6.4$ kgf. **689.** $\sigma_{III} = 47.4\frac{P}{d^2}$. **690.** $\sigma_{III} = 1240$ kgf/cm^2. **691.** $\sigma_{III} = 1510$ kgf/cm^2. **692.** $\sigma_{III} = 89\frac{P}{d^2}$. **693.** $96\frac{P}{b^2} \leqslant [\sigma]$. **694.** $8.14\frac{P}{b^2} \leqslant [\sigma]$. **695.** 1570 kgf/cm^2 < 1600 kgf/cm^2. **696.** $b = 12$ cm; $h = 8.3$ cm. **697.** $d_1 \cong 8.5$ cm; $d_2 \cong 8.5$ cm; $d_3 \cong 6.1$ cm. **698.** $d_1 = 6$ cm; $b = 3.63$ cm; $d_2 = 5.71$ cm; $c = 2.72$ cm. **699.** 715 N; $\tau_{max} = 685$ MN/m^2. **700.** $\frac{\max\tau_1}{\max\tau_2} = 0.74$; $\frac{\delta_1}{\delta_2} = 0.88$. **701.** $\max\tau_1 = 955$ kgf/cm^2; $\max\tau_2 = 2550$ kgf/cm^2; $\delta = 26$ cm. **702.** $\tau_{max} = 4760$ kgf/cm^2; $n = 15$; $\delta_0 = 8.1$ cm. **703.** $\max\tau_1 = 128$ kgf/cm^2. $\max\tau_2 = 417$ kgf/cm^2. **704.** $P \cong 800$ kgf. **705.** By 10%. **706.** $P_0 = 902$ N. **707.** $d = 1$ cm; $n = 10$. **708.** $P = 109$ kgf. **709.** $P = 50$ kgf. **710.** 43 kgf; 20.5 kgf/cm^2. **711.** 132.5 tnf; 2630 kgf/cm^2. **712.** (a) 1070 kgf; 713 kgf/cm^2; (b) 3460 kgf; 2310 kgf/cm^2. **713.** (a) 171 tnf;

4530 kgf/cm^2; (b) 124 tnf; 3300 kgf/cm^2. **714.** 1210 kgf; 192 kgf/cm^2. **715.** 5.47 tnf; 17.1 kgf/cm^2. **716.** 7.4. **717.** 2. **718.** $\frac{F}{2}\sqrt{\frac{\pi E\cos\alpha}{P}}$. **719.** 9.5 m. **720.** $\frac{F}{2}\sqrt{\frac{\pi E}{qa}}$. **721.** (a) 31.9 cm; (b) 17.8 cm. **722.** 82°. **723.** (a) 37.5°; (b) 10.1°. **724.** 63°. **725.** $d=13.5$ cm. **726.** $d=6$ cm. **727.** $d=6$ cm. **728.** No. 22a. **729.** Overstress 33%. **730.** Overstress 2.4%. **731.** Understress 24%. **732.** Understress 4.8%. **733.** Understress 14%. **734.** 9.1 tnf. **735.** 54.6 tnf. **736.** 177.6 tnf. **737.** $b\cong 11.3$ cm. **738.** $b=2.86a$. **739.** $b=2a$. **740.** No. 30. **741.** No. 20 (16). **742.** $b=10$ cm. **743.** 20 cm. **744.** $d=8$ cm. **745.** No. 14; $B=14$ cm; $l_0=40$ cm. **746.** No. 16; $B=22$ cm; $l_0=70$ cm. **747.** No. 10 (10); $B=39$ cm; $l_0=60$ cm (one of possible cases at $\varphi=0.81$). **748.** No. 10 (7); $B=34$ cm; $l_0=60$ cm (one of possible cases at $\varphi=0.81$). **749.** 4.91 cm; 1462 kgf/cm^2; **750.** 10.96 cm; 107.4 kgf/cm^2. **751.** 1.30 cm; 67.1 kgf/cm^2. **752.** 0.303 cm; 1240 kgf/cm^2; 2.52; 1.84. **753.** 229 cm; 1275 kgf/cm^2; 2.27; 1.78. **754.** $80\times 50\times 6$ (one of possible cases). **755.** $d=16$ cm.

Note. In the answers to Problems 756-765 the values of $N_{\max}$, $Q_{\max}$ and $M_{\max}$ are given. **756.** $2P$; $2P$; $\frac{3}{2}Pa$. **757.** P; $3P$; $3Pa$. **758.** $P\sqrt{2}$; $P\sqrt{2}$; $2Pa$. **759.** $qa\sqrt{2}$; $qa\sqrt{2}$; $2.414qa^2$. **760.** $\frac{qa}{2}$; $\frac{qa}{2}$; $\frac{qa^2}{3}$. **761.** $\frac{3}{2}P$; $\frac{3}{2}P$; $2Pa$. **762.** $2qa$; $3qa$; $\frac{9}{2}qa^2$. **763.** $2qa$; qa; $2qa^2$; **764.** qa; $2qa$; $5.713qa^2$. **765.** P; P; $\frac{5}{2}Pa$. **766.** 127; -206; 111 kgf/cm^2. **767.** 105; -87; -48 MN/m^2. **768.** 1050; -700; -382 kgf/cm^2. **769.** 805; -403 kgf/cm^2. **770.** $P=208$ tnf. **771.** $M_1=123$ kgf-m; $M_2=303$ kgf-m; $M_3=128$ kgf-m; $M_4=134$ kgf-m. **772.** $782<1600$ kgf/cm^2. **773.** $360<1600$ kgf/cm^2. **774.** $\frac{17Ph}{EF}$; $\frac{1.73Ph}{EF}$. **775.** $2.12\frac{Pa}{EF}$; $3.6\frac{Pa}{EF}$. **776.** $\frac{qr^2}{EF}$. **777.** $34.4\frac{Ma}{Gd^4}$. **778.** $\frac{28}{3}\frac{Ma}{GI_t}$. **779.** $\frac{41}{24}\frac{qa^4}{EI}$; $\frac{7}{6}\frac{Pa^2}{EI}$. **780.** $\frac{qa^4}{4EI}$; $\frac{qa^3}{12EI}$. **781.** $\frac{qa^4}{15EI}$; $\frac{17qa^3}{180EI}$. **782.** $\frac{2}{3}\cdot\frac{qa^4}{EI}$; $-k\frac{qa^2}{GF}$. **783.** $\frac{768P}{bE}$.

In the answers to Problems 784-790, 797-819 and 822-833 coefficients at $\frac{Ma^2}{EI}$, $\frac{Pa^3}{EI}$ or $\frac{qa^4}{EI}$ are given for linear displacements, and at $\frac{Ma}{EI}$, $\frac{Pa^2}{EI}$ or $\frac{qa^3}{EI}$ for angular displacements. **784.** $16\left(\frac{1}{3}-\frac{I}{a^2F}\right)$; 8; 2. **785.** $2\left(\frac{8}{3}+\frac{I}{a^2F}\right)$; 1; 1. **786.** $\frac{1}{2}\left(1+\frac{I}{2a^2F}\right)$; $\frac{1}{4}$. **787.** $2\left(11+\frac{I}{a^2F}\right)$; $2\left(9.1+\frac{I}{a^2F}\right)$. **788.** $4\left(3+\frac{I}{a^2F}\right)$. **789.** $2+\frac{3I}{a^2F}$. **790.** $\frac{1}{2}$. **791.** $103.7\frac{Pa^3}{Ed^4}$. **792.** $85\frac{Pa^3}{Ed^4}$. **793.** $39.1\frac{Pa^3}{Ed^4}$. **794.** $\frac{80Pa^3}{Ed^4}\times\left(1+\frac{d^2}{63a^2}\right)$. **795.** $\frac{170Pa^3}{Eb^4}\left(1+\frac{b^2}{170a^2}\right)$. **796.** $\frac{56.9Pa^3}{Eb^4}\times\left(1+\frac{b^2}{56.9a^2}\right)$. **797.** 13.6; $\frac{1}{2}$; 6.85. **798.** $\frac{7}{6}$; 1.45;

$\frac{5}{3}$. **799.** 39; 19.1; 20.1. **800.** 0.36; $\frac{1}{2}$; 0.57; **801.** 0.071; 0; 0.142. **802.** $\frac{1}{12}$; 0.010; 0.028. **803.** 0.47; 0.85; 0.24; **804.** 8; 37.4; 4π. **805.** $\frac{1}{12}$; 0.0927; 0.0181. **806.** $\frac{1}{2}$; 1.225. **807.** $\frac{2}{3}$; 0.392. **808.** 5.14; 3.14. **809.** $\frac{3\pi}{2}$; $\frac{1}{2}$. **810.** 1, 1; $\frac{1}{2}$. **811.** 0.312. **812.** 0.071. **813.** 0.355. **814.** $0.0198\left[1+3.57\frac{b^2}{a^2}\right]$. **815.** 12.4. **816.** $\frac{28}{3}$. **817.** 3π. **818.** $\frac{1}{2}$. **819.** 7.24. **820.** 45°. **821.** 48°15′. **822.** $\frac{5}{12}$; $\frac{1}{12}$. **823.** $\frac{1}{2}$; $\frac{1}{2}$. **824.** $\frac{2}{9}$; $\frac{2}{9}$. **825.** $\frac{28}{3}+\frac{I}{a^2F}$; $\frac{44}{9}$. **826.** $\frac{7}{3}+2\frac{I}{a^2F}$; 2.5. **827.** $\frac{23}{3}$; 3. **828.** 14, 8.5. **829.** 0.5; 0. **830.** 3; 1. **831.** $\frac{5}{6}$. **832.** $\frac{7}{2}$. **833.** $\frac{32}{3}$. **834.** 0; $\frac{P}{2}\sin\beta$; $\frac{P}{2}\cos\beta$. **835.** $\sim -0.739P$; $\sim 0.369P$. **836.** ma. **837.** $2.7M$. **838.** $\frac{qa}{8}$; $\frac{7}{48}\frac{qa^4}{EI}$. **839.** $\frac{7}{8}qa$; $\frac{5}{24}\frac{qa^4}{EI}$. **840.** $\frac{6M}{7a}$; $\frac{Ma^2}{21EI}$. **841.** $\frac{25}{16}qa$; $\frac{qa^4}{48EI}$; $M_A=\frac{23}{48}qa^2$. **842.** $\frac{P}{10}$; $\frac{19Pa^3}{60EI}$. **843.** $\frac{qa}{3}$; $\frac{16}{9}\cdot\frac{qa^4}{EI}$. **844.** $\frac{3}{8}\cdot\frac{M}{a}$; $\frac{Ma^2}{2EI}$. **845.** $\frac{3}{4}qa$; $\frac{3}{4}\cdot\frac{qa^4}{EI}$. **846.** $\frac{qa}{7}$; $\sim 0.137\frac{qa^4}{EI}$. **847.** $\frac{3}{10}P$; $\frac{11}{15}\cdot\frac{Pa^3}{EI}$. **848.** $\frac{15}{28}P$. **849.** $\frac{45}{44}qa$. **850.** $\sim 0.162P$. **851.** $\frac{3}{2}qa$. **852.** $A_y=0$; $A_x=\frac{P}{4}$. **853.** $A_y=A_x=\frac{9M}{20a}$. **854.** $A_y=\frac{32}{25}qa$; $A_x=\frac{14}{25}qa$. **855.** $A_y=\frac{125}{54}qa$; $A_x=\frac{91}{54}qa$. **856.** $A_y=\frac{14}{19}P$; $A_x=\frac{9}{38}P$. **857.** $A_y=A_x=\frac{qa}{6}$. **858.** $\frac{3}{5}Pa$. **859.** $\frac{3}{4}Pa$. **860.** $0.1642qa^2$. **861.** $0.0634qa^2$. **862.** $A_y=0$; $A_x=\frac{3P}{16}$; $M_A=\frac{Pa}{4}$. **863.** $A_y=\frac{2}{11}qa$; $A_x=\frac{57}{55}qa$. **864.** $A_y=\frac{P}{2}$; $A_x=\frac{3P}{62}$. **865.** $A_y=\frac{40}{71}qa$; $A_x=\frac{76}{71}qa$. **866.** $A_y=\frac{11}{20}qa$; $A_x=\frac{qa}{40}$; $M_A=\frac{qa^2}{10}$. **867.** $A_y=\frac{P}{8}$; $A_x=\frac{15}{16}P$. **868.** $\frac{44}{23}P$; $\frac{68}{69}\cdot\frac{Pa^3}{EI}$. **869.** $\frac{10}{27}P$; $\frac{191}{81}\cdot\frac{Pa^3}{EI}$. **870.** $\frac{19}{47}P$; $\sim 0.0722\frac{Pa^3}{EI}$. **871.** $0.75E\alpha\Delta t\frac{h}{a}$. **872.** $\frac{3}{11}E\alpha\Delta t\frac{h}{a}$. **873.** $0.3E\alpha\Delta t\frac{h}{a}$. **874.** $0.6E\alpha\Delta t\frac{h}{a}$. **875.** $\frac{24}{31}E\alpha\Delta t\frac{h}{a}$. **876.** $\frac{3}{11}\cdot\frac{\Delta Eh}{a^2}$. **877.** $\frac{3\Delta Eh}{16a^2}$. **878.** $\frac{\Delta Eh}{7a^2}$. **879.** $\frac{3\Delta Eh}{8a^2}$. **880.** $\frac{P}{3}$; $\frac{\pi+1}{3}\cdot\frac{Pa^3}{EI}$. **881.** $0.0372qa$; $0.1188\frac{qa^4}{EI}$. **882.** $\frac{5}{6}qa$; $\frac{\pi}{24}\cdot\frac{qa^4}{EI}$. **883.** $0.0638P$; $0.1052\frac{Pa^3}{EI}$. **884.** $0.1867qa$; $0.115\frac{qa^4}{EI}$. **885.** $R_y=\frac{qa}{2}$; $R_x=0.590qa$. **886.** $R_y=\frac{P}{2}$; $R_x=0.786P$. **887.** $R_y=0$; $R_x=\frac{4qa}{\pi}$. **888.** $R_y=\pi qa$; $R_x=3qa$. **889.** $R_y=\frac{P}{2}$;

$R_x=0.722P$; $M_R=1.077Pa$. **890.** $3.44\frac{Pa^3}{EI}$. **891.** $\frac{2\pi}{9}\cdot\frac{Pa^3}{EI}$. **892.** $0.586\times\frac{Pa^3}{EI}$. **893.** $0.25\frac{Pa^3}{EI}$. **894.** $\frac{2\pi}{5}\cdot\frac{Pa^3}{EI}$. **895.** $0.0994\frac{Pa^3}{EI}$. **896.** $0.369Pa$. **897.** $0.273qa^2$. **898.** $0.629Pa$. **899.** $1.152Pa$. **900.** $A_y=0.216P$; $A_x=0.433P$. **901.** $A_y=0.961P$; $A_x=0.774P$. **902.** $A_y=\frac{3}{4}qa$; $A_x=\frac{qa}{4}$. **903.** $A_y=0.271qa$; $A_x=0.277qa$. **904.** $A_y=\frac{P}{2}$; $A_x=0.0522P$; $N=0.437P$. **905.** $A_y=0.519qa$; $A_x=0.339qa$. **906.** $A_y=0.214qa$; $A_x=0.701qa$; $M_A=0.569qa^2$. **907.** 2.414; 2.515. **908.** 2.75; 9.40. **909.** 2.48; 8.52. **910.** 3.30; 15.26. **911.** $M_u=\frac{M}{2}$; $M_t=\frac{M}{9\pi}$; $\delta=0.298\frac{Ma^2}{EI}$. **912.** $M_u=\frac{Pa}{2}$; $M_t=0.565Pa$; $\delta=0.625\times\frac{Pa^3}{EI}$. **913.** $M_u=2qa^2$; $M_t=0.934qa^2$; $\delta=1.435\frac{qa^4}{EI}$. **914.** $d\cong 1.362\times\sqrt[3]{\frac{Pa}{[\sigma]}}$. **915** $b\cong 1.371\sqrt[3]{\frac{qa^2}{[\sigma]}}$. **916.** $r_2=17.3$ cm; $\Delta r_1=0.23$ mm. **917.** $p_2=111$ MN/m²; $\Delta r_2=-0.19$ mm. **918.** (a) $r_2=14.1$ cm. $\Delta r_1=0.23$ mm; $\Delta r_2=0.19$ mm; (b) $p_1=24$ MN/m². **919.** $p_0=\frac{a\beta E(r_2^2-r_1^2)}{2r_1[r_2^2+r_1^2+\mu(r_2^2-r_1^2)]}$; $\max|\sigma_r|=p_0$; $\max\sigma_t=p_0\frac{r_2^2+r_1^2}{r_2^2-r_1^2}$. **920.** $\beta=\frac{\Delta}{ar_1r_2}[r_2^2+r_1^2+\mu(r_2^2-r_1^2)]\cong\cong 0.017$ rad $\cong 1°$. **921.** $\beta=0.028\cong 1.6°$. **922.** $p_0=17.4$ kgf/cm²; $\sigma_{eqV}=17.4$ kgf/cm². **923.** $p=8.6$ kgf/cm²; $p_0=3.8$ kgf/cm². **924.** $p_0=\frac{\mu q}{\frac{r_2^2+r_1^2}{r_2^2-r_1^2}-\mu}$. **925.** $p_0=0$.

Problem No.	σ_r, kgf/cm²			σ_t, kgf/cm²				p_0, kgf/cm²
	Tube			Tube				
	I		II	I		II		
	Points			Points				
	1	2	3	1	2	2	3	
926	—2000	−577	0	1800	371	962	385	577
927	—2000	−248	0	2670	920	413	165	248
928	0	−173	0	—460	—290	290	115	173

929. $p_0=\frac{\Delta}{2r_2A}E_IE_{II}(1-n_1^2)(1-n_2^2)$, where $n_1=\frac{r_1}{r^2}$, $n_2=\frac{r_2}{r_3}$, $A=(1-n_1^2n_2^2)\times(E_I+E_{II})+(n_1^2-n_2^2)(E_{II}-E_I)-(1-n_1^2)(1-n_2^2)\times(\mu_IE_{II}-\mu_{II}E_I)$. **930.** $p_0=\frac{\Delta E}{2r_2\left(\frac{r_2}{\delta}+\frac{r_2^2+r_1^2}{r_2^2-r_1^2}-\mu\right)}$. **934.** $\max\sigma_I=3\gamma l$; $\max\sigma_{II}=66\gamma l$. **935.** $\sigma_I=$

Problem No.	σ_r, kgf/cm²			σ_t, kgf/cm²				Reduction of the design stress
	Tube			Tube				
	I		II	I		II		
	Points			Points				
	1	2	3	1	2	2	3	
931	−2000	−825	0	917	−309	1291	517	30%
932	−3000	−1500	0	1000	−500	2500	1000	—
	$r_2=20$ cm;			$r_3=40$ cm; $\Delta=0.6$ mm				

- 933	Tube	I		II		III	
	Points	1	2	2	3	3	4
$\sigma_{eq_{III}}$, kgf/cm²	One-piece tube	5717	3650	3650	1860	1860	914
	Composite tube	3615	2304	3624	1852	3365	1654

$=216$ MN/m²; $\sigma_{II}=10.8$ MN/m². **936.** $\sigma_I=181$ kgf/cm²; $\sigma_{II}=322$ kgf/cm². **937.** (a) $\sigma_I=\sigma_{III}=\sigma_{II}\cos^2\alpha=\frac{Q\cos^2\alpha}{(1+2\cos^3\alpha)F}\left(1+\frac{\omega^2 l}{g}\right)$; (b) $\sigma_{III}=\frac{\sigma_I}{\sqrt{3}}=$ $=\frac{\sigma_{II}}{1+\sqrt{3}}=\frac{Q}{(3+\sqrt{3})F}\left(1+\frac{\omega^2 l}{g}\right)$. **938.** ~ 66.8 MN/m². **939.** $\frac{32Q\omega^2 l^2}{g\pi d^3}$. **940.** $d=14.3$ cm; $b=12$ cm. **941.** 250 r. p. m. **942.** $\frac{4\gamma r l^2}{g\pi d^3}\sqrt{\varepsilon_0^2+\omega_0^4}$. **943.** $\sim 0.3\sqrt{\frac{Qn^2rl}{g[\sigma]}}$. **944.** $\max\sigma_I=1420$ kgf/cm², $\max\sigma_{II}=2220$ kgf/cm². **945.** ~ 1350 kgf/cm². **946.** $\frac{Q\omega^2\rho^4}{2gFI}(3\pi+8)$. **947.** 195 r.p.m.; ~ 1.59 cm. **948.** $\frac{16Q\omega^2R^2}{g\pi d^3}$. **949.** $4\frac{\gamma\omega^2R^3}{gd}$. **950.** $4.9\frac{Q\omega^2R^2}{g\pi d^3}$; $28.5\frac{Q\omega^2R^4}{gEd^4}$. **951.** $\frac{\pi b}{32r}\sqrt{\frac{E\pi bga}{Qr}}$. **952.** $\frac{\pi d^2}{32a}\times$ $\times\sqrt{\frac{\pi Eg}{2Qa}}$. **953.** 672 r. p. m. **954.** (a) 50 1/sec, (b) 5.6 1/sec. **955.** $\sqrt{\frac{2gEF}{7Qa}}$. **956.** $2\pi\tan\alpha\sqrt{\frac{Q}{Cg}}$. **957.** $\frac{4\pi}{3}\sqrt{\frac{Qa^3}{EIg}}$. **958.** $4\pi\sqrt{\frac{Qa^3}{3EIg}}$. **959.** 1/40 sec. **960.** $\frac{\pi l}{2}\sqrt{\frac{Ql}{6EIg}}$. **961.** 0.106 sec. **962.** $1.17\sqrt{\frac{EIg}{Qa^3}}$. **963.** 0.075 sec. **964.** $\frac{d^2}{12R}\sqrt{\frac{\pi Gg}{2Qa}}$. **965.** 2.82 1/sec. **966.** $\frac{61}{35}$. **967.** $\frac{41}{70}$. **968.** $\frac{333}{2240}$. **969.** $\frac{1}{3}$. **970.** $\frac{4}{3}$. **971.** 0.144 sec. **972.** 0.147 sec. **973.** $2\pi\sqrt{\frac{6Ql^3}{bh^3Eg}\left(1+\frac{Q_b}{15Q}\right)}$. **974.**

$$\frac{2\pi}{\sqrt{g\left(\frac{2C}{Q+\frac{1}{3}\gamma Fl}-\frac{1}{l}\right)}}.$$

975. $2\pi\sqrt{\frac{I_m a}{2GI_p}\left(1+\frac{2\gamma I_p a}{3I_m g}\right)}$. **976.** 2000 r.p.m.

977. $A=\frac{P_0Fq}{Cg-Q\omega_0^2}$. **978.** 1.65 mm. **979.** 0.8 mm. **980.** 0.54 rad. **981.** 20 cm; 2160 kgf/cm^2. **982.** 550 kgf/cm^2. **983.** 510 kgf/cm^2. **984.** 2.1 kgf. **985.** 12.8 cm. **986.** $\sigma_{d_I}=\frac{1}{2}\sigma_{d_{II}}=\sqrt{\frac{QhE}{3Fa}}$; $\delta_d^B=\sqrt{\frac{4Qha}{3EF}}$. **987.** $\sqrt{\frac{18Qh}{C_I+4C_{II}}}$. **988.** (a) $\frac{Q}{C}\left(1+\sqrt{1+\frac{2Ch}{Q}}\right)$; (b) $v\sqrt{\frac{Q}{Cg}}$; (c) $\frac{Q}{C}\sin\alpha\left(1+\sqrt{1+\frac{2Ch}{Q\sin\alpha}}\right)$. **989.** $\frac{[\sigma]^2W^2a}{2QEI}$. **990.** $\sqrt{\frac{20Qha^3}{EI}}$. **991.** $av\sqrt{\frac{Qa}{2gEI}}$. **992.** 33.2 kgf. **993.** $\frac{[\sigma]^2W^2l}{6QEI}$. **994.** $\frac{5.2}{W}\sqrt{\frac{QhEI}{a}}$. **995.** $\frac{1.65}{W}\sqrt{\frac{QhEI}{a}}$. **996.** $7.46\sqrt{\frac{Qha^3}{EI}}$. **997.** $\frac{1}{W}\sqrt{\frac{1.6QhEI}{\pi a}}$. **998.** $v\sqrt{\frac{0.0379Qa^3}{gEI}}$. **999.** $\frac{1.88}{W}\sqrt{\frac{QhEI}{a}}$. **1000.** 9.3 MN/m^2.

1001. $\sqrt{\frac{3.84Qha^3}{EI}}$. **1002.** $\sqrt{\frac{2Qha^3}{3GI_p}}$. **1003.** $2\sqrt{\frac{Qha^3}{3GI_p}}$. **1005.** (1) $a=\frac{l}{2}$; (2) $a=(2-\sqrt{2})\,l$. **1006.** (a) $\frac{4\omega}{d}\sqrt{\frac{I_mG}{\pi l}}$; (b) $\frac{4\omega}{d}\sqrt{\frac{2I_mG}{\pi l}}$. **1007.** $\frac{4\omega}{d}\sqrt{\frac{I_mG}{\pi a}}$. **1008.** 1260 kgf/cm^2. **1009.** $\sqrt{\frac{6QE}{5F}}$. **1010.** 337 kgf/cm^2; 0.78 mm. **1011.** 336 kgf/cm^2. **1012.** $h=\frac{W^2l\,[\sigma]^2}{6QEI}\left(1+\frac{13}{35}\cdot\frac{Q_0}{Q}\right)$; more accurately $h=\frac{W^2l\,[\sigma]^2}{6QEI}\left(1+\frac{13}{35}\cdot\frac{Q_0}{Q}\right)\times\left(1-\frac{Ql}{4W[\sigma]}\right)$. **1013.** $\omega\sqrt{\frac{2I_mG}{V\left(\frac{3}{2}+\frac{I_B}{3I_m}\right)}}$, where $V=\frac{\pi d^2}{4}l$. **1014.** $\sigma_{d_I}=366$ kgf/cm^2; $\sigma_{d_{II}}=512$ kgf/cm^2; $\sigma_{d_{III}}=720$ kgf/cm^2. **1015.** 1 MN/m^2. **1016.** ~12 mm. **1017.** ~1.74. **1018.** 6.25 kN. **1019.** The shaft made of steel, Grade Ст. 5, is stronger. **1020.** $\sigma_{-0.6}\cong 80$ kgf/mm^2; $\sigma_{0.6}\cong 80$ kgf/cm^2. **1021.** ~26 kgf/mm^2. **1022.** ~1.38. **1023.** ~ −0.5. **1024.** ~7.8 kgf/mm^2. **1025.** ~118 kgf-m. **1026.** ~1.3. **1027.** ~9.2 kN. **1028.** ~1.47.

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